Approximation in Mechanism Design

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http://www.eecs.northwestern.edu/~hartline/amd.pdf

Mechanism Design

Basic Mechanism Design Question: How should an economic system be designed so that selfish agent behavior leads to good outcomes?

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General Theme: resource allocation.

Overview

Part I: Optimal Mechanism Design

- single-item auction.
- objectives: social welfare vs. seller profit.
- characterization of Bayes-Nash equilibrium.
- consequences: solving, uniqueness, and optimizing over BNE.

Part II: Approximation in Mechanism Design

- single-item auctions.
- multi-dimensional auctions.
- prior-free auctions.
- computationally tractable mechanisms.

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Part II: Approximation in Mechanism Design

- single-item auctions. (Chapter 4)
- multi-dimensional auctions. (Chapter 7)
- prior-free auctions. (Chapters 5 & 6)
- computationally tractable mechanisms. (Chapter 8)

Mechanism Design Problem: *Single-item Auction*

Given:

- one item for sale.
- n bidders (with unknown private values for item, $v_1, \ldots, v_n$)
- Bidders' objective: maximize *utility* $=$ value $-$ price paid.

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- Auction to solicit bids and choose winner and payments.

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Possible Auction Objectives:

- Maximize *social surplus*, i.e., the value of the winner.
- Maximize *seller profit*, i.e., the payment of the winner.

Objective 1: maximize social surplus

Example Auctions

First-price Auction

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Questions:

- what are equilibrium strategies?
- what is equilibrium outcome?
- which has higher surplus in equilibrium?
- which has higher profit in equilibrium?

Second-price Auction Equilibrium Analysis

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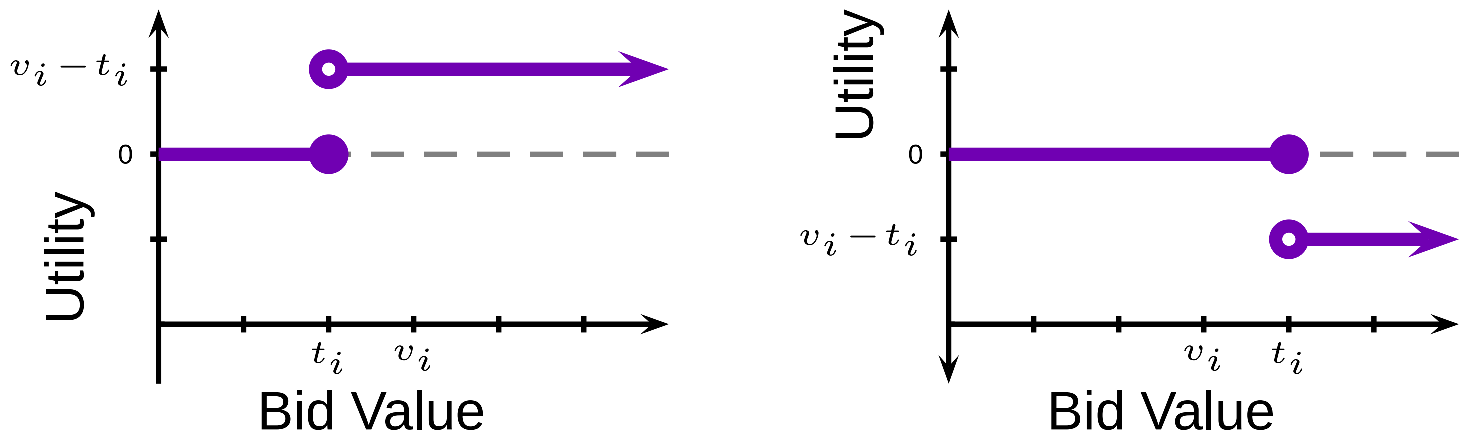

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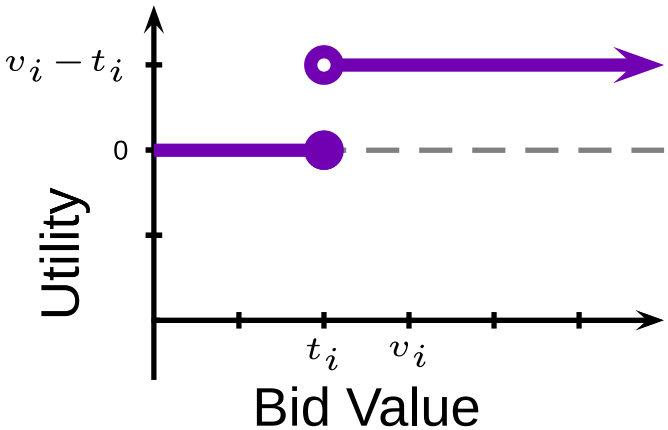

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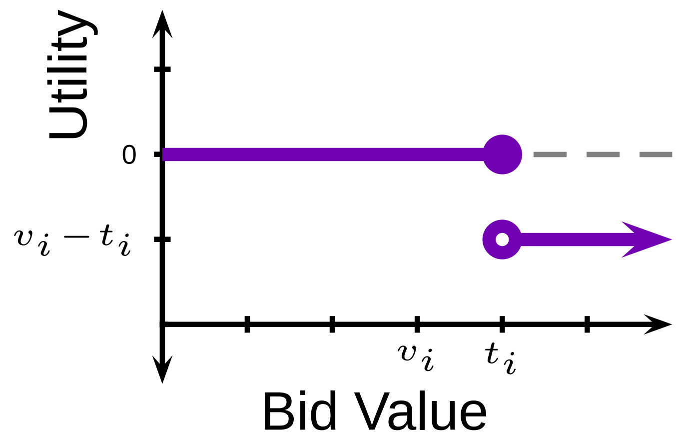

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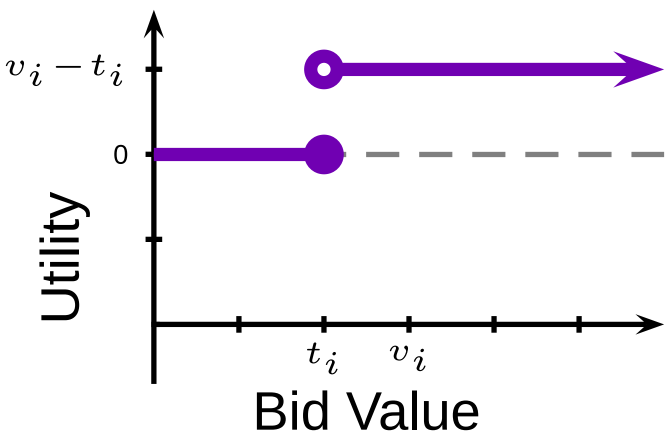

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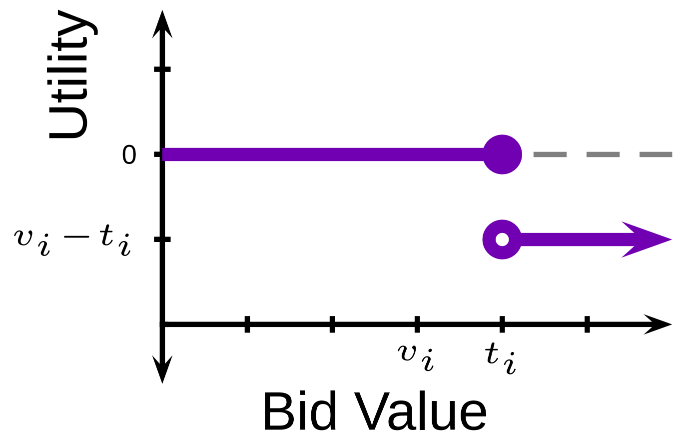

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What about first-price auction?

Recall First-price Auction

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How would you bid?

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Note: first-price auction has no DSE.

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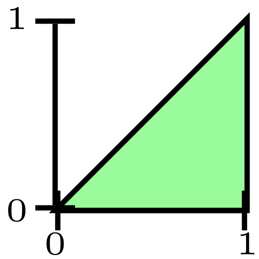

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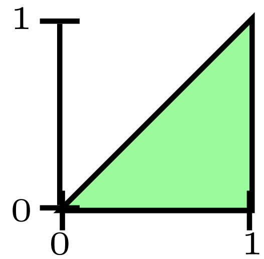

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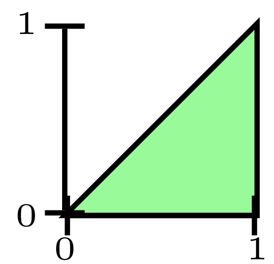

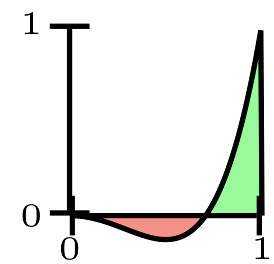

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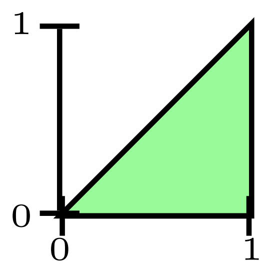

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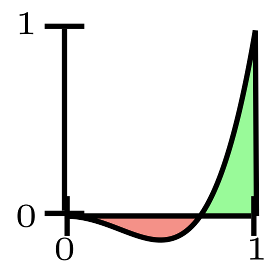

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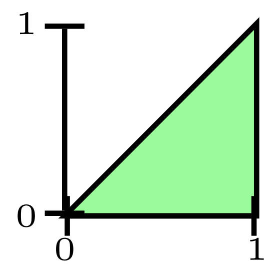

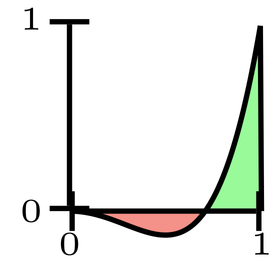

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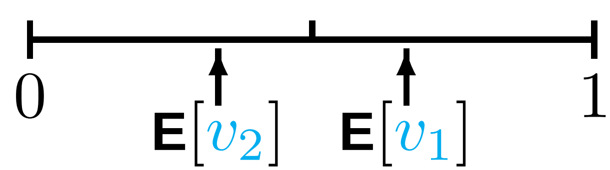

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Conclusion 2: bidder with highest value wins

Conclusion 3: first-price auction maximizes social surplus!

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Definition: a *strategy profile* is in *Bayes-Nash Equilibrium (BNE)* if for all i, $s_i(v_i)$ is best response when others play $s_j(v_j)$ and $v_j \sim F_j$.

