Fair division, Part 1

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Fair division: generalities

equals should be treated equally, and unequals unequally, according to relevant similarities and differences

what are the relevant differences?

horizontal equity: *equal treatment of equals*

differences

- call for compensation when agents are not responsible for creating them,

- call for reward/penalty when agents are responsible for creating them

example: divide a single resource, concave utilities: expensive tastes, versus handicaps versus talents: who gets a bigger share?

information about own characteristics

as in general mechanism design

$\longrightarrow$ public information gives full flexibility to the benevolent dictator (BD), a.k.a. planner, central authority, system manager

$\longrightarrow$ dispersed private information yield incentives to strategic distortions by agents, and limits the freedom of the BD

consequence: cardinal measures of utility are only meaningful under public information; private information on preferences forces the BD to use ordinal data only *efficiency*: economics' trademark

a requirement neutral w.r.t. fairness

contents of Part 1

1. *welfarism*: end state justice based on cardinal utilities for which agents are not responsible; example parents to children, triage doctor, relief distribution; the collective utility model

2. *division of manna*: full responsibility for(private) individual ordinal preferences; resources are common property (inheritance, bankruptcy, divorce)

the Arrow Debreu model of pure consumption, with elicitation of preferences

special subdomains of preferences: homogeneous, linear, Cobb Douglas and Leontief preferences

variants: non disposable single commodity; assignment with lotteries

contents of Part 2

cost (or surplus) sharing: full responsibility for individual contributions to cost/surplus

3.TU cooperative games: counterfactual Stand Alone costs or surplus determine the individual shares; the core and the Shapley value

applications to connectivity games and division of manna with cash transfers elastic demands (inputs): responsibility for own utility *and* own demand; look for fair, incentive compatible cost sharing rules with efficient equilibrium

4. exploitation of a commons with supermodular costs: incremental and serial sharing rules, versus average cost game; price of anarchy

5. exploitation of a commons with submodular costs: cross monotonic sharing rules; binary demands, optimality of the Shapley value; weaker results in the variable demands case; approximate budget balance

1 Welfarism

(see [14] for a survey)

end state distributive justice based on cardinal utilities for which agents are not responsible; example parents to children, triage doctor, relief distribution

maximizing a collective utility; a reductionist model requiring public information basic tradeoff: the egalitarian/utilitarian dilemma egalitarian $\rightarrow$ leximin social welfare ordering (SWO):

$u \succsim_{lxmin} v \overset{def}{\Leftrightarrow} u^* \succsim_{lex} v^*$ where

- $\succsim_{lex}$ is the lexicographic ordering of $\mathbb{R}^n$
- rearranging increasingly individual utilities: $\mathbb{R}^N \ni u \rightarrow u^* \in \mathbb{R}^n$

utilitarian $\rightarrow$ collective utility function (CUF) $W(u) = \sum_N u_i$, with some tie-breaking *examples*

$u_1 = 2u_2$

location of a facility on a line (or general graph) when each agent wants the facility as close as possible to home general CUFs: $W(u_1, .., u_n)$ symmetric, monotone, scale invariant

a rich subfamily: $W^p(u) = sign(p) \sum_N u_i^p$ for any given $p \in \mathbb{R}$

$\longrightarrow p = 1$: utilitarian

$\longrightarrow p = -\infty$: leximin SWO

$\longrightarrow p = 0$: $W(u) = \sum_N \ln(u_i)$, the Nash CUF

the subfamily and its benchmark elements are characterized by properties of informational parsimony

Pigou Dalton transfer: from u to v such that for some k and ε:

$v^* = (u_1^*, \cdots, u_{k-1}^*, u_k^* + \varepsilon, u_{k+1}^* - \varepsilon, u_{k+2}^*, \cdots, u_n^*)$, and $u_k^* + \varepsilon \leq u_{k+1}^* - \varepsilon$

an equalizing move: it improves the leximin SWO, the Nash CUF, all W^p such that $p \leq 1$, and is neutral w.r.t. the utilitarian CUF

