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Lecture 9

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1 Allocation Of Indivisible Goods - Introduction

In the previous lecture, we talked about fair division of divisible goods between n agents. In this week's lecture, we will examine the allocation of indivisible goods between the agents, such that the social welfare is maximized, and in an envy free fashion (as we will define later on).

2 Efficient Allocations

2.1 Basic Definitions:

Definition 1 *We will denote the group of goods for allocation by $G = \{g_1...g_k\}$.*

Definition 2 *We will denote the group of agents by $N = \{1...n\}$.*

Definition 3 (Evaluation Function) *For each of the agents, we will define an Evaluation Function $V_i : 2^{|G|} \to \mathbb{R}$, that will assign a value for every subset of items.*

Definition 4 (Social Welfare) *Let $A = (A_1, ..., A_n)$ be an allocation of the resources, such that $\forall i\ A_i \subseteq G$ and $\forall i, j\ A_i \cap A_j = \emptyset$. We will define the Social Welfare of the allocation as $SW(A) := \sum_{i \in N} V_i(A_i)$.*

Definition 5 (Efficient Allocation) *An allocation A is efficient if it maximizes the social welfare.*

There are two common approaches to finding an efficient allocation:

1. The more common approach is a standard "computer science" one - Finding an efficient allocation is NP-Hard, so we are left with seeking approximations. These approaches also seek "truthful" algorithms - Algorithms in which it does not pay the agents to cheat.
2. Distributed computing approach which is what this lecture is about.

2.2 Deals Between Agents:

Definition 6 (Deal) *A deal $\delta = (A, A')$ between agents is an exchange of goods, where A is the allocation before the deal and A' is the allocation after the deal.*

Definition 7 (Payment) *For each deal δ we will define a payment vector $\overrightarrow{\mathbf{P}} \in R^n$ as* $\overrightarrow{\mathbf{P}} = (p_1, ..., p_n) \mid \sum_{i \in N} p_i = 0$

Definition 8 (Individually Rational (IR) Deal) *We will say a deal $\delta = (A, A')$ is IR if:* $\exists \overrightarrow{\mathbf{P}} \in R^n\ \ \forall i \in (1, .., n)\ \ V_i(A') - V_i(A) \geq p_i$ *And at least for one agent the inequality is strict.*

Example 1

$$V_1(\emptyset) = 0 \ , \ V_1(\{chair\}) = 2 \ , \ V_1(\{table\}) = 3 \ , \ V_1(\{chair, table\}) = 7$$

$$V_2(\emptyset) = 0 \ , \ V_2(\{chair\}) = 3 \ , \ V_2(\{table\}) = 3 \ , \ V_2(\{chair, table\}) = 8$$

We shall start with an allocation $A = (\{chair, table\}, \emptyset)$ *- both the chair and the table are allocated to Agent 1.*

We do not have an IR deal that passes one object to agent 2, but we do have an IR deal that passes both objects - If both objects are passed from agent 1 to agent 2 for a payment of 7.5, each of the agents gains 0.5. We can see that by making this deal, the social welfare is increased by 1 (From 7 to 8).

Lemma 9 *A deal* $\delta = (A, A')$ *is IR* iff *the social welfare of* A' *is greater than the social welfare of* A : $SW(A') > SW(A)$.

Proof
$\Rightarrow$: Assume that δ is IR, i.e. $\exists \overrightarrow{\mathbf{P}} \in R^n \ \ \forall i \in (1,..,n) \ \ V_i(A') - V_i(A) \geq p_i$ and for at least one agent the inequality is strict (In order for someone to profit from the deal so the deal would occur).
We shall sum over all agents and get:

$$\sum_i V_i(A_i) - \sum_i V_i(A'_i) > 0 \implies_{(by\ \ def.)} \ SW(A') > SW(A)$$

$\Leftarrow$: Assume that $SW(A') > SW(A)$.
We shall assign the payments vector $\overrightarrow{\mathbf{P}}$ by:

$$\forall i \in (1,..,n) \ \ : \ \ p_i = V_i(A'_i) - V_i(A_i) - \frac{SW(A') - SW(A)}{n}$$

If follows that each agent pays what they earned and earns their equal share of the total profit.