Surplus Maximization Conclusions

Conclusions:

- second-price auction maximizes surplus in DSE regardless of distribution.
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Questions?

Objective 2: maximize seller profit

(other objectives are similar)

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Example Scenario: two bidders, uniform values

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What is profit of second-price auction?

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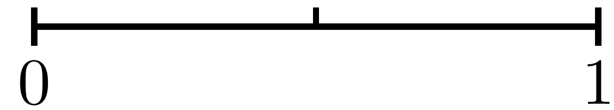

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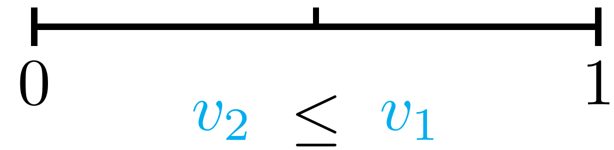

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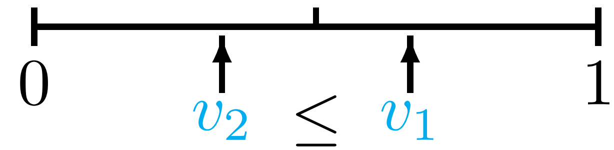

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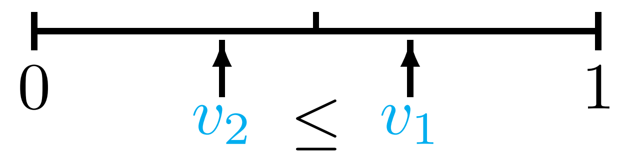

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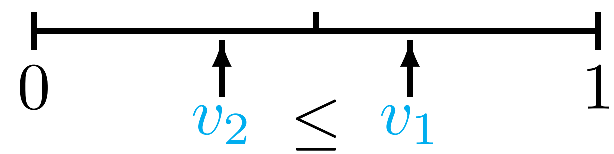

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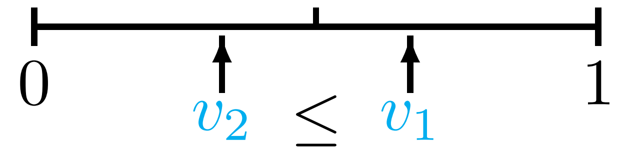

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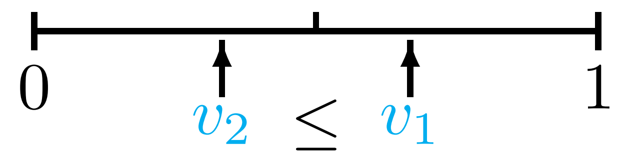

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What is profit of first-price auction?

- $\mathbf{E}[\text{Profit}] = \mathbf{E}[v_1]/2 = 1/3.$

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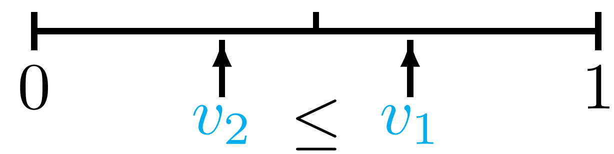
Example Scenario: two bidders, uniform values

What is profit of second-price auction?



- draw values from unit interval.
- Sort values.
- In expectation, values evenly divide unit interval.
- $\mathbf{E}[\text{Profit}] = \mathbf{E}[v_2] = 1/3.$

What is profit of first-price auction?

- $\mathbf{E}[\text{Profit}] = \mathbf{E}[v_1]/2 = 1/3.$

Surprising Result: second-price and first-price auctions have same expected profit.

Can we get more profit?

Second-price with reserve price

Second-price Auction with reserve r

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Case 1: $\frac{1}{2} > v_1 \geq v_2$		
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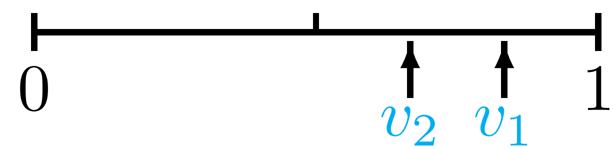

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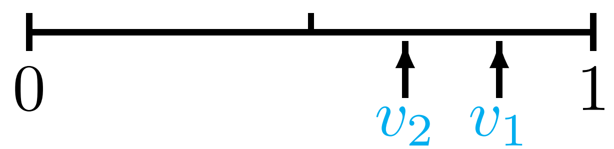

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E[profit of 2nd-price with reserve] $= \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{12}$

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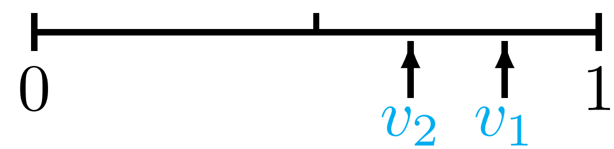

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$\geq$ **E**[profit of 2nd-price] $= \frac{1}{3}$.

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Observations:

- pretending to value the good increases seller profit.
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Questions?

Bayes-Nash Equilibrium Characterization and Consequences

- solving for BNE
- uniqueness of BNE
- optimizing over BNE

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- $\mathbf{x}$ is an allocation, x_i the allocation for i.
- $\mathbf{x}(\mathbf{v})$ is BNE allocation of mech. on valuations $\mathbf{v}$.
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Analogously, define $\mathbf{p}$, $\mathbf{p}(\mathbf{v})$, and $p_i(v_i)$ for payments.

Characterization of BNE

Thm: a mechanism and strategy profile is in BNE iff

1. *monotonicity (M)*: $x_i(v_i)$ is monotone in v_i.
2. *payment identity (PI)*: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$.

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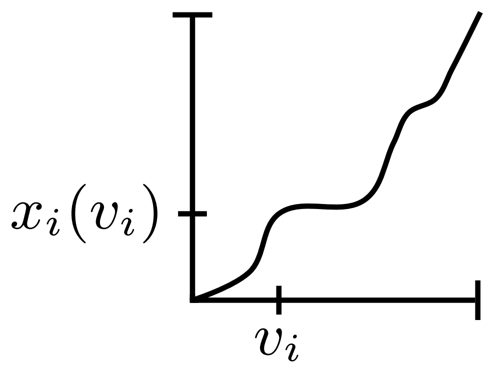

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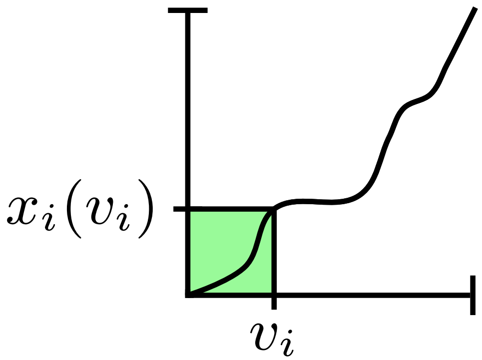

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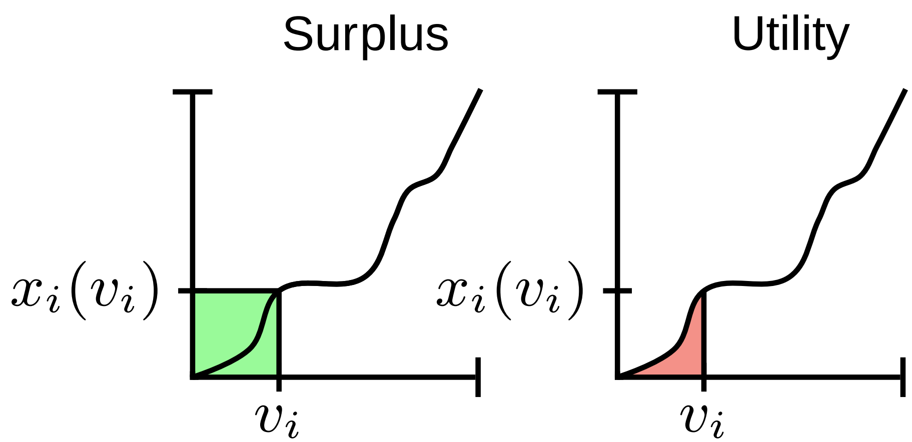

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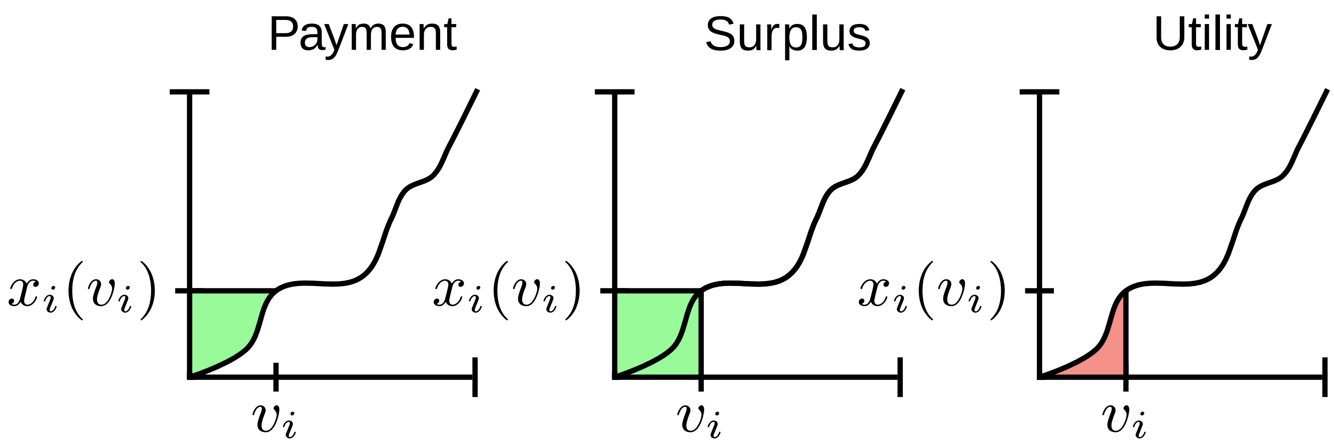