$\rightarrow$ example: W^2 favors inequality

u *Lorenz dominates (LD)* v iff: $v_1^* \leq u_1^*, v_{12}^* \leq u_{12}^*, \cdots, v_N^* \leq u_N^*$ with at least one strict inequality (notation $u_S = \sum_S u_i$)

u is Lorenz optimal in the set $\mathcal{F}$ of feasible utility profiles, iff it is not Lorenz dominated in $\mathcal{F}$

Lemma 1). *u LD v if and only if we can go from v to u by a sequence of PD transfers; 2). u is Lorenz optimal in $\mathcal{F}$ if any PD transfer from u leaves $\mathcal{F}$*

examples: figures when $\mathcal{F}$ is convex and n=2; location of a facility

- a Lorenz dominant profile is the endpoint of all sequences of PD transfers

- a Lorenz dominant profile u maximizes **all** separably additive and concave CUFs $W(u) = \sum_N w(u_i)$, in particular the leximin SWO, all W^p CUFs, hence Nash, utilitarian,..

$\longrightarrow$ *it is the compelling welfarist solution*

$\longrightarrow$ existence of a compact subset of Lorenz optimal profiles is guaranteed if $\mathcal{F}$ is compact

$\longrightarrow$ existence of a Lorenz dominant profile is not

(see examples $n = 2$)

Proposition: *fix V a sub (super)modular set function $V : 2^N \diagdown \varnothing \longrightarrow \mathbb{R}$, and $\mathcal{F} = \{u | u_N = V(N)$ and $u_S \leq V(S)$ for all $S \subset N\}$, (resp. $\geq$); then the greedy algorithm starting with the largest solution to* $\arg\min \frac{V(S)}{|S|}$ *(resp. to* $\arg\max \frac{V(S)}{|S|}$*) picks the Lorenz dominant element of $\mathcal{F}$*

2 Division of manna

(see [15] or [22] for a survey)

$\longrightarrow$ to share a bundle of desirable goods/ commodities, consumed privately

$\longrightarrow$ full responsibility for own "tastes" (preferences reflect no needs)

$\longrightarrow$ no individual responsibility for the creation of the resources: common property regime

$\longrightarrow$ private ordinal preferences (we still speak of utility for convenience) *goal*: to design a division rule achieving

Efficiency (aka Pareto optimality) (EFF)

Strategyproofness (SP): truthful report of one's preferences (dominant strategy for a prior-free context)

Fairness: sanctioned by a handful of tests starting with

Equal Treatment of Equals (ETE): same utility for same preferences

strengthened as

Anonymity (ANO): symmetric treatment of all players (names do not matter) *four plus one tests of fairness*

two *single profile* tests

Unanimity Lower Bound (ULB): my utility should never be less than the utility I would enjoy if every preferences was like my own (and we were treated equally)

No Envy (NE): I cannot strictly prefer the share of another agent to my own share

in the standard Arrow-Debreu model below, the unanimity utility level corresponds precisely to the consumption of 1/n-th of the resources (this is not always true in other models)

ETE, ULB, and NE are not logically independent:

$\longrightarrow$ NE implies ETE

$\longrightarrow$ NE implies ULB in the Arrow Debreu model with only two agents

there is also a link between $\{$ETE $+$ SP$\}$ and NE two plus one *multi-profile* tests

Resource Monotonicity (RM): when the manna increases, ceteris paribus, the utility of every agent increases weakly

Population Monotonicity (PM): when a new agent is added to the participants , ceteris paribus, the utility of every agent decreases weakly

both RM and PM convey the spirit of a community eating the resources jointly (egalite et fraternite), while UNA and NE formalize precise individual rights

Consistency (CSY): when an agent leaves, and takes away the share assigned to him, the rule assigns the same shares in the residual problem (with one less agent and fewer resources) as in the original problem

unlike the other 6 axioms CSY conveys no intuitive account of fairness; it simply checks that "every part of a fair division is fair"

→ each multiprofile test by itself is compatible with grossly unfair rules

the *fixed priority rules* is RM, PM, and CSY: fix a priority ordering of all the potential agents, and for each problem involving the agents in N, give all the resources to the agent in N with highest priority

this is not true for ETE, ANO, ULB, or NE: each property by itself, guarantees some level of fairness

an incentive (as opposed to a normative) interpretation of RM and PM

→ absent RM, I may omit to discover new resources that would benefit the community

