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Lecture 8

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1 Introduction: Fair Division (Divisible Goods)

In this lecture we will deal with a known problem of fair division of divisible goods. In general, in fair division problems there is a resource *(Goods)* that has to be divided "fairly" amongst n players.

We can look at the Divisible Goods problem as a Cake Division problem: There is a cake with all kinds of delicious sweets on it, and there are n players with different preferences,(chocolate cream cherries..) and we want to divide the cake between the players so that each player will be satisfied as much as possible.

And now to the formal description:

- The cake is represented by the unit interval [0,1]
- The preferences of the i'th Player are represented by a function $V_i : [0,1] \longrightarrow \mathbf{R}$. This function maps a union of intervals to a value according to the i-th preference. The function has the following properties:

  1 For all $X \subseteq [0,1] \quad v_i(X) \geq 0$
  2 For all $X, Y \subseteq [0,1]$ so that $X \cap Y = \phi \quad v_i(X \cup Y) = v_i(X) + v_i(Y)$
  3 For all $x \in [0,1] \quad v_i([x,x]) = 0$
  4 $v_i([0,1]) = 1$

- The fairness requirement can be described in two ways:
  - **Definition 1** *A division $S_1, ..., S_n$ is called a* ***Proportional Division*** *if for all* $0 \leq i \leq 1 \quad v_i(S_i) \geq \frac{1}{n}$
  - **Definition 2** *A division $S_1, ..., S_n$ is called an* ***Envy-free Division*** *if for all* $0 \leq i, j \leq 1 \quad v_i(S_i) \geq v_i(S_j)$

2 Division Protocols

We will now describe several fair division protocols. For each protocol we will examine the following properties:

1. Proportional Division - does the protocol produce a proportional division ?
2. Envy-free Division - does the protocol produce an envy-free division ?
3. Contiguous Division - are the pieces of the cake contiguous?
4. Number of Cuts - how many cuts are made by using the protocol?

2.1 Cut-and-Choose Protocol

A very simple and intuitive protocol for n=2,

1. Player 1 cuts the cake into two pieces of equal value according to v_1
2. Player 2 chooses the piece he prefers according to v_2

Lemma 3 *The Cut-and-Choose protocol is envy-free*

Proof Player 1 does not envy Player 2 because the two pieces are equal according to v_1 . Player 2 does not envy Player 1 because he got to choose the piece he prefers according to v_2 $\blacksquare$

Lemma 4 *For $n \geq 2$ envy-free division $\Longrightarrow$ proportional division.*

Proof Given a player i, there is a player j such that $v_i(S_j) \geq \frac{1}{n}$,the division is envy-free therefore $v_i(S_i) \geq v_i(S_j) \geq \frac{1}{n}$ $\blacksquare$

Note that the other direction is not true (it's not true that for $n \geq 2$. every proportional division is also an envy-free division).
Let us see a counter example for the case of $n = 3$:

The valuation functions v_2 and v_3 are distributed uniformly.
The valuation function v_1 is defined:
$v_1([0, \frac{1}{3}]) = \frac{1}{3}$
$v_1([\frac{1}{3}, \frac{2}{3}]) = \frac{2}{3}$
$v_1([\frac{2}{3}, 1]) = 0$

Then the following partition:
$[0, \frac{1}{3}]$ to Player 1
$[\frac{1}{3}, \frac{2}{3}]$ to Player 2
$[\frac{2}{3}, 1]$ to Player 3
is proportional but not envy-free, since Player 1 envies Player 2's choice.

2.1.1 Protocol Properties

+ The protocol is envy-free and proportional (from the last lemma)
+ The pieces are contiguous
+ The number of cuts is minimal - one cut.

2.2 Protocol for $n = 3$ (Steinhaus, 1943)

This protocol generalizes the simple "Cut-and-Choose" protocol.

1. Player 1 cuts the cake into three pieces, that are considered even according to v_1.

2. Player 2 is given the choice of either passing, or labeling two of the pieces as "bad".
   If two or more pieces are worth at least $\frac{1}{3}$ according to v_2, then Players 3, 2, and 1, in that order, choose a piece (that they consider to be of size at least $\frac{1}{3}$) and we are done.
   Otherwise, Player 2 labels two pieces that are worth strictly less than $\frac{1}{3}$, according to v_2, as "bad".

3. If Player 2 did not pass at step 2, then Player 3 is given the same two options that Player 2 had at step 2. He ignores Player 2's labels.
   If Player 3 passed, then Players 2, 3, and 1, in that order, choose a piece (that they consider to be of size at least $\frac{1}{3}$) and we are done.

4. If neither Player 2 or Player 3 passed, then Player 1 is required to take a piece that both Player 2 and Player 3 labeled as "bad".
   The other two pieces are reassembled, and Players 2 & 3 use the Cut-and-Choose Protocol to divide the reassembled piece.