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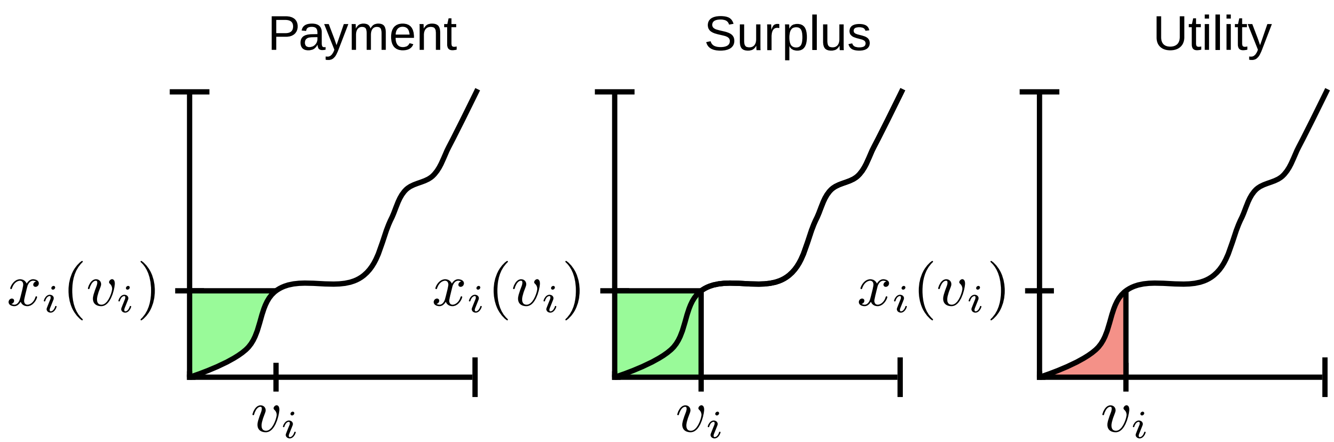

Consequence: *(revenue equivalence)* in BNE, auctions with same outcome have same revenue (e.g., first and second-price auctions)

Questions?

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3. Verify guess and BNE: $b(v)$ continuous, strictly increasing, symmetric.

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Non-essential Assumption: bid functions are *continuous* and *strictly increasing*.

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Proof of Corollary:

- player 1 & 2 face random reserve "$\max(b_3, \ldots, b_n)$"
- by theorem, their bid function is symmetric.
- same for player 1 and i.
- so all bid functions are symmetric.

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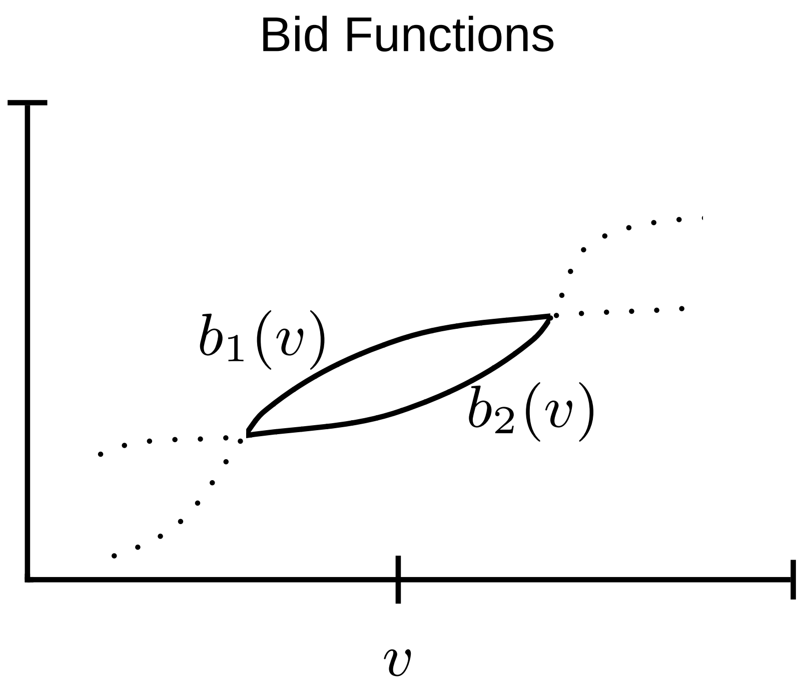

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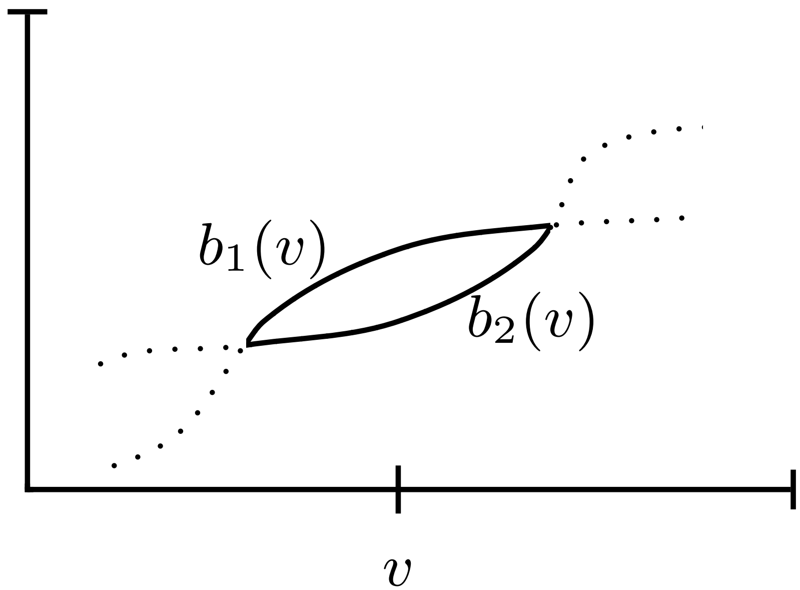

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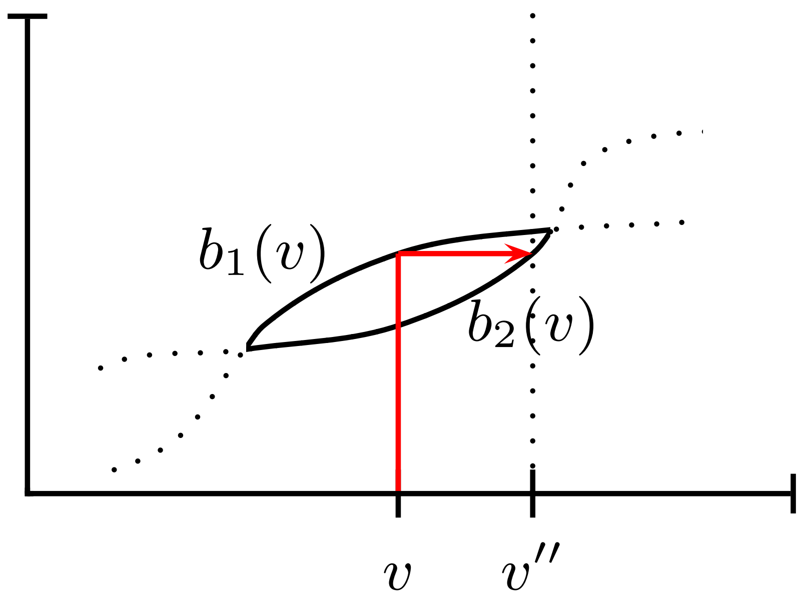

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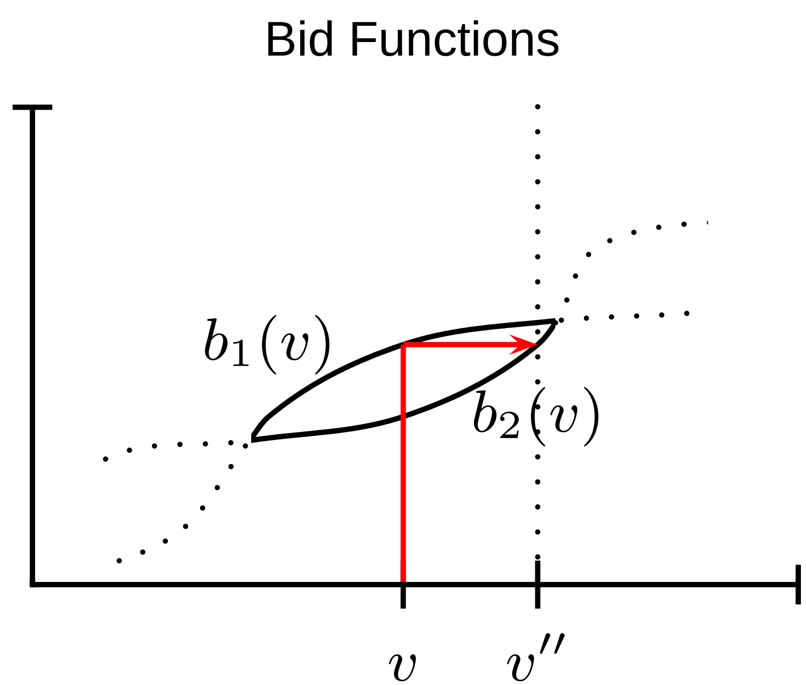

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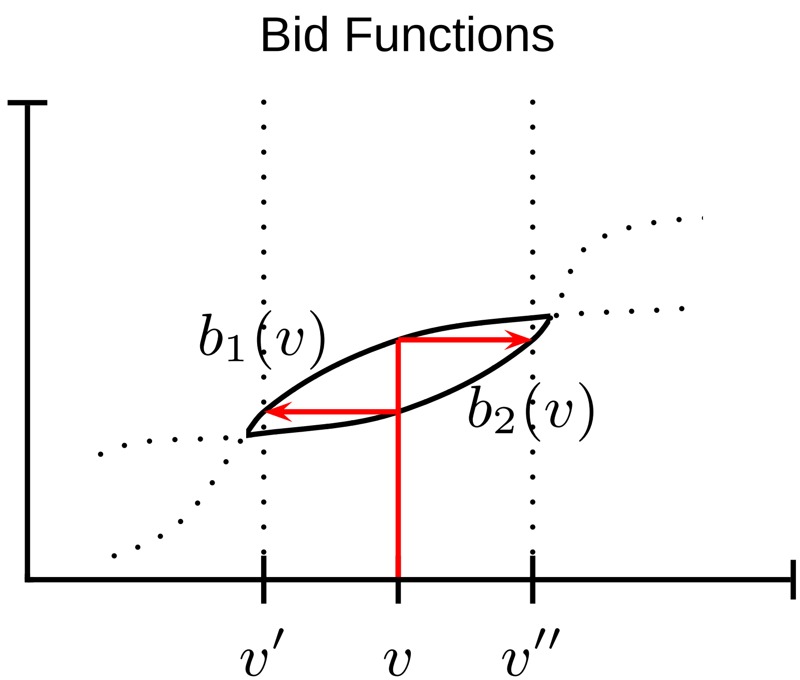

