Auctions as Games: Equilibria and Efficiency
Near-Optimal Mechanisms

Éva Tardos, Cornell
Yesterday: Simple Auction Games

- item bidding games: second price simultaneous item auction
- Very simple valuations: unit demand or even single parameter
- Ad Auctions: Generalized Second Price

Today:

- More auction types
- More expressive valuations
Summary of problems

Full information single minded bidders

- \( v_{ij} = \text{buyer i's value for house j} \)
- \( v_i(S) = \max_{j \in S} v_{ij} \)

Bidding \( b_{ij} > v_{ij} \) is dominated. 

**assume not done**

\( GSP \) (AdAuction), also single parameter:

- \( v_{kj} = v_k \gamma_k \alpha_j \)
Summary of techniques

- Price of anarchy 2 based on: no-regret for bidding $b_{ij_i^*} = v_{ij_i^*}$ and $b_{ij} = 0 \ \forall j \neq j_i^*$

- Bound also applies to learning outcomes (see more Avrim Blum)

- Bayesian game (valuations from correlated distribution $\mathcal{F}$) price of anarchy of 4 based on no-regret for bidding $\frac{1}{2}v_i$
  - GSP
  - Single value auctions
First Price vs Second Price?

Proof based on “player i has no regret about bidding \( \frac{1}{2} v_i \)” applies just as well for first price.

If player wins: price \( \leq b_i \leq \frac{1}{2} v_i \), hence utility at least \( \frac{1}{2} v_i \).

- If he looses, all his items of interest, went to players with bid (and hence value) at least \( \frac{1}{2} v_i \).

If i has value of opt, i or k has high value at Nash.
First Price vs Second Price?

Proof based on “no-regret for bidding \( b_{ij}^* = v_{ij}^* \) and \( b_{ij} = 0 \ \forall j \neq j_i^* \)” no good, but similar proof applies with \( b_{ij}^* = \frac{1}{2} v_{ij}^* \) and \( b_{ij} = 0 \ \forall j \neq j_i^* \)

• If player wins: price \( \leq b_{ij}^* \leq \frac{1}{2} v_{ij}^* \) hence utility at least \( \frac{1}{2} v_{ij}^* \)

• If he looses, his items of interest went to players with bid (and hence value) at least \( \frac{1}{2} v_{ij}^* \)
First Price Pure Nash


Proof each item $i$ was sold for a price $p_i$.

- price $p$ is market equilibrium: all players maximizing $v_i(S) - \sum_{i \in S} p_i$ players

  otherwise bid $p_i^+$ for items in $i \in S$

- market equilibrium is socially optimal

  \begin{align*}
  \{S_1, \ldots, S_k\} \text{ Nash and } \{S_1^*, \ldots, S_k^*\} \text{ alternate soln.}
  \\
  v_i(S_i) - \sum_{i \in S} p_i \geq v_i(S_i^*) - \sum_{i \in S^*} p_i
  \\
  \text{sum over all } i \quad \sum_i v_i(S_i) \geq \sum_i v_i(S_i^*)
  \end{align*}
Sequential Game (eBay)

How important is simultaneous play?

Buyers

Sellers

10

9

5
Canon EOS 5D Mark II 21.1 MP Digital SLR Camera - Black (Body Only)

User reviews (147) ★★★★★

Options Black, Body Only ($2,138.95)

11 results found for Used

Preferences: Auction

In Digital Cameras

Condition
- Used
  Choose more... | Clear

Bundled Items
- Case or Bag (1)
- Extra Battery (3)
- Lens (1)
- Memory Card (4)
- Strap (Neck or Wrist) (2)
  Choose more...

Price
- $ to $

Seller
- eBay Top-rated sellers
- Specify sellers...