→ absent PM, I may omit to reveal that one of us has no right to share the resources

3 Arrow Debreu (AD) consumption economies

the canonical microeconomic assumptions

$N \ni i$: agents, $|N| = n$

$A \ni a$: goods, $|A| = K$

$\omega \in \mathbb{R}_+^K$: resources to divide (infinitely divisible)

$\succsim_i$: agent i's preferences: monotone, convex, continuous, hence representable by a continuous utility function

$\longrightarrow$ an allocation $(z^i, i \in N)$ is feasible if $z^i \in \mathbb{R}_+^K$ and $\sum_N z^i = \omega$

$\longrightarrow$ it is efficient iff the upper contour sets of $\succsim_i$ at z^i are supported by a common hyperplane the equal division rule ($z^i = \frac{1}{n}\omega$ for all i) meets all axioms above, except EFF

first impossibility results: fairness $\leftrightarrow$ *efficiency tradeoff*

- EFF $\cap$ ULB $\cap$ RM =EFF $\cap$ NE$\cap$ RM =$\varnothing$ ([16])

the easy proof rests on preferences with strong complementarities, close to Leontief preferences

second impossibilities: tradeoff efficiency $\leftrightarrow$ *strategyproofness* $\leftrightarrow$ *fairness*

- EFF $\cap$ ULB $\cap$ SP = EFF $\cap$ ETE $\cap$ SP =$\varnothing$ ([5]; [4])

the proof is **much** harder

the fixed priority rules are EFF $\cap$ SP $\cap$ RM $\cap$ PM $\cap$ CSY, and violates ETE, hence ANO and NE as well

$\longrightarrow$ **from now on we only consider division rules meeting EFF and ANO**

the next two rules are the main contributions of microeconomic analysis to fair division

Competitive Equilibrium from Equal Incomes (CEEI): find a feasible allocation $(z^i, i \in N)$ and a price vector $p \in \mathbb{R}_+^K$ s.t. $p \cdot \omega = n$ and $z^i = \arg\max_{z:p\cdot z\leq 1} \succsim_i$ for all i

existence requires convexity of preferences (as well as continuity and monotonicity); uniqueness is not always guaranteed, except in the large subdomains of homogenous preferences (below), or under gross substitutability; efficiency is hardwired

$\longrightarrow$ the CEEI rule meets both single-profile tests ULB and NE (irrespective of tie-breaking)

[when agents are negligible and their preferences are connected CEEI is characterized by EFF $\cap$ NE]

$\longrightarrow$ the CEEI rule is CSY but fails RM and PM on the AD domain

ω-**Egalitarian Equivalent** rule (ω-EE): find an efficient allocation $(z^i, i \in N)$ and a number $\lambda, \frac{1}{n} \leq \lambda \leq 1$, such that $z^i \simeq_i \lambda\omega$ for all i

existence, and uniqueness of utilities holds even with non convex preferences (continuity and monotonicity are still needed)

for the third solution we fix a numeraire vector $\delta \gg 0$ *in* $\mathbb{R}^K_+$

δ-**Egalitarian Equivalent** rule (δ-EE): find an efficient allocation $(z^i, i \in N)$ and a number $\lambda \geq 0$, such that $z^i \simeq_i \lambda\delta$ for all i

same remarks about existence

$\longrightarrow$ the ω-EE rule meets ULB and PM; it fails RM and CSY

$\longrightarrow$ the δ-EE rule is RM and PM; it fails ULB and CSY

both rules fail NE and can even lead to *Domination:* $z^i \gg z^j$ for some agents i, j

we dismiss δ-EE rules in the sequel because a) they fail both critical single profile tests, b) the choice of δ is entirely arbitrary *an example*: **two goods X,Y, four agents with linear preferences**

utilities $5x+y, 3x+2y, 2x+3y, x+5y$; resources $\omega=(4,4)$

ω-EE allocation: $y^1=y^2=x^3=x^4=0$, and

$$5x^1=3x^2=3y^3=5y^4=24\lambda$$

$$\Rightarrow x^1=y^4=\frac{3}{2};\ x^2=y^3=\frac{5}{2}$$

exhibiting Domination

compare CEEI: $x^1=x^2=y^3=y^4=2$

an example: **two goods X,Y, and linear preferences**

fix n and $\theta, 0 \leq \theta \leq 1$, such that θn is an integer; set $\theta' = 1 - \theta$