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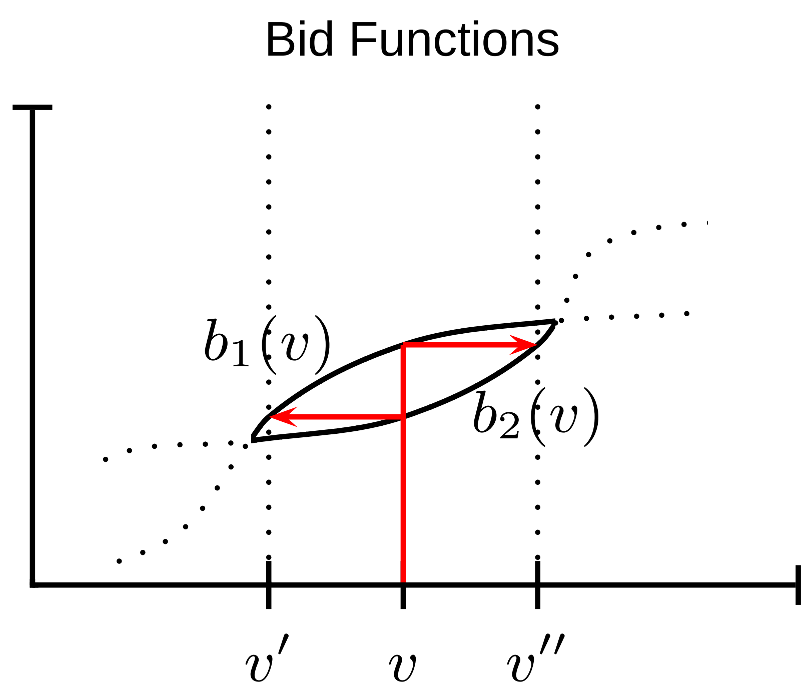

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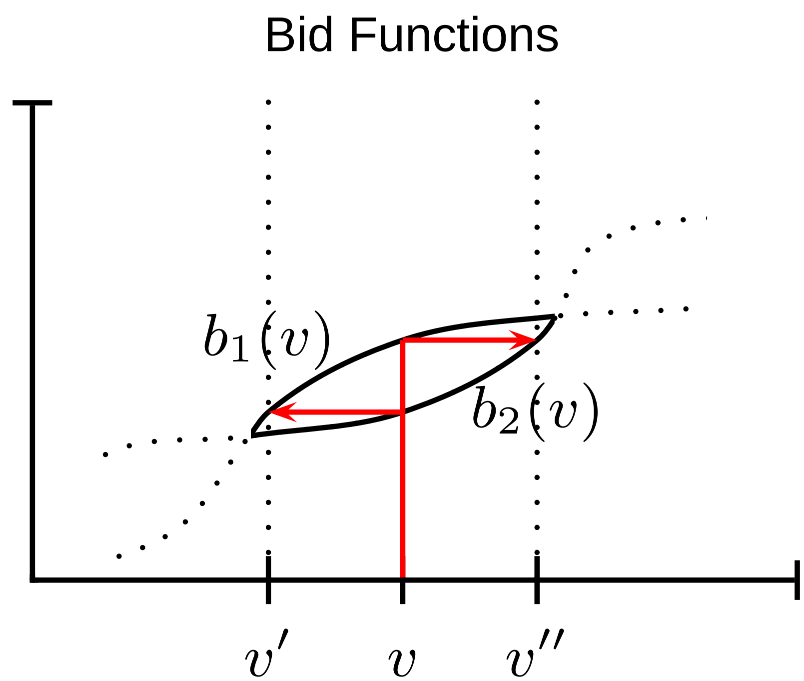

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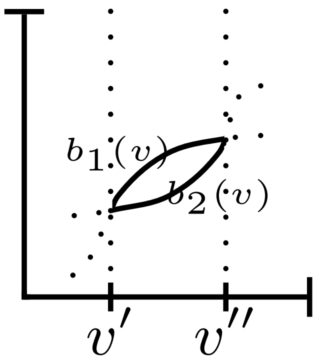
Bid Functions



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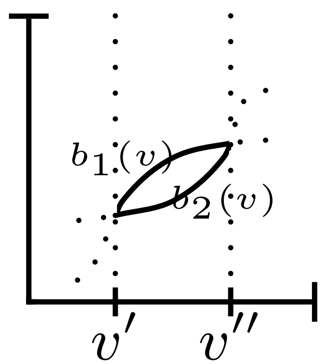

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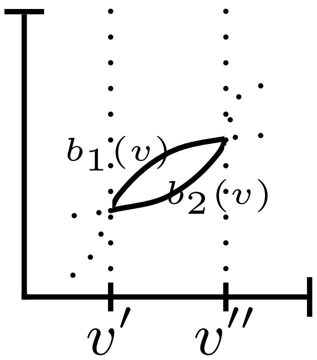

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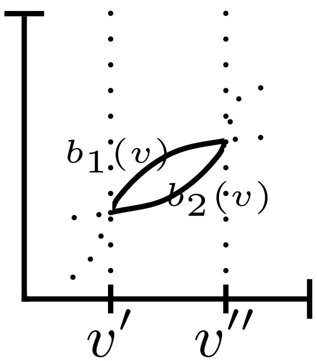

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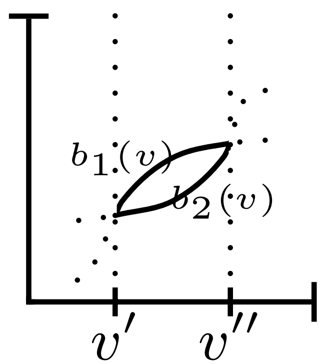

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- but $u_1(v'') - u_1(v') = u_2(v'') - u_2(v')$ by Claim 1.

Questions?

Optimizing BNE

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Proof: expected virtual valuation of winner $=$ expected payment.

Proof of Lemma

Recall Lemma: In BNE, $\mathbf{E}[p_i(v_i)] = \mathbf{E}\left[\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right) x_i(v_i)\right]$.

Proof Sketch:

- Use characterization: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(v)dv$.
- Use definition of expectation (integrate payment $\times$ density).
- Swap order of integration.
- Simplify.

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What is optimal single-item auction for $U[0,1]$?

Optimal Auction for $U[0, 1]$

Optimal auction for $U[0, 1]$:

- $F(v_i) = v_i$.
- $f(v_i) = 1$.
- So, $\phi(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)} = 2v_i - 1$.
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Optimal Mechanisms Conclusions

Conclusions:

- expected virtual value $=$ expected revenue
- optimal mechanism maximizes virtual surplus.
- optimal auction depends on distribution.
- i.i.d., regular distributions: second-price with reserve is optimal.
- theory is “descriptive”.

Questions?

Bayes-Nash Equilibrium Characterization Proof

Proof Overview

Thm: a mechanism and strategy profile is in BNE iff

1. *monotonicity (M)*: $x_i(v_i)$ is monotone in v_i.
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$\Longrightarrow$ 1. BNE $\Leftarrow$ M & PI

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3. BNE $\Rightarrow$ PI

BNE $\Leftarrow$ M & PI

Claim: BNE $\Leftarrow$ M & PI

Case 1: mimicking $z > v_i$

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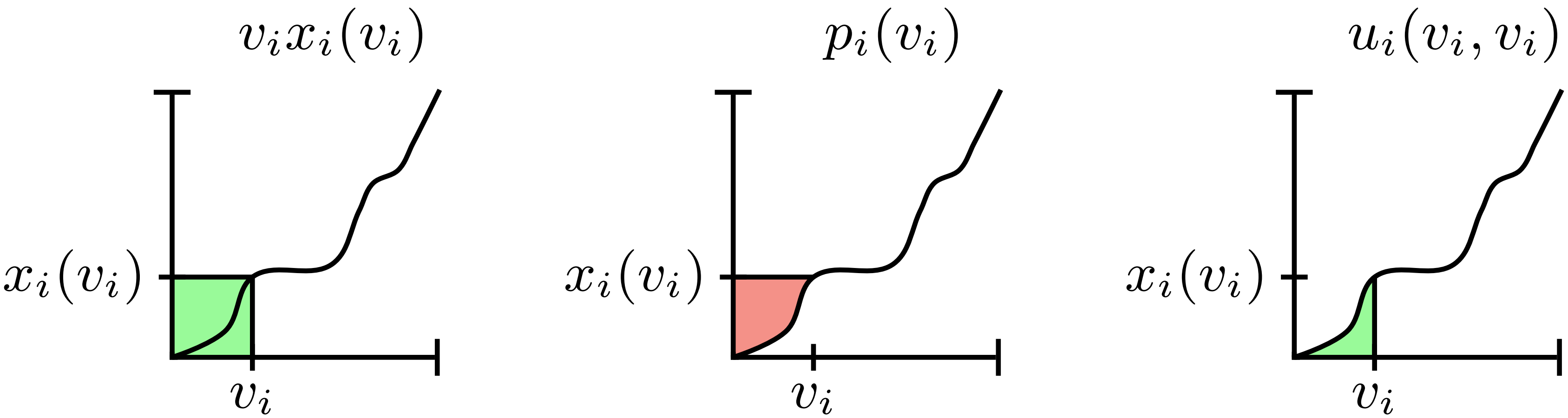