Buying formats
- Auction
- Buy It Now
  Choose more... | Clear

Canon EOS 5D Mark II 21.1 MP Digital SLR Camera - Black...
Condition: Used
0 bids $1,999.00 3h 8m

Canon EOS 5D Mark II 21.1 Megapixel
One-day shipping available
Condition: Used
20 bids $1,800.00 11h 37m

Canon EOS 5D Mark II 21.1 MP Digital SLR Camera
Expected shipping available
Condition: Used
1 bid $1,799.00 19h 16m

Canon EOS 5D Mark II 21.1 MP Digital SLR Camera - Black...
Mint Condition: Low Actuations: Like New
Condition: Used
4 bids $1,860.00 1d 7h 30m

Canon EOS 5D Mark II 21.1 MP Digital SLR Camera - Black...
CANON MARCO EF100mm LENS INCLUDED (1099.99 VALUE)
Condition: Used
0 bids $2,399.99 1d 8h 9m
### Canon EOS 5D Mark II 21.1 MP Digital SLR Camera - Black (Body Only)

**User reviews:** (147) ⭐⭐⭐⭐

**Options:** Black, Body Only ($2,138.95)

**11 results found for Used**

**Preferences:** Auction

<table>
<thead>
<tr>
<th>Item Description</th>
<th>Bids</th>
<th>Price</th>
<th>Time Left</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canon EOS 5D Mark II 21.1 MP Digital SLR Camera - Black...</td>
<td>0</td>
<td>$1,999.00</td>
<td>3h 8m</td>
</tr>
<tr>
<td>Condition: Used</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bundle Items:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case or Bag (1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Extra Battery (3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lens (1)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Memory Card (4)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Strap (Neck or Wrist) (2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Price</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$ to $</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Seller</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>eBay Top-rated sellers</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Specify sellers...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buying formats</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Auction</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Buy It Now</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
PF1201 | Important Mobilier, Sculptures et Objets d'Art

Welcome, Guest | Paddle: W_ | Saleroom Notices

**Upcoming Lots**

**LOT 96**
BERGÈRE EN HÊTRE AU NATUREL SCULPTÉ D'ÉPOQUE...
SOLD: 6,000 EUR

**LOT 97**
BERGÈRE EN HÊTRE RELAQUÉ D'ÉPOQUE LOUIS XV...
EST: 3,000-5,000 EUR

**LOT 98**
CANAPÉ CORBEILLE EN BOIS REDORÉ D'ÉPOQUE LOU...
EST: 10,000-15,000 EUR

**LOT 99**
PAIRE D'ENCOIGNURES EN PLACAGE DE SATINÉ, MA...
EST: 20,000-30,000 EUR

**LOT 100**
SUITE DE QUATRE FAUTEUILS À CHÂSSIS EN BOIS SCULP...
EST: 70,000-100,000 EUR

**LOT 101**
IMPORTANT GROUPE EN BRONZE REPRÉSENTANT PÊTUS ET À...
EST: 80,000-120,000 EUR

**Bid**

**LOT 97**
Bergère en hêtre relaqué d'époque Louis XV
EST: 3,000 EUR - 5,000 EUR

Next bid 2,400 EUR

**CURRENT BID**
2,200 EUR

**YOUR BID STATUS**
Viewing only

**Current Lot**

A bergère chair and a set of four armchairs are displayed.

A woman is shown in the video, indicating an ongoing auction.
Second Price and Sequential Auctions

• Second price allows signaling
• Bidding above value is not dominated
• Can have unbounded price of anarchy both with
  – Additive valuations
  – Unit demand valuations (even after iterated elimination of dominated strategies)
Bad example for 2\textsuperscript{nd} price

<table>
<thead>
<tr>
<th></th>
<th>( k )</th>
<th>( k )</th>
<th>( k )</th>
<th>( k )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \ldots )</td>
<td>0</td>
</tr>
<tr>
<td>( B )</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td>( \ldots )</td>
<td>20</td>
</tr>
<tr>
<td>( C )</td>
<td>20</td>
<td>20</td>
<td>10</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( D )</td>
<td>1</td>
<td>( \epsilon )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \ldots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Z )</td>
<td>1</td>
<td></td>
<td>( \epsilon )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Sequential game