θn agents of type "X" have utilities $2x + y$ when consuming (x, y)

$\theta' n$ agents of type "Y" have utilities $x' + 2y'$ when consuming (x', y')

the endowment is $\omega = (n, n)$

efficiency rules out at least one of $x' > 0$ and $y > 0$

the ω-EE allocation is symmetric (same allocation for agents of same type) and solves

$$2x + y = 3\lambda;\ \ x' + 2y' = 3\lambda$$

$$\theta x + \theta' x' = 1;\ \ \theta y + \theta' y' = 1$$

assume without loss $\theta \leq \frac{1}{2}$; the solution is $\lambda = \frac{2}{1+\theta'}$

$$x = \frac{3}{1+\theta'}, y = 0;\ \ x' = \frac{4\theta' - 2}{\theta'(1+\theta')}, y' = \frac{1}{\theta'}$$

the CEEI allocation hinges around the price (p_X, p_Y) normalized so that $p_X + p_Y = 1$

if $p_X \leq 2p_Y$ and $p_Y \leq 2p_X$, type X agents spend all their money to get $\frac{1}{p_X}$ units of good X, while type Y agents similarly buy $\frac{1}{p_Y}$ units of good Y; this is feasible only if $p_X = \theta, p_Y = \theta'$; so if $\frac{1}{3} \leq \theta \leq \frac{2}{3}$, the allocation is

$$x = \frac{1}{\theta}, y = 0;\ x' = 0, y' = \frac{1}{\theta'}$$

if $\theta \leq \frac{1}{3}$ the type Y agents must eat some of each good, which is only possible at the price $p_X = \frac{1}{3}, p_Y = \frac{2}{3}$ where they are indifferent about buying either good; then

$$x = 3, y = 0;\ x' = \frac{3\theta' - 2}{\theta'}, y' = \frac{1}{\theta'}$$

CEEI and ω-EE take radically different views of scarce preferences

assume θ goes from $\frac{1}{2}$ to 0, so the type X become increasingly scarce

$\longrightarrow$ the utility of both types X and types Y under ω-EE is $\frac{6}{2-\theta}$, **de**creasing from 4 to 3

$\longrightarrow$ under CEEI, while θ decreases to $\frac{1}{3}$, the utility $\frac{2}{\theta}$ of type X **in**creases from 4 to 6, the utility $\frac{2}{1-\theta}$ of type Y decreases from 4 to 3; both utilities remain flat for $\frac{1}{3} \geq \theta \geq 0$

misreporting opportunities are more severe under the ω-EE rule

ω-EE: if the number of agents is large, a type X agent i benefits by reporting utility $x + y$: the parameter λ does not change much and i gets x_i, y_i s.t. $x_i + y_i \simeq 2\lambda$ and $y_i = 0$; so $x_i \simeq 2\lambda$ improves upon $x = \frac{3}{2}\lambda$

CEEI: misreport does not pay when $\frac{1}{3} \leq \theta \leq \frac{2}{3}$ if a single message does not alter the price much; if $\theta \leq \frac{1}{3}$ a type Y agents' misreport only has a second order impact on his utility

3.1 subdomains of Arrow Debreu preferences

the largest and most natural, containing all the applications

homothetic preferences: $z \succsim z' \Rightarrow \lambda z \succsim \lambda z'$ for all $z, z' \in \mathbb{R}_+^K$ and all $\lambda > 0$

representable by utility u homogenous of degree one

Theorem *(Eisenberg, Chipman, Moore)*: *under homothetic preferences, the CEEI allocation maximizes the Nash CUF* $\sum_N \ln\{u_i(z^i)\}$ *over all feasible allocations* $(z^i, i \in N)$

the proof is elegantly simple, see Chapter 14 in [23]