It is evident that:

$$\sum_i p_i = \sum_i V_i(A'_i) - \sum_i V_i(A_i) - SW(A') + SW(A) = SW(A') - SW(A) - SW(A') + SW(A) = 0$$

We shall prove that the deal is IR:

$$p_i = V_i(A'_i) - V_i(A_i) - \frac{SW(A') - SW(A)}{n} \Rightarrow p_i < V_i(A'_i) - V_i(A_i)$$

(Since from our assumption $SW(A') \geq SW(A)$ and therefore the fraction $\frac{SW(A') - SW(A)}{n}$ is positive). ■

Lemma 10 *Given an allocation A, the number of IR deals starting from A is finite.*
Proof Since every deal switches to a different allocation with a better social welfare (As proved in Lemma 10), and the number of possible allocations is finite (and specifically is $n^{|G|}$), the process will stop. ■

Theorem 11 *Every series of IR deals will eventually reach an efficient allocation.*
Proof From Lemma 11, each series of IR deals will reach an allocation from which there are no possible IR deals. We'll call that allocation A^*. Assume that A^* isn't an efficient allocation. i.e., there is an allocation A such that $SW(A) > SW(A^*)$.

According to Lemma 10, the deal (A, A^*) is IR (Since it increases the social welfare), and thus we reach a contradiction to our assumption that there are no more IR deals. ■

Remark The process may require an exponential number of deals, but empirical results show that in the average case it converges quickly. An open question is what are the starting conditions for which the process converges quickly.

3 Envy Free Allocations

In the previous lecture we talked about envy free allocations of continuous goods, for example, a cake, and showed that there is an envy free allocation, to continuous parts. In this lecture we'll talk about envy free allocations of discrete goods. In this case, we don't always have an envy free allocation.

Example 2

$N = \{1, 2\}$
$G = \{g\}$
$V_1(\{g\}) = V_2(\{g\}) = 1$

We can't give the object g to any of the agents since the other will be envious.

To assure an envy free allocation we shall introduce the notion of "Payment Balance" (PB).

Definition 12 (Payment Balance (PB)) *For each allocation A, we shall assign a PB π. $\pi : N \to \mathbb{R}$ such that $\sum_i \pi_i = 0$, where π_i is the payment by agent i.*

Definition 13 (State) *A state is a pair (π, A) where π is a PB and A is an allocation.*

Definition 14 (Envy Free State) *A state is defined as an envy free state if:*
$\forall i, j \;\; V_i(A_i) - \pi_i \geq V_i(A_j) - \pi_j$
(Each agent would rather receive his part and payment).

Example 3 *In the above example, we shall allocate the object g to Agent 1, and he shall pay 0.5 to Agent 2.*
We will get the state:

$A_1 = \{g\} \qquad \pi_1 = -0.5$
$A_2 = \emptyset \qquad \pi_2 = 0.5$

4 Efficient Envy Free Allocation

Our goal now will be to reach an Efficient Envy Free (EEF) state.

Definition 15 (Super Additive Function) *$F : \Omega \to \mathbb{R}$ is super additive if :*
$\forall_{B_1,B_2 \subseteq \Omega} \quad (B_1 \cap B_2 = \emptyset) \to (F(B_1) + F(B_2) \leq F(B_1 \cup B_2))$