• Items are not available at the same time: sellers arrive sequentially

• Players are strategic and make decisions reasoning about the decisions of other players in the future

• Each player has unit demand valuation \( v_{ij} \) on the items

• First price auction
  – Full Information (Paes Leme, Syrgkanis, T. SODA'12)
  – Bayesian (Syrgkanis, T. EC'12)
Incomplete Information and Efficiency

\( V_1 \sim U[0,1] \)

\( V_2 \sim U[0,1] \)

\( V_3 \sim U[0,1] \)
Incomplete Information and Efficiency

$V_1 \sim U[0,1]$

$V_2 \sim U[0,1]$

$V_3 \sim U[0,1]$
Incomplete Information and Efficiency

$V_1 \sim U[0,1]$

$V_2 \sim U[0, v_1]$

$V_3 \sim U[0,1]$
Incomplete Information and Efficiency

Player 2 bids more aggressively $\Rightarrow$ outcome inefficient
Example

$V_1 = 1$

Now I win for price of 1. Maybe better to wait...

$V_2 = 100$

And win C for free.

$V_3 = 100$

Now I will pay 99. At the last auction I will pay 100.

$V_4 = 99$

Suboptimal Outcome
A bidding strategy is a bid for each item for each possible history of play on previous items.

- Can depend only on information known to player:
  - Identity of winner, maybe also winner's price.

Solution concept:

**Subgame Perfect Equilibrium**

= **Nash** in each subgame
Bayesian Sequential Auction games

Valuations $v$ drawn from distribution $\mathcal{F}$

For simplicity assume for now

- single value $v_i$ for items of interest
- $(v_1, \ldots, v_n) \in \mathcal{F}$ drawn from a joint distribution

- $\text{OPT}_{i_j^*}$ random
- Depends on information $i$ doesn't have!
- Deviating in early auctions may change behavior of others later
Sequential Bayesian Price of Anarchy

**Theorem** In first price sequential auction for unit demand single parameter bidders from correlated distributions. The total value \( v(N) = \sum_{i \in N} v_i \) at a Bayesian Nash equilibrium Distribution \( D \) of \( N = \{(i,j_i)\} \) is at least \( \frac{1}{4} \)th of optimum expected value of OPT (assuming \( b_i \leq v_i \forall \ i \)).

Proof based player \( i \) bidding \( \frac{1}{2}v_i \) on all items of interest.

Deviation only noticeable if winning!

- If player wins: hence utility = \( \frac{1}{2}v_i \)
- If he looses, his items of interest valued at least \( \frac{1}{2}v_i \) by others.

In either case \( \frac{1}{2}v_{ij_i^*} \geq v_{ij_i} + v(j_i^*) \)

Sum over player, and take expectation over \( v \in \mathcal{F} \)

\[ \frac{1}{2} \text{OPT} \geq E(v(N) + E(v(N)) \]
Bayesian Price of Anarchy

**Theorem** Unit demand single parameter bidders, the total expected value $E(v(N)) = E(\sum_{i \in N} v_i)$ at an equilibrium distribution $N = \{(i,j)\}$ (assuming $b_i \leq v_i \forall i$) is at least $\frac{1}{4}$ of the expected optimum $OPT = E(\max_{M} \sum_{i \in M} v_i)$

**proof** “player i has no regret about bidding $\frac{1}{2} v_i$ on all items of interest”

Simple strategy: no regret about this one strategy is all that we need for quality bound!