$\Rightarrow$ the CEEI solution is unique utility-wise, and even allocation-wise if the functions u_i are log-concave

moreover if the CEEI rule is RM on some homogenous subdomain, it is also PM on that domain we look at three subdomains of homogenous preferences, useful in applications because each preference is described by a vector $\beta \in \mathbb{R}^K_+$ normalized by $\sum_A \beta_a = 1$

- Cobb-Douglas: $u(z) = \sum_A \beta_a \ln(z_a)$

- linear: $u(z) = \sum_A \beta_a z_a$

- Leontief: $u(z) = \min_a\{\frac{z_a}{\beta_a}\}$ (where $\beta \gg 0$)

linear preferences have maximal substitutability, Leontief ones have maximal complementarity, with Cobb-Douglas preferences somehwere in between

3.1.1 Cobb-Douglas preferences

the CEEI allocation is computed in closed form

$$\text{price } p_a = \frac{\beta_a^N}{\omega_a};\ z^i = \arg \max_{z:p\cdot z\leq 1} \{\sum_A \beta_a^i \ln(z_a^i)\} = (\frac{\beta_a^i}{\beta_a^N}\omega_a, a \in A)$$

implying at once that the CEEI rule is RM, hence PM as well

the ω-EE allocation cannot be computed in closed form; its computational complexity appears to be high

the ω-EE rule is RM, and PM as always

$\rightarrow$ the two solutions have very similar properties (CSY is the only exception), in particular neither is SP on the Cobb Douglas domain

3.1.2 linear preferences

Proposition *the CEEI rule is RM, hence PM as well*

the proof is not simple, and neither is the computation of the solution

Open question: is the ω-EE rule also RM in the linear domain?

neither solution is SP on the linear domain ([6])

3.1.3 linear + dichotomous preferences

agent i likes the commodities in A_i as equally good, others are equally bad: $u_i(z^i) = z^i_{A_i}$; assume $\cup_N A_i = A$; notation $A_S = \cup_S A_i$

efficiency: all goods are eaten and i consumes only goods in A_i

utility profile $(u_i, i \in N)$ is feasible *iff* $u_N = \omega_N$ and $u_S \leq \omega_{A_S}$ for all $S \subset N$

Proposition: *the CEEI utility profile is the Lorenz dominant feasible profile; the CEEI rule is RM, PM, and (Group)SP*

$\longrightarrow$ the ω-EE utility profile becomes similarly the Lorenz dominant feasible profile of relative utilities $(\frac{u_i}{\omega_{A_i}}, i \in N)$; it maximizes the weighted Nash CUF $\sum_N \omega_{A_i} \ln\{u_i\}$ in the feasible set

$\longrightarrow$ the ω-EE rule is RM and PM, but not SP

3.1.4 cake cutting

Ω a compact set in $\mathbb{R}^L$: the cake

agent i's utility for a (Lebesgue-measurable) piece of cake A: $\int_A u_i(x)dx$ (or simply $\int_A u_i$)

the density u_i is strictly positive and continuous on Ω, and normalized as $\int_\Omega u_i = 1$

agent i's share is A_i, where $\{A_i, i \in N\}$ is a partition of Ω

a partition is efficient if and only if

$$\min_{A_i} \frac{u_i}{u_j} \geq \max_{A_j} \frac{u_i}{u_j} \text{ for all } i, j$$

hence $\frac{u_i}{u_j}$ is constant on any contact line of A_i and A_j

consequence of additivity of utilities: NE $\Rightarrow$ ULB

ω-EE allocation: each agent receives the same fraction of total utility (normalized to 1), therefore $\int_{A_i} u_i = \int_{A_j} u_j$ for all i, j

the CEEI partition maximizes the Nash CUF; the KT conditions read

$$\frac{u_i(x)}{\int_{A_i} u_i} \geq \frac{u_j(x)}{\int_{A_j} u_j} \text{ for all } i \text{ and all } x \in A_i$$

write i's net utility $U_i = \int_{A_i} u_i$; the KT conditions amount to

$$\min_{A_i} \frac{u_i}{u_j} \geq \frac{U_i}{U_j} \geq \max_{A_j} \frac{u_i}{u_j} \text{ for all } i, j$$

the price is simply $p(x) = \frac{u_i(x)}{\int_{A_i} u_i}$ for $x \in A_i$

cake cutting as a limit case of the linear preferences AD model

if each density u_i takes only finitely many distinct values (therefore discontinuous), cake division is an instance of the AD model with linear preferences

conjecture: a limit argument carries the properties of the linear model to cake division:

$\Rightarrow$ CEEI is RM and PM, ω-EE is PM and perhaps RM as well *cake cutting* is the subject of a large mathematical literature: e.g., [11],[10], see [12] for a survey

and (recently) algorithmic literature: [**?**],[16]

its own terminology

ULB $\leftrightarrow$ proportional: $\int_{A_i} u_i \geq \frac{1}{n}$

EE $\leftrightarrow$ equitable: $\int_{A_i} u_i = \int_{A_j} u_j$ for all i, j

goal: find simple "cutting" or "knife-stopping" algorithms to implement a non envious allocation, or an equitable allocation strategy-proof cake-cutting methods

Lemma ([10]) there always exists a perfect division of the cake: $\int_{A_i} u_i = \int_{A_j} u_i$ for all i, j ($\Rightarrow \int_{A_i} u_i = \frac{1}{n}$ for all i)

mechanism: elicit utilities, then compute a perfect division, then assign shares randomly without bias

$\longrightarrow$ this requires risk-averse preferences

$\longrightarrow$ far from efficient allocation *cake cutting with dichotomous preferences*

a limit case of the linear + dichotomous domain above

many SP mechanisms to explore: [?]

3.1.5 Leontief preferences

the definition of the ω-EE allocation is altered to rule out waste, then it is computed in almost closed form

$$\{z^i = \mu_i \beta^i \text{ and } \mu_i = \lambda u_i(\omega)\} \Rightarrow \lambda\{\sum_N u_j(\omega)\beta_a^j\} \leq \omega_a$$

the optimal λ is $\min_a \frac{\omega_a}{\sum_N u_j(\omega)\beta_a^j}$ therefore

$$z^i = \min_a \frac{u_i(\omega)\omega_a}{\sum_N u_j(\omega)\beta_a^j}\beta^i, \text{ and } u_i(z^i) = \min_a \frac{u_i(\omega)\omega_a}{\sum_N u_j(\omega)\beta_a^j}$$

Theorem ([2],[3]): *the non wasteful ω-EE rule is GSP, NE, RM, PM, and CSY*

the only missing axiom is ULB

it is possible to define rules meeting GSP, ULB, and PM

$\rightarrow$ compare CEEI: not SP and neither RM nor PM

many more mechanisms meet the axioms in the theorem; they respect the spirit of ω-EE to equalizeutilities along a benchmark ([3])

4 one non disposable commodity

a variant of the AD model: satiated preferences, no free disposal

examples: sharing a workload, a risky investment, a fixed amount of a fixed price commodity

$\omega \in \mathbb{R}_+$: amount of resource to divide (infinitely divisible)

$\succsim_i$: agent i's preferences over $[0, \omega]$: single-peaked, i.e., unique maximum π^i, strictly increasing (decreasing) before (after) π^i

$\rightarrow$ feasible allocation $(z^i \in \mathbb{R}_+, i \in N)$,$\sum_N z^i = \omega$

$\longrightarrow$ efficient allocation:

if $\sum_N \pi^i \geq \omega$ then $z^i \leq \pi^i$ (excess demand)

if $\sum_N \pi^i \leq \omega$ then $z^i \geq \pi^i$ (excess supply) the uniform solution

$$\text{if } \sum_N \pi^i \geq \omega \text{ then } z^i = \min\{\lambda, \pi^i\} \text{ where } \sum_N \min\{\lambda, \pi^i\} = \omega$$

$$\text{if } \sum_N \pi^i \leq \omega \text{ then } z^i = \max\{\lambda, \pi^i\} \text{ where } \sum_N \max\{\lambda, \pi^i\} = \omega$$

CEEI-like interpretation: if excess demand, price 1 and budget λ, disposable; if excess supply, price 1, unbounded budget, must spend at least λ

Resource/Population Monotonicity need adapting: more resources/ fewer agents is good news if excess demand, bad news if excess supply