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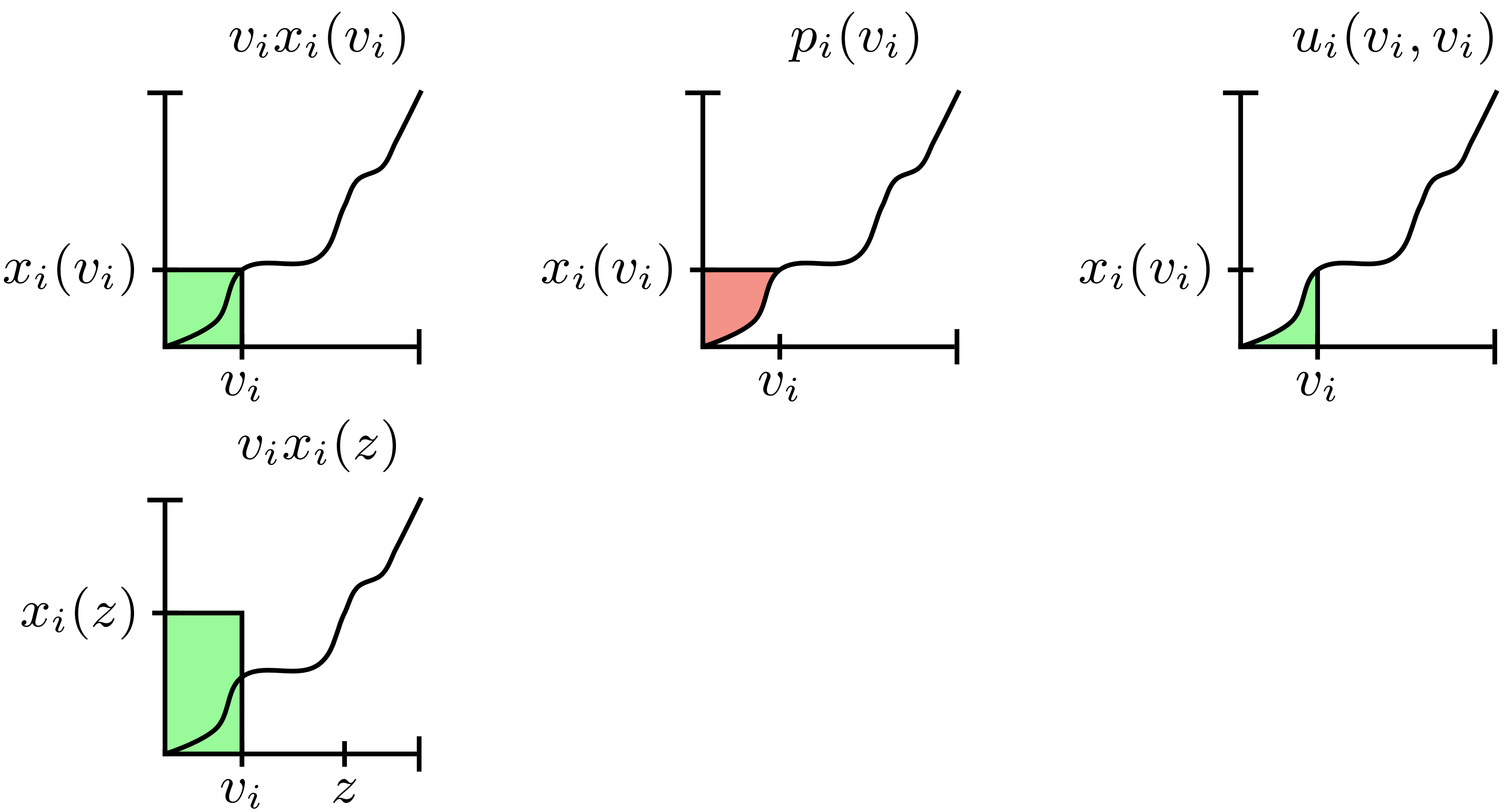

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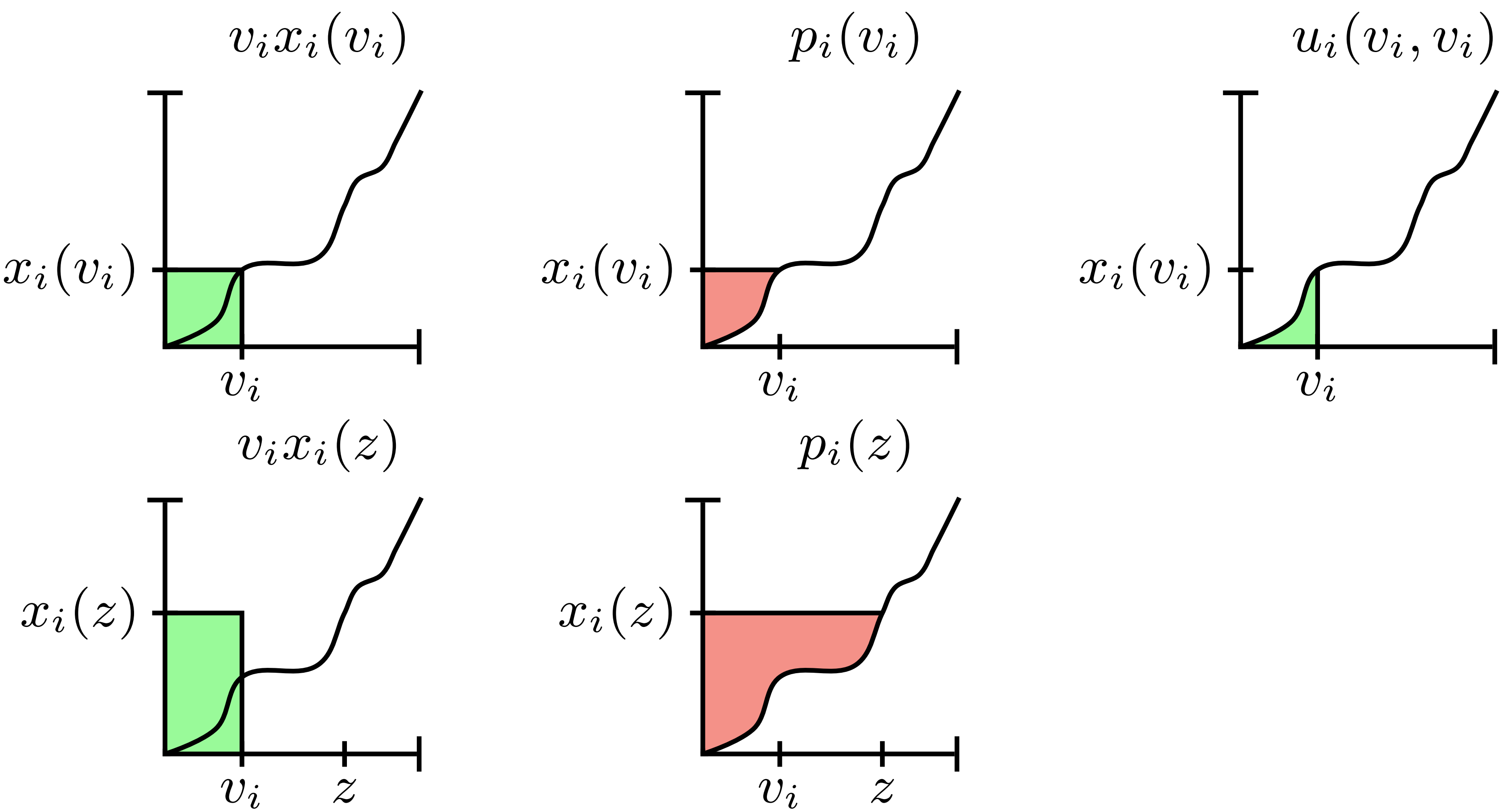

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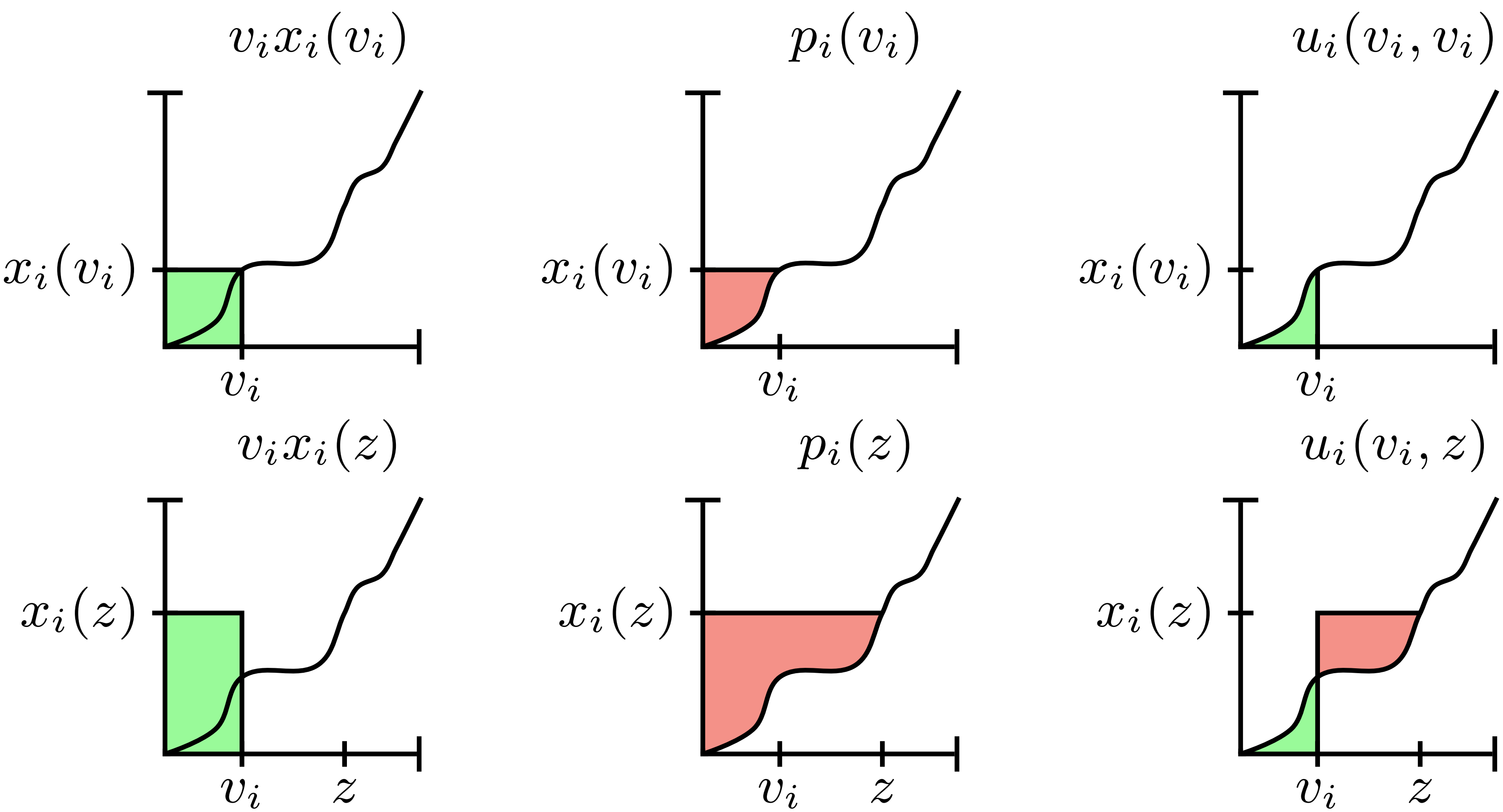