2.2.1 Protocol Properties

\+ Guarantees a proportional division of the cake.
  Proof In case the protocol ends at step 2, each Player receives a piece of size at least $\frac{1}{3}$ according to his valuation function. This is true of Player 3 because he chooses first, of Player 2 because he thinks at least two pieces are that large, and so at least one of them will still be available after Player 3 chooses his piece, and of Player one since he cut the cake in this manner in step 1.
  In case the protocol ends at step 3, the proof is similar.
  In case the protocol ends at step 4, Player 1 gets the piece that both Players 2 and 3 labeled as "bad" (There must exist such a piece since each of them labeled two pieces out of three). Since Player 1 cut the cake in step 1, this piece is worth exactly $\frac{1}{3}$, according to v_1. Note that the remaining pieces are worth more than $\frac{2}{3}$, according to v_2 and v_3. The protocol ends with Cut-and-Choose between Players 2 and 3, which we have proved to be proportional. ■

− Is not envy-free.
  Example In case Players 2 and 3 play Cut-and-Choose at step 4, and Player 2 cuts 2 pieces which are considered uneven according to v_1, then Player 1 will envy the Player who gets the bigger piece (according to v_1).

− The resulting pieces do not have to be contiguous.

\+ Requires at most 3 cuts.

2.3 Protocol for arbitrary n (Dubins-Spanier, 1961)

1. A referee moves a knife slowly across the cake, from left to right.
2. Any player may shout "stop" when the piece to the left of the knife becomes of size $\frac{1}{n}$, according to his valuation function. Whoever does so receives the piece to the left of the knife, and is done. When a piece has been cut off, we continue with the remaining players.
3. When there is just one player left, he takes the rest of the cake.

2.3.1 Protocol Properties

+ Is proportional.
  Proof Each of the first $n-1$ players gets a piece once it becomes of size $\frac{1}{n}$, according to his valuation function. Since the last player hasn't shouted "stop", none of the $n-1$ pieces which were given to the rest of the players were worth at least $\frac{1}{n}$, according to his valuation function, thus he considers the remaining piece of size at least $1 - ((n-1) \cdot \frac{1}{n}) = \frac{1}{n}$. ■

− Is not envy-free.
  The last chooser is best off (he is the only one that can get more than $\frac{1}{n}$).

+ Uses a minimal number of cuts (exactly $n-1$).

+ Produces contiguous slices.

− Requires the active help of a referee, and a moving knife.

2.4 Envy-Free Protocol for $n = 3$ (Selfridge-Conway, 1960)

0 Stage Zero:

  0.1 Player 1 divides the cake into three equal parts according to his valuation function v_1

  0.2 Player 2 trims the largest piece (according to his valuation function v_2) so that it will become equal to the second largest piece. We shall denote the trimmed part as **cake 2** and the rest as **cake 1**.

1 Stage One (Dividing Cake 1):

  1.1 Player 3 chooses a piece of cake 1.

  1.2 If Player 3 chose the trimmed piece, Player 2 chooses freely from the remaining pieces. Else, Player 2 takes the trimmed piece.

  1.3 Player 1 takes the remaining piece of cake 1.

  We shall denote the player that took the trimmed piece with $T \in \{2, 3\}$ and the player $\in \{2, 3\}$ that took a non-trimmed piece with NT.

2 Stage Two (Dividing Cake 2):

  2.1 Player NT divides cake 2 into three equal parts according to v_{NT}.

  2.2 Player T chooses first, then Player 1 and then Player NT.

Theorem 5 *The Selfridge-Conway Protocol is envy-free.*

Proof We will show that the partition of cake 1 and cake 2 are envy-free.

- Partition of **Cake 1**:
  - Player 3 chooses first and so he does not envy the other players' choices.
  - Player 2 has two pieces he values most, one of them being the trimmed piece. If Player 3 chooses the trimmed piece then Player 2 chooses its second largest piece. Else, Player 2 gets the trimmed piece (which is one of the two largest pieces according to v_2)
  - Player 1 does not take the trimmed piece (which is worth at least $\frac{1}{3}$ according to v_1), but instead he takes one of the other pieces which are worth *exactly* $\frac{1}{3}$ to him.
- Partition of **Cake 2**:
  - Player T chooses first and so he does not envy the other players' choices.
  - Player NT has divided cake 2 into three equal parts and so any choice he makes is envy-free.
  - Player 1 cannot envy Player T since his combined share of cakes 1 & 2 is worth at most $\frac{1}{3}$ according to v_1. Also, Player 1 cannot envy Player NT since he makes his choice before NT does.

■

2.4.1 Protocol Properties

+ Envy-Free (⇒ Proportional).

− Non-contiguous slices.

0 Number of slices is at most 5.

2.5 A Deceitful Protocol

There exists a one-to-one function from all possible valuation functions on to [0,1] (denoted by f).

- **for** {i=1,...,n-1}
  - The protocol asks Player i to slice a piece that is worth $\frac{1}{2}$.
  - Player i slices at point α which encodes v_i according to f.
- Player n computes an envy-free partition (offline) and executes the slicing.

Remark The protocol is deceitful in the sense that it uses cut queries, but the players do not answer truthfully to them. Instead, their answer is $\alpha = f(v_i) \in [0,1]$ and this information enables the calculatation of an envy-free division in linear communication complexity.

References

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