Applies for learning outcome, and Bayesian Nash with correlated bidder types.
Full info Sequential Auction with unit demand bidders

**Thm:** Value of any Nash at least $\frac{1}{2}$ of optimum

\[ v_{ij^*(i)} - p_{j^*(i)} - P^- \leq v_{ij(i)} - P^- \]

Summing for all $i$: \( OPT \leq 2 \text{SPE} \)
Bayesian Sequential Auction?

\[ \nu_{ij^*(i)} - p_{j^*(i)} - P^- \leq \nu_{ij(i)} - P^- \]

Summing for all \( i \): \[ \text{OPT} \leq 2 \text{SPE} \]
Complications of Incomplete Information

- $j_i^*(\nu)$ depends on other players’ values which you don’t know

- Bidding becomes correlated at later stages of the game since players condition on history
Simultaneous Item Auctions

Theorem [Christodoulou, Kovacs, Schapira ICALP’08]
Unit demand bidders, assuming values drawn independently $v_i$ from $\mathcal{F}_i$, and $b_{ij} \leq v_{ij} \forall i \& j$

the total expected value $E(v(N))=E\left(\sum_{i \in N} v_{ij_i}\right)$ at an equilibrium distribution $N = \{(i,j)\}$ is at least $\frac{1}{2}$ of the expected optimum $\text{OPT}=E\left(\max_M \sum_{(i,j) \in M} v_{ij}\right)$.

Proof? The assigned item in optimum $j^*_i$ depends on $v_{-i}$ hence not known to $i$.
Not a possible bid to consider
Simultaneous Item Auctions (proof)

Sample valuations of other players $w_{-i}$ from $\mathcal{F}_{-i}$.
Use $(v_i, w_{-i})$ to determine $j_i^*$

- bid $b_{ij_i} = v_{ij_i}$ and $b_{ij} = 0 \forall j \neq j_i^*

- Nash's value of $j_i^*$ is $v(j_i^*)$. Exp. cost of item $j_i^*$
  $\leq E_v (v(j_i^*)|v_i)$

- i's utility for given $v_i$
  $E_w(v_{ij_i}) - E_wE_{v_{-i}}(v(j_i^*)|v_i)$

- Use Nash for i
  $E_{v_{-i}}(v_{ij_i}) \geq E_w(v_{ij_i}) - E_wE_{v_{-i}}(v(j_i^*)|v_i)$
Simultaneous Item Auctions (proof2)

Use Nash for i

\[ E_{v_i}(v_{ij_i}) \geq E_w(v_{ij_i}^*) - E_w E_{v-i}(v(j_i^*)|v_i) \]

- Take expectation over
  \[ E_v(v_{ij_i}) \geq E_v E_w(v_{ij_i}^*) - E_w E_v(v(j_i^*)) \]
  
  - Lhs sum over i: \[ \sum_i E_v(v_{ij_i}) = Nash(SW) \]
  
  - Rhs term 1: \[ E_v E_w(v_{ij_i}^*) = E_{v_i} E_{w-i}(v_{ij_i}^*) = E_v(v_{ij_i}^*) \]
  
  - Sum over i: \[ \sum_i E_v E_w(v_{ij_i}^*) = OPT(SW) \] (use indep)
  
  - Last term sum over i:
    \[ \sum_i E_w E_v(v(j_i^*)) = \sum_j E_w E_v(v(j)) \]
    \[ = \sum_j E_v(v(j)) = Nash(SW) \]
Bayesian second Price of Anarchy

**Theorem** [Christodoulou, Kovacs, Schapira ICALP’08]

Unit demand bidders, assuming values drawn independently $v_i$ from $\mathcal{F}_i$, and $b_{ij} \leq v_{ij} \ \forall \ i \& j$

the total expected value $E(v(N)) = E\left(\sum_{i \in N} v_{i,ij}\right)$ at an equilibrium distribution $N = \{(i,j)\}$ is at least $\frac{1}{2}$ of the expected optimum $OPT = E\left(\max_M \sum_{(i,j) \in M} v_{ij}\right)$.