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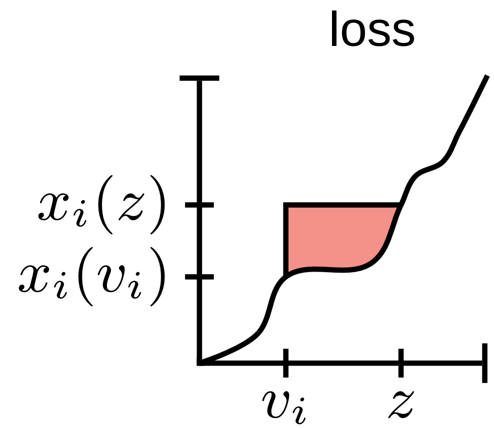

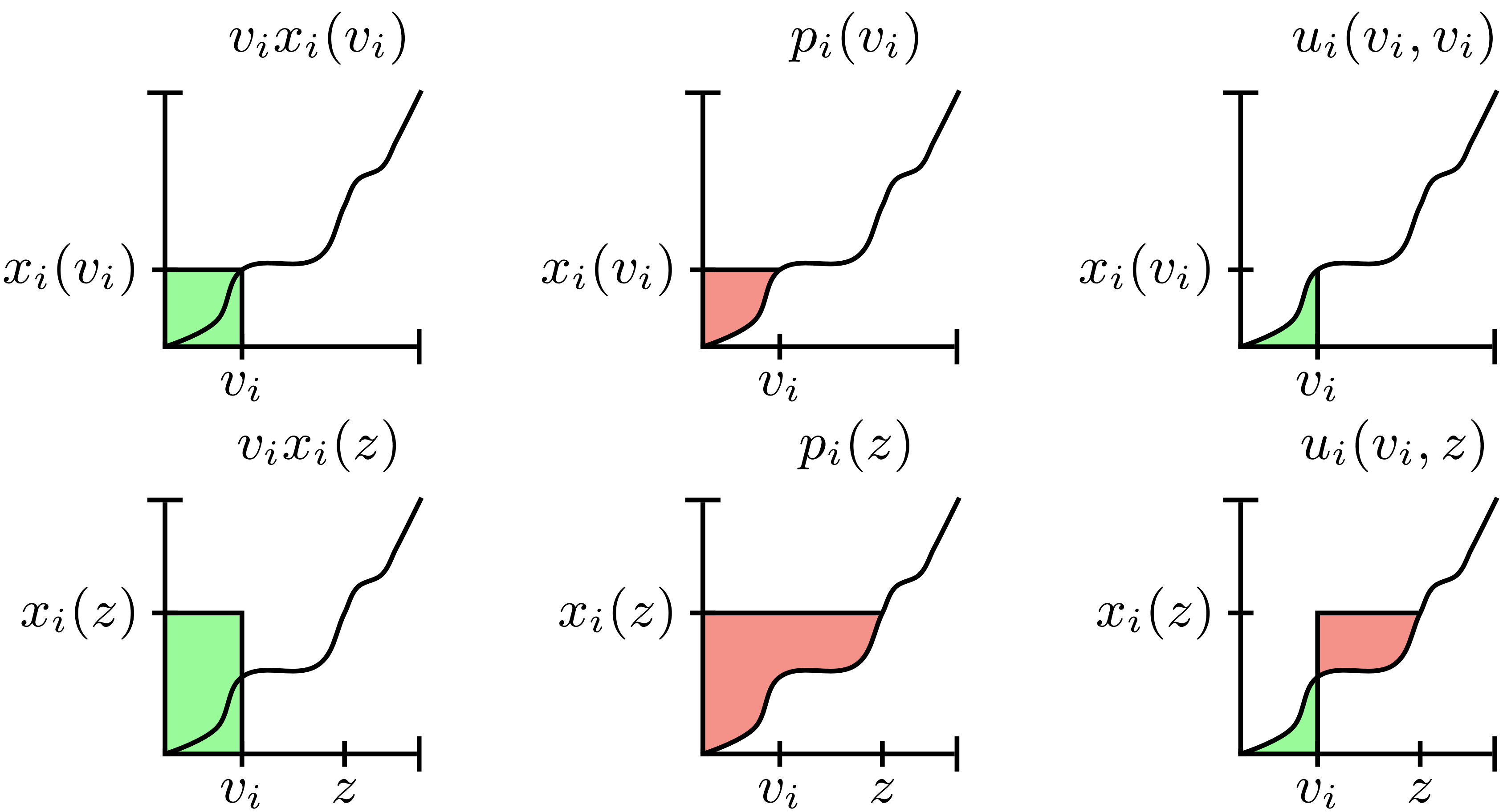

BNE $\Leftarrow$ M & PI (cont)

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Case 2: mimicking $z < v_i$

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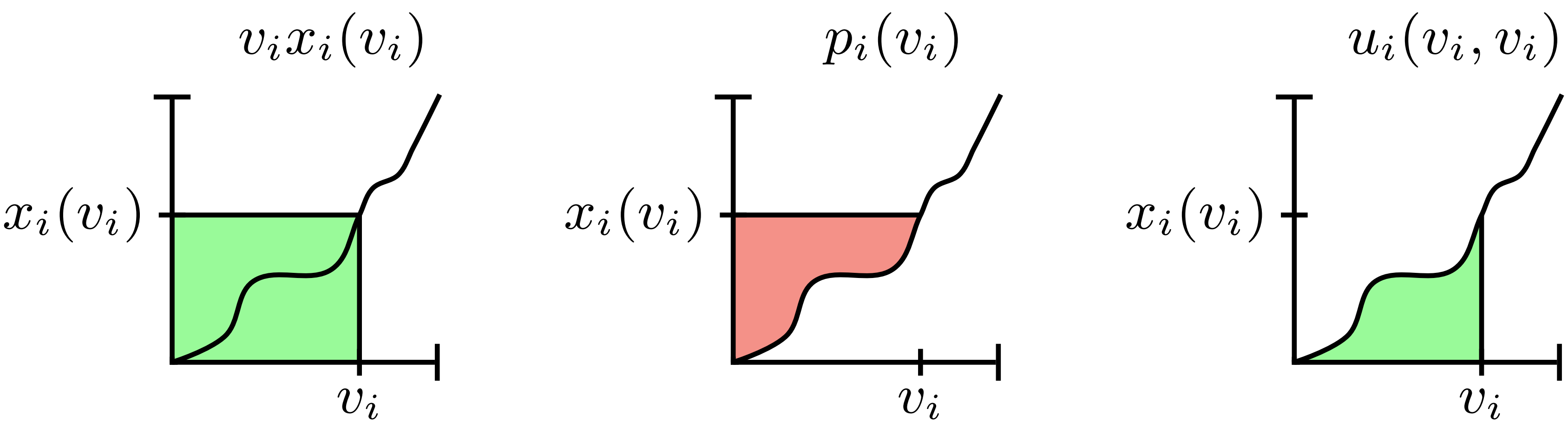

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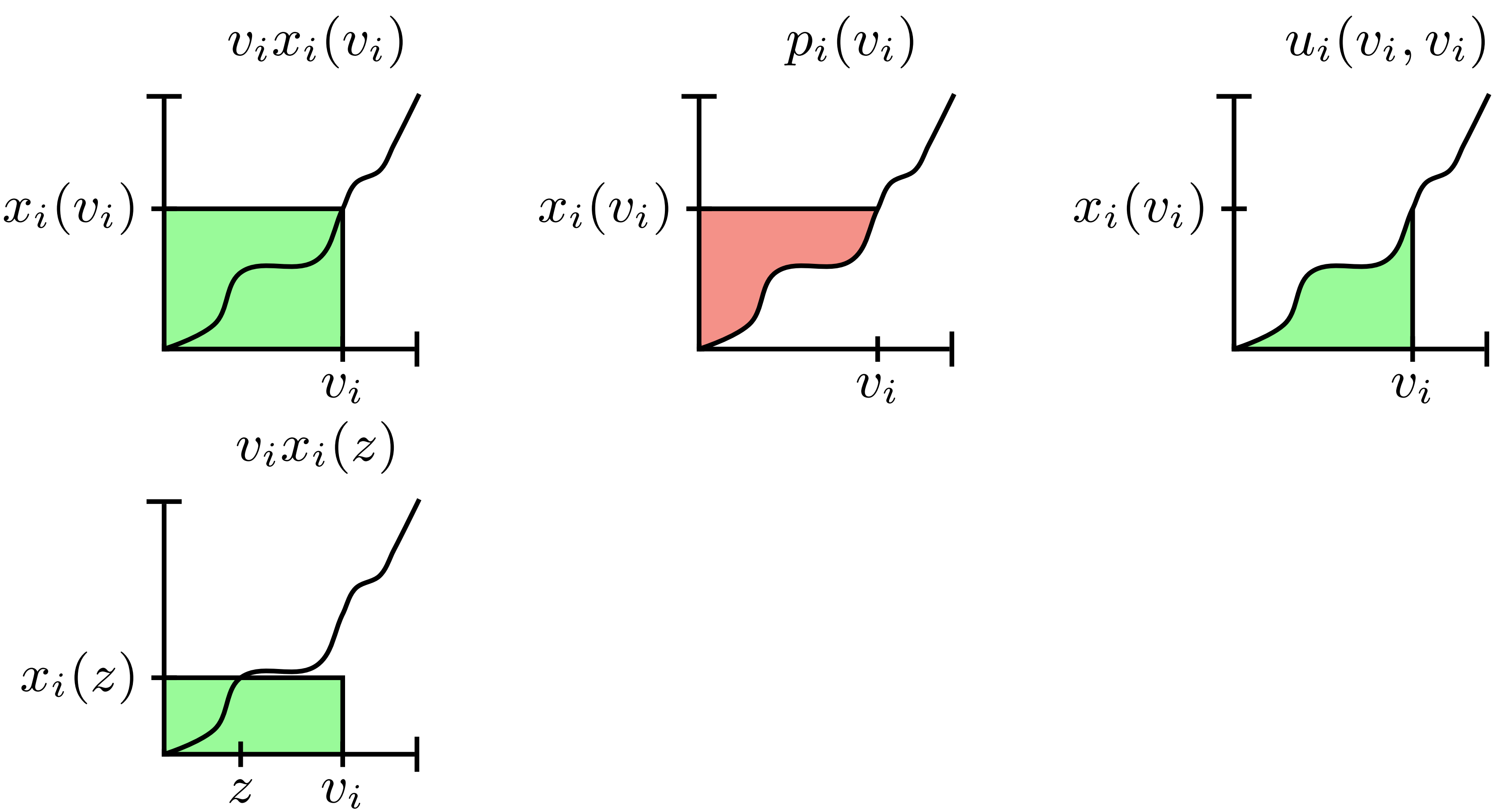