Theorem 16 *[1] If $V_1 ... V_n$ are super additive evaluation functions, there exists an EEF state for every start allocation A.*
Proof We shall start with an efficient allocation A^*. Our goal is to define π^* such that (A^*, π^*) is EEF.
Let's define:

$$\pi_i^* = V_i(A_i^*) - \frac{SW(A^*)}{n}$$

Note that:

$$\sum_i \pi_i^* = \sum_i V_i(A_i^*) - SW(A^*) = 0$$

Let i, j be two agents. Since A^* is efficient, we get:

$$V_i(A_i^* \cup A_j^*) \leq V_i(A_i^*) + V_j(A_j^*)$$

From our assumption that V_i is super additive, we get:

$$V_i(A_i^* \cup A_j^*) \geq V_i(A_i^*) + V_i(A_j^*)$$

Combining those facts together:

$$V_i(A_i^*) + V_j(A_j^*) \geq V_i(A_i^*) + V_i(A_j^*)$$
$$\Rightarrow V_i(A_i^*) - V_i(A_i^*) \geq V_i(A_j^*) - V_j(A_j^*)$$
$$\Rightarrow V_i(A_i^*) - V_i(A_i^*) + \frac{SW(A^*)}{n} \geq V_i(A_j^*) - V_j(A_j^*) + \frac{SW(A^*)}{n}$$
$$\Rightarrow V_i(A_i^*) - \left[V_i(A_i^*) - \frac{SW(A^*)}{n}\right] \geq V_i(A_j^*) - \left[V_j(A_j^*) - \frac{SW(A^*)}{n}\right]$$
$$\Rightarrow V_i(A_i^*) - \pi_i^* \geq V_i(A_j^*) - \pi_j^*$$

i.e., by definition, the state (A^*, π^*) is envy free.
Since we defined A^* as an efficient allocation, the state (A^*, π^*) is EEF. ■

Example 4 (IR deals cannot always reach an EEF state)
$G = \{g\}$, $V_1(\{g\}) = 4$, $V_2(\{g\}) = 7$
The initial allocation : $A^0 = (\{g\}, \emptyset)$

There is only one possible deal which is IR, and that would be to pass g from Agent 1 to Agent 2, for which Agent 2 will need to pay more than 4. But, in that case, Agent 2 will envy Agent 1, since he would rather receive 4 (Which will bring his profit to 4) than receive g and pay 5 (Which will bring his profit to 2).

Definition 17 (Initial Payments) *Since the above mentioned problem originates from the fact that Agent 1 has been given an item without paying, we shall enforce initial payments which will balance the initial profit of each agent from the allocation of the goods:*

$$\pi_i^0 = V_i(A_i^0) - \frac{SW(A^0)}{n}$$

Definition 18 (Globally Uniform Payment Function (GUPF)) *For a deal $\delta = (A, A')$ which is IR we shall assign a GUPF:*

$$p_i = V_i(A_i') - V_i(A_i) - \frac{SW(A') - SW(A)}{n}$$

Theorem 19 *[1] If all evaluation functions are super additive and the initial payments are enforced as above, then every sequence of IR deals with payments as in the GUPF creates an EEF state.*

Proof From Theorem 12, each sequence of IR deals will end in an efficient allocation A^*, so all that is left to show is that the sum of payments in the deals creates an envy free state.

We shall prove this by induction. Define $A^0, ..., A^k$ an allocation sequence that reaches an efficient allocation. We need to show that for each stage l, the payment is $\pi_i^l = V_i(A_i^l) - \frac{SW(A^l)}{n}$. If that is true, then the last allocation, $A^k = A^*$ has PB $\pi_i^* = V_i(A_i^*) - \frac{SW(A^*)}{n}$, and from Theorem 19 we get that (A^*, π^*) is envy free.

Induction base: Follows from the definition of π^0.

Induction step: We shall assume correctness up to an allocation A, and prove for (A, A').

$$\pi_i = V_i(A_i) - \frac{SW(A)}{n}$$

$$\pi_i' = V_i(A_i') - V_i(A_i) - \frac{SW(A') - SW(A)}{n} + \pi_i$$

(This follows from definition 22)

$$\pi_i' = V_i(A_i') - V_i(A_i) - \frac{SW(A') - SW(A)}{n} + V_i(A_i) - \frac{SW(A)}{n}$$

$$= V_i(A_i') - \frac{SW(A')}{n}$$

■

References

[1] Y. Chevaleyre, U. Endriss, S. Estivie, and N. Maudet. Reaching envy-free states in distributed negotiation settings. In *IJCAI*, 2007.