**Proof:** In expectation over $v$ and $w$

$\text{Nash(SW)} \geq \text{OPT(SW)} - \text{Nash(SW)}$
Bayesian Sequential Auction

Try similar idea (idea 1):
Sample valuations of other players $w_i$ from $\mathcal{F}_i$, Use $(v_i, w_i)$ to determine $j = j_i^*$

- Bid as before till $j$ comes up, then bid $\frac{1}{2}v_{ij}$ for $j$
Bayesian Sequential Auction (idea 1)

• If $i$ wins item $j$ then he gets utility at least:
  \[ v_{ij} - \frac{v_{ij}}{2} - p_{ij}^-(v, v_{-i}) = \frac{v_{ij}}{2} - p_{ij}^- (v, v_{-i}) \]

• If he doesn’t then the winning bid must be at least:
  \[ p_{j}^-i (v_i, v_{-i}) \geq \frac{v_{ij}}{2} \]

• In any case utility from the deviation is at least:
  \[ \frac{v_{ij}}{2} - P_i(v_i, v_{-i}) - p_{j}^-i (v_i, v_{-i}) \]
Correlated Bidding

- $p_j^{-i}(v_i, v_{-i})$ depends implicitly on your bid through the history of play

- When player $i$ arrives at $j_i^*(v_i, w_{-i})$ he doesn’t “face” the expected equilibrium price but a “biased” price

- Will not allow us to claim that:
  - “either bidder already gest high value or expected price of some item is high”
The Bluffing Deviation

- Player draws a random sample $w_i$ from his value and a random sample $w_{-i}$ of the other players' values.
- He plays as if he was of type $w_i$ until item
  \[ j = j_i^*(v_i, w_{-i}) \]
- Then he bids
  \[ \frac{v_{ij}}{2} \]
The Bluffing Deviation

The utility from the deviation is at least:

$$\frac{v_{ij}}{2} - P_{i}(w_{i}, v_{-i}) - p_{j}^{-i}(w_{i}, v_{-i})$$

Summing for all players and taking expectation

Note: price for $j$ independent of $v_{i}$

$$\frac{1}{2}OPT - Rev(SPE) - Rev(SPE) \leq Util(SPE)$$

$$\frac{1}{2}OPT \leq 2SPE$$
Simple Auction Games

Examples of simple games
• Item bidding first and second price
• Generalized Second Price

Simple valuations: unit demand

Results: Bounding outcome quality
  – Nash,
  – Bayesian Nash,
  – learning outcomes
Overbidding assumptions

• We used: unit demand bidders
  – assume \( b_{ij} \leq v_{ij} \)
  – Bidding \( b_{ij} > v_{ij} \) is dominated by \( b_{ij} = v_{ij} \)

• more general 2\textsuperscript{nd} price results use
  – assume \( \sum_{j \in S} b_{ij} \leq v_i(S) \)
  – A best respond in this class always exists!

• First price: no such assumption is needed

• Sequential Auction: overbidding may be very useful/natural
**The Dining Bidder Example**

<table>
<thead>
<tr>
<th></th>
<th>Bread</th>
<th>...</th>
<th>Meat</th>
<th>Grilled Meat</th>
<th>$k\varepsilon$</th>
<th>$10 - \frac{k\varepsilon}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A</strong></td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>0</td>
<td></td>
<td>$10 - \frac{k\varepsilon}{2}$</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>1</td>
<td>...</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$10 - \frac{k\varepsilon}{2}$</td>
</tr>
<tr>
<td><strong>C</strong></td>
<td>$\varepsilon$</td>
<td>...</td>
<td>$\varepsilon$</td>
<td>20</td>
<td>20 - fish - bread</td>
<td></td>
</tr>
<tr>
<td><strong>D</strong></td>
<td>$\varepsilon$</td>
<td>...</td>
<td>$\varepsilon$</td>
<td>$20 - bread - \frac{\varepsilon}{2}$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
References and Better results

- [Christodoulou, Kovacs, Schapira ICALP’08] Price of anarchy of 2 assuming conservative bidding, and fractionally subadditive valuations, independent types
- [Bhawalkar, Roughgarden SODA’10] subadditive valuations,
- [Hassidim, Kaplan, Mansour, Nisan EC’11] First Price Auction mixed Nash
- [Paes Leme, Syrgkanis, T, SODA’12] Price of Anarchy for sequential auction
- [Syrgkanis, T EC’12] Bayesian Price of Anarchy for sequential auction, better bounds of 3 and 3.16