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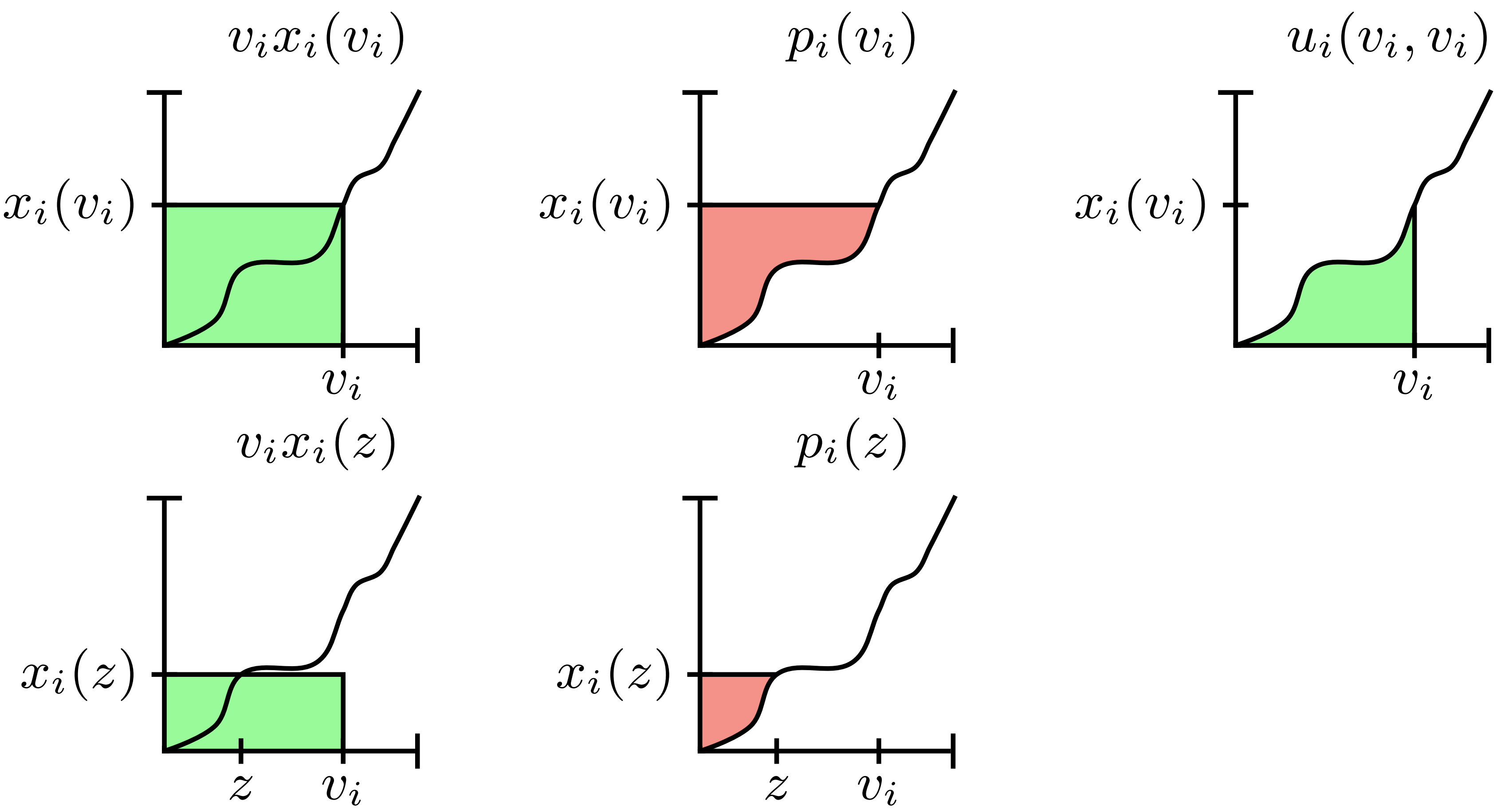

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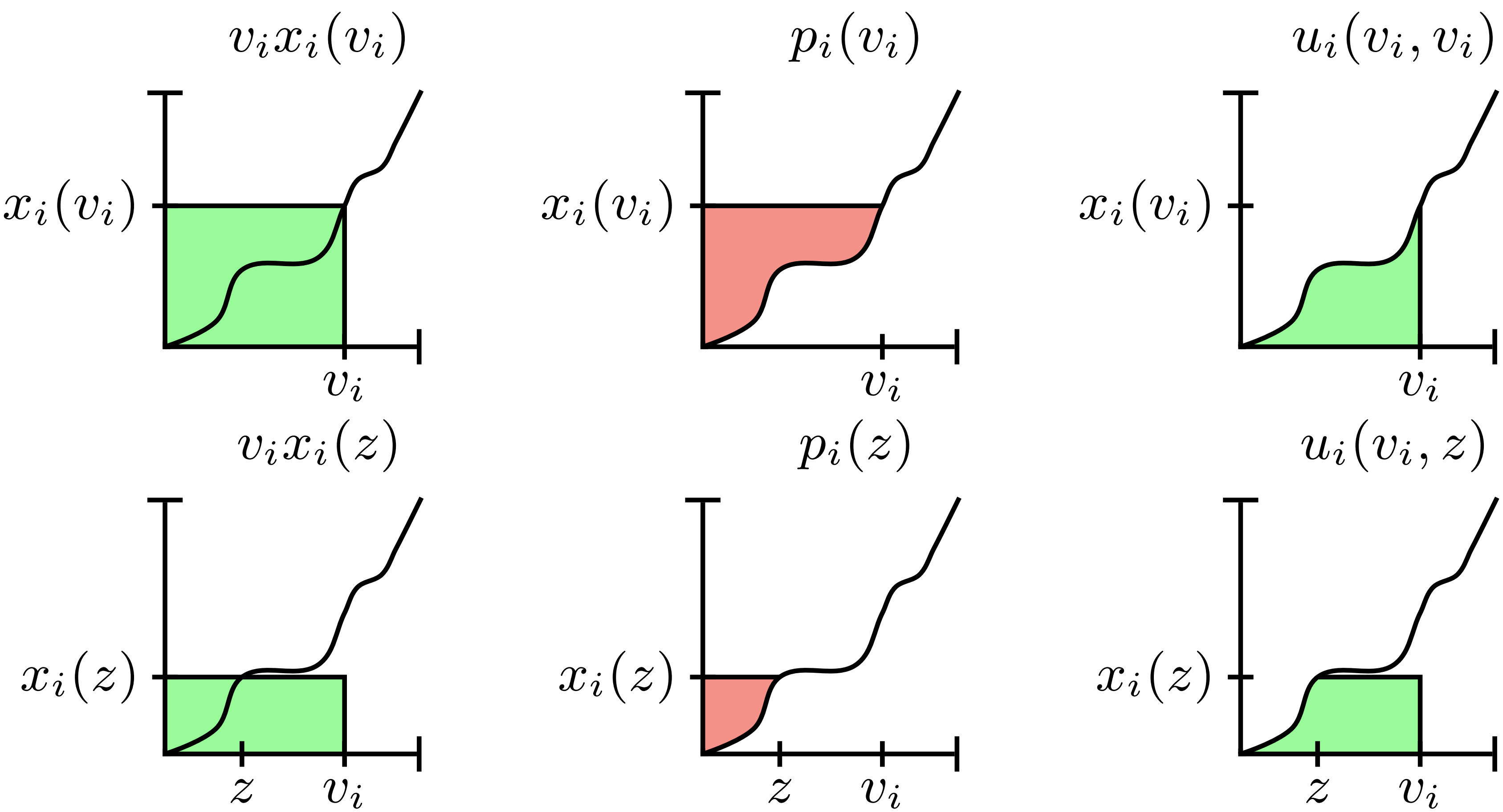

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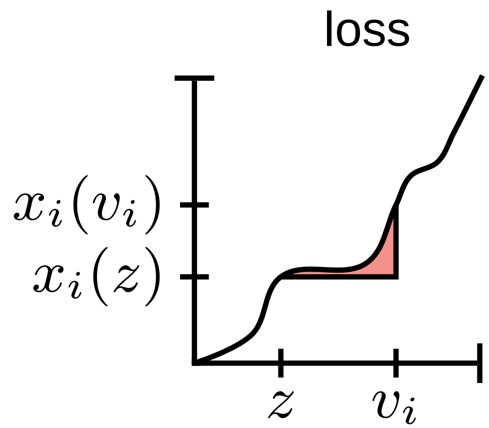