*Resource Monotonicity** (RM*): more resources means either weakly good news for everyone, or weakly bad news for everyone

*Population Monotonicity** (PM*): one more agent means either weakly good news for all current agents, or weakly bad news for all

Theorem ([21],[20]) *the uniform solution is SP, NE, RM*, PM*, and CSY; it is characterized by the combination of EFF, ETE, and SP*

$\longrightarrow$ the uniform rule is the compelling fair division rule

$\longrightarrow$ the division in proportion to peaks plays no special role because agents are responsible for their preferences (compare with the claims problem, where a claim π^i is an objective "right", and proportional division is a major player)

5 assignment

a variant of the AD model with several comparable commodities (similar jobs), and fixed individual total shares of commodity

special case: random assignmemt of indivisible goods (one item per agent)

$N \ni i$: agents, $|N| = n$

$A \ni a$: goods, $|A| = K$

$\omega \in \mathbb{R}_+^K$: resources to divide (infinitely divisible)

agent i has a quota q^i

$\longrightarrow$ an allocation $(z^i, i \in N)$ is feasible if

$$z^i \in \mathbb{R}_+^K; \ \sum_N z^i = \omega; \ \sum_A z_a^i = q^i$$

because we focus on anonymous division rules, we assume $q^i = \frac{1}{n} \sum_A \omega_a$

the random assignment model: $|A| = n, \omega_a = 1$ for all a, z_a^i is the probability that i gets object a

Birkhof's theorem $\Rightarrow$ $\{$random assignment of the indivisible goods$\}$ $\Leftrightarrow$ $\{$deterministic assignment of the divisible goods$\}$

we discuss several assumptions on individual preferences

5.1 linear preferences (random assignment: vonNeuman-Morgenstern utilities)

CEEI rule: find a price $p \in \mathbb{R}^K_+$ and a feasible $(z^i, i \in N)$ such that

$$z^i \in \text{arg} \max_{p \cdot z \leq 1, z_A = q^i} \{\beta \cdot z\} \text{ for all } i$$

ω-EE rule: find a positive number λ and an efficient feasible $(z^i, i \in N)$ such that $\beta \cdot z^i = \lambda(\beta \cdot \omega)$ for all i

the Eisenberg Chipman Moore theorem still holds: the CEEI solution maximizes the Nash product, is unique utility-wise and allocation-wise

$\longrightarrow$ all easy properties are preserved: CEEI meets ULB, NE, CSY

$\longrightarrow$ ω-EE meets ULB, PM, but generates Domination and fails CSY

$\longrightarrow$ neither solution is SP

Open question: does the CEEI rule meets RM (hence PM)?

Open question: is the ω-EE rule also RM in the linear domain?

5.2 ordinal preferences

in practical instances of the random assignment problem (school choice, campus rooms, time slots, similar jobs), we can only elicit from each agent i her ordinal ranking $\succ_i$ of the various goods; this yields a partial ordering $\succ_i^{sd}$ of her allocations

if top $= a \succ_i b \succ_i c \succ_i \cdots$

$z \succ_i^{sd} z' \overset{def}{\Leftrightarrow} z_a \geq z'_a,\ z_a + z_b \geq z'_a + z'_b,$ with at least one strict inequality

(sd : stochastic dominance for the probabilistic interpretation; otherwise Lorenz dominance) the feasible allocation $(z^i, i \in N)$ is *ordinally efficient* iff there is no feasible allocation $(z'^i, i \in N)$ such that $z \succsim_i^{sd} z'$ for all i, with at least one strict relation

$\longrightarrow$ for random assignment, this notion is stronger than ex post efficiency, and weaker than ex ante efficiency

$\longrightarrow$ for deterministic assignments, alternative interpretation (Schulman Vazirani): individual preferences are *lexicographic* in $\mathbb{R}_+^K$ when coordinates are ranked according to $\succ_i$; (ordinary) efficiency w.r.t. lexicographic preferences $\Longleftrightarrow$ ordinal efficiency the ω-EE allocation cannot be adapted in the absence of a complete preference relation; same remark for the Nash CUF

the **Probabilistic Serial** (PS) allocation has two equivalent definitions:

- eating algorithm: agents eat at the same speed from their best commodity among those not yet exhausted

- leximin optimum of the Lorenz profile

it can be interpreted as a version of CEEI (Kesten) *leximin definition of PS*

the Lorenz curve of z at $\succ_i$ is $Lc(z, \succ_i) = (z_a, z_a + z_b, z_a + z_b + z_c, \cdots)$ where top$=a \succ_i b \succ_i c \succ_i \cdots$

the *Lorenz profile* $Lc(z, \succ)$ is the concatenation of the Lorenz curves $Lc(z^i, \succ_i)$ at the allocation $z = (z^i, i \in N)$ and preference profile $\succ = (\succ_i, i \in N)$

Proposition: the Lorenz profile $Lc(z^*, \succ)$ of the PS allocation z^* is leximin optimal:

$$Lc(z^*, \succ) \succsim_{lxmin} Lc(z, \succ) \text{ for all feasible allocations } z$$

this definition holds even if preferences exhibit some indifferences; the eating algorithm is harder to adjust to indifferences (Katta Sethuraman) example: random assignment with 3 agents and 3 objects

$$\begin{array}{ccc} a \succ_1 & b \succ_1 & c \\ a \succ_2 & c \succ_2 & b \\ b \succ_3 & a, & c \end{array}$$

$$PS = \begin{array}{ccc} a & b & c \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{array}$$

Resource/Population Monotonicity: more resources, or one agent less means a larger quota for everyone

if $z_A \geq z'_A$ we say that z is sd preferred to z' iff $z_{/z'} \succsim_i^{sd} z'$, where $z_{/z'}$ collects the z'_A best units of commodities for agent i; we still write $z \succsim_i^{sd} z'$

the definition of RM, PM is then the same

Strategyproofness: if i gets z^i by telling the truth $\succsim_i$, and z'^i by telling a lie, sd-SP requires $z \succsim_i^{sd} z'$, whereas weak-SP only asks $\rceil z' \succsim_i^{sd} z$

Theorem ([18])

i) *the PS meets ordinal-EFF; (sd) ULB, NE, RM, PM; and weak-SP*

ii) *for* $n \geq 4$*, there is no assignment rule meeting ordinal-EFF, ETE, and sd-SP* the **Random Priority** assignment is simply the average of the fixed priority assignments (first in line takes his best q^i units, next one takes his best q^i in what is left, etc..); it is a popular method, easier to implement than PS, but much harder to "compute"

RP has stronger incentives properties than PS

$\longrightarrow$ *the RP rule meets sd–ULB, sd-SP, and weak-NE*

back to the example

$$RP = \begin{array}{ccc} a & b & c \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{5}{6} & \frac{1}{6} \end{array} \qquad PS = \begin{array}{ccc} a & b & c \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{array}$$

RP has weaker efficiency properties than PS: *not ordinally efficient*

$$\begin{array}{llll} a \succ_1 & b \succ_1 & c \succ_1 & d \\ a \succ_1 & b \succ_1 & c \succ_1 & d \\ b \succ_1 & a \succ_1 & d \succ_1 & c \\ b \succ_1 & a \succ_1 & d \succ_1 & c \end{array}$$

scheduling example with deadline (opting out)

4 agents with deadlines respectively $t = 1, 2, 3, 4$

$$RP = \begin{matrix} \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{1}{24} \end{matrix} \qquad PS = \begin{matrix} \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} & 0 \end{matrix}$$

$\longrightarrow$ PS stochastichally dominates RP

$\longrightarrow$ PS and RP are assymptotically equivalent

5.3 dichotomous preferences

the particular case of ordinal preferences where commodities (objects) are viewed as good or bad (two indifference classes) $\Rightarrow$ the relation $\succsim_i^{sd}$ is complete, represented by the canonical utility $u_i(z) = \sum_{a \text{ is good for } i} z_a$

Proposition ([19]): *CEEI allocations and PS allocations (leximin definition) coincide; their unique utility profile is Lorenz dominant in the feasible set; the corresponding rule is (are) (group) SP*

$\longrightarrow$ similar to the case of manna with linear dichotomous preferences

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