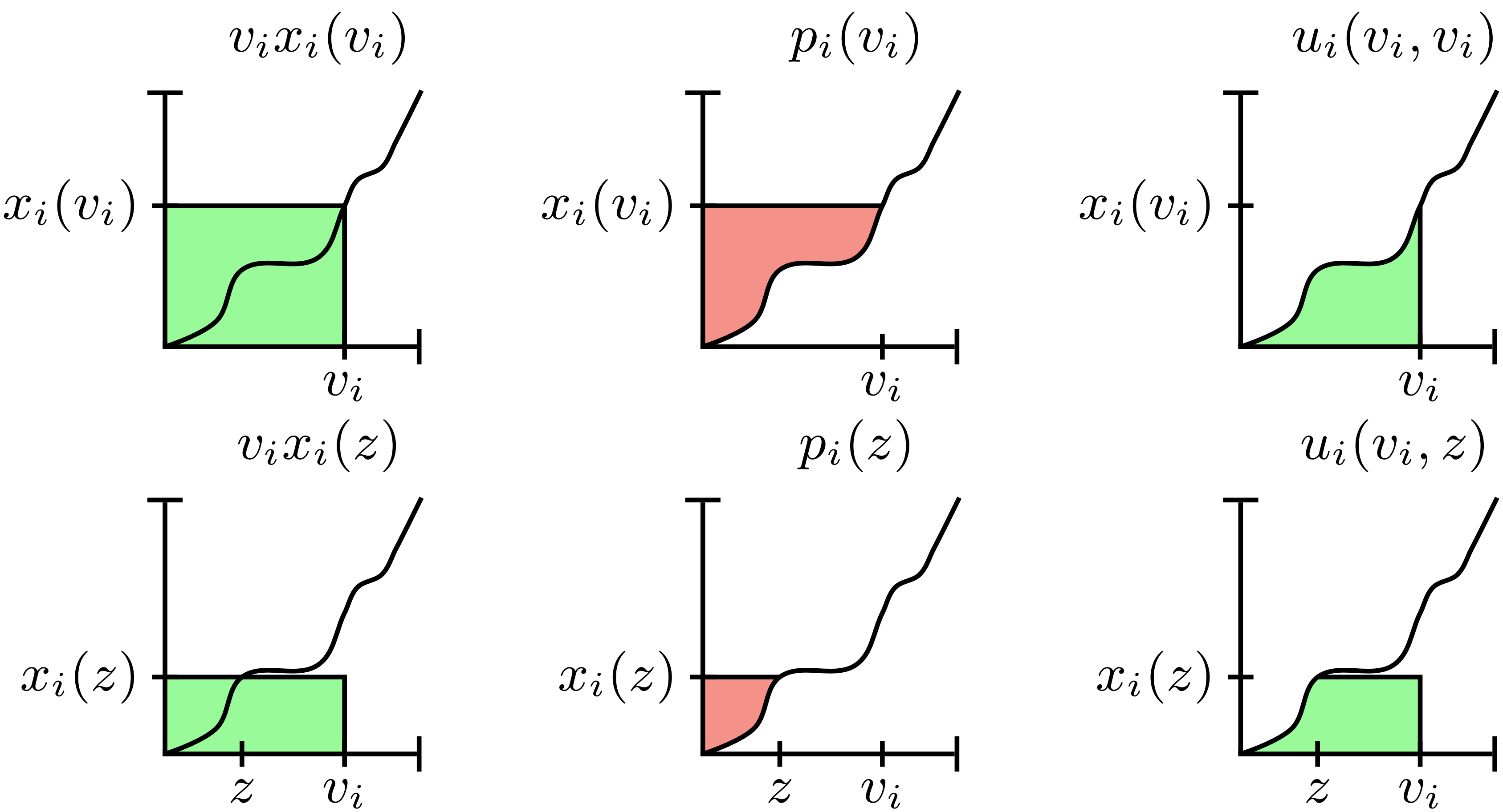

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- Regroup:

$$(z'' - z')(x_i(z'') - x_i(z')) \geq 0$$

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$$z'x_i(z') - p_i(z') \geq z'x_i(z'') - p_i(z'')$$

- Add and cancel payments:

$$z''x_i(z'') + z'x_i(z') \geq z''x_i(z') + z'x_i(z'')$$

- Regroup:

$$(z'' - z')(x_i(z'') - x_i(z')) \geq 0$$

- So $x_i(z)$ is monotone:

$$z'' - z' > 0 \Rightarrow x(z'') \geq x(z')$$

Proof Overview

Thm: a mechanism and strategy profile is in BNE iff

1. *monotonicity (M)*: $x_i(v_i)$ is monotone in v_i.
2. *payment identity (PI)*: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$.

and usually $p_i(0) = 0$.

Proof Overview:

1. BNE $\Leftarrow$ M & PI
2. BNE $\Rightarrow$ M
3. BNE $\Rightarrow$ PI

BNE $\Rightarrow$ PI

Claim: BNE $\Rightarrow$ PI.

BNE $\Rightarrow$ PI

Claim: BNE $\Rightarrow$ PI.

- BNE $\Rightarrow u_i(v_i, v_i) \geq u_i(v_i, z)$

BNE ⇒ PI

Claim: BNE ⇒ PI.

- BNE ⇒ $u_i(v_i, v_i) \geq u_i(v_i, z)$
- Take $v_i = z'$ and $z = z''$ and vice versa:

$$z''x_i(z'') - p_i(z'') \geq z''x_i(z') - p_i(z')$$
$$z'x_i(z') - p_i(z') \geq z'x_i(z'') - p_i(z'')$$

BNE $\Rightarrow$ PI

Claim: BNE $\Rightarrow$ PI.

- BNE $\Rightarrow u_i(v_i, v_i) \geq u_i(v_i, z)$
- Take $v_i = z'$ and $z = z''$ and vice versa:

$$z''x_i(z'') - p_i(z'') \geq z''x_i(z') - p_i(z')$$
$$z'x_i(z') - p_i(z') \geq z'x_i(z'') - p_i(z'')$$

- solve for $p_i(z'') - p_i(z')$:

$$z''x_i(z'') - z''x_i(z') \geq p_i(z'') - p_i(z') \geq z'x_i(z'') - z'x_i(z')$$

- Picture:

Claim: BNE ⇒ PI.

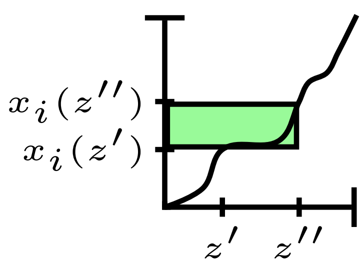
- BNE ⇒ $u_i(v_i, v_i) \geq u_i(v_i, z)$
- Take $v_i = z'$ and $z = z''$ and vice versa:

$$z''x_i(z'') - p_i(z'') \geq z''x_i(z') - p_i(z')$$
$$z'x_i(z') - p_i(z') \geq z'x_i(z'') - p_i(z'')$$

- solve for $p_i(z'') - p_i(z')$:

$$z''x_i(z'') - z''x_i(z') \geq p_i(z'') - p_i(z') \geq z'x_i(z'') - z'x_i(z')$$

- Picture:



upper bound

BNE ⇒ PI

Claim: BNE ⇒ PI.

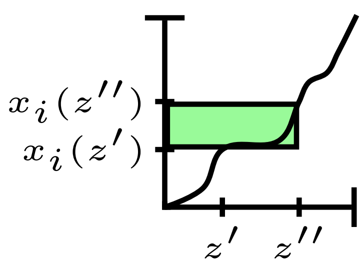
- BNE ⇒ $u_i(v_i, v_i) \geq u_i(v_i, z)$
- Take $v_i = z'$ and $z = z''$ and vice versa:

$$z'' x_i(z'') - p_i(z'') \geq z'' x_i(z') - p_i(z')$$
$$z' x_i(z') - p_i(z') \geq z' x_i(z'') - p_i(z'')$$

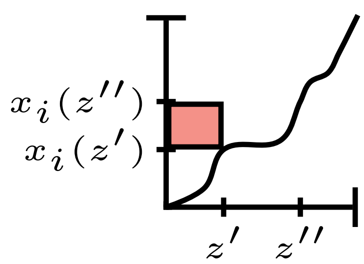
- solve for $p_i(z'') - p_i(z')$:

$$z'' x_i(z'') - z'' x_i(z') \geq p_i(z'') - p_i(z') \geq z' x_i(z'') - z' x_i(z')$$

- Picture:



upper bound



lower bound

BNE $\Rightarrow$ PI

Claim: BNE $\Rightarrow$ PI.

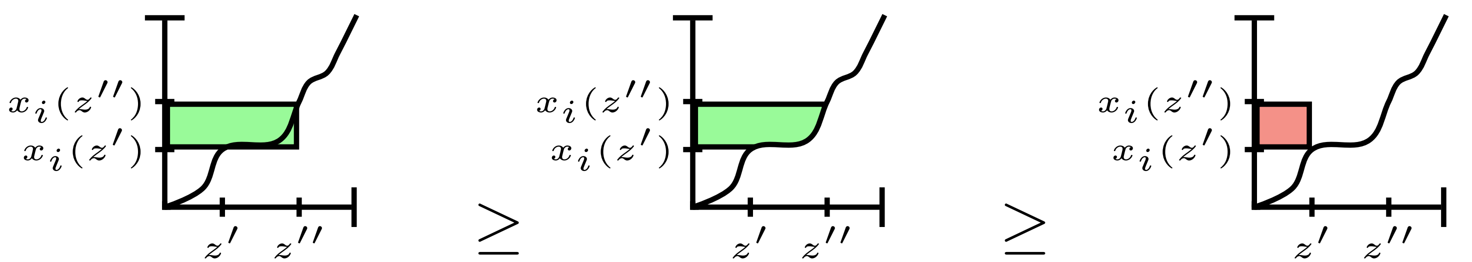
- BNE $\Rightarrow u_i(v_i, v_i) \geq u_i(v_i, z)$
- Take $v_i = z'$ and $z = z''$ and vice versa:

$$z''x_i(z'') - p_i(z'') \geq z''x_i(z') - p_i(z')$$
$$z'x_i(z') - p_i(z') \geq z'x_i(z'') - p_i(z'')$$

- solve for $p_i(z'') - p_i(z')$:

$$z''x_i(z'') - z''x_i(z') \geq p_i(z'') - p_i(z') \geq z'x_i(z'') - z'x_i(z')$$

- Picture:



upper bound $\geq$ only solution $\geq$ lower bound

Thm: a mechanism and strategy profile is in BNE iff

1. *monotonicity (M)*: $x_i(v_i)$ is monotone in v_i.
2. *payment identity (PI)*: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$.

and usually $p_i(0) = 0$.

Questions?

Research Directions

Research Directions:

- are there simple mechanisms that are approximately optimal?
  (e.g., price of anarchy or price of stability)
- is the optimal mechanism tractible to compute (even if it is complex)?
- what are optimal auctions for multi-dimensional agent preferences?
- what are the optimal auctions for non-linear agent preferences, e.g., from budgets or risk-aversion?
- are there good mechanisms that are less dependent on distributional assumptions?

BNE and Auction Theory Homework

1. For two agents with values $U[0, 1]$ and $U[0, 2]$, respectively:
   (a) show that the first-price auction is not socially optimal in BNE.
   (b) give an auction with "pay your bid if you win" semantics that is.
2. What is the virtual value function for an agent with value $U[0, 2]$?
3. What is revenue optimal single-item auction for:
   (a) two agents with values $U[0, 2]$? n agents?
   (b) two agents with values $U[a, b]$?
   (c) two values $U[0, 1]$ and $U[0, 2]$, respectively?
4. For n agents with values $U[0, 1]$ and a *public good*, i.e., where either all or none of the agents can be served,
   (a) What is the revenue optimal auction?
   (b) What is the expected revenue of the optimal auction? (use big-oh notation)

`http://www.eecs.northwestern.edu/~hartline/amd.pdf`