A Hierarchy of Proof Rules for Checking Differential Invariance of Algebraic Sets

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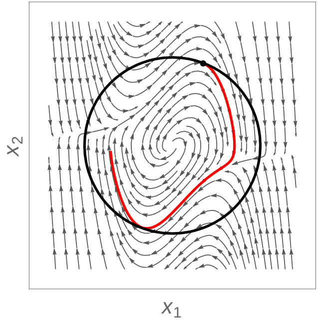
VMCAI, Mumbai, India
January 13th, 2015

Problem: Checking the Invariance of Algebraic Sets



Ordinary Differential Equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x + (1 - x^2)y \end{pmatrix} = \mathbf{p}$$

(Real) Algebraic Sets

$$V_{\mathbb{R}}(h) = \{(x, y) \in \mathbb{R} \mid \underbrace{x^2 + y^2 - 1 = 0}_{h(x,y)=0}\}$$

Context

Motivations

- Theorem Proving for Hybrid Systems
- Stability and Safety Analysis of Dynamical Systems
- Qualitative Analysis of Differential Equations

Current Status

- Invariance of algebraic sets is **decidable**
- A decision procedure exists
- Many sufficient conditions are known

Contributions

Hierarchy of the different proof rules

- Compare the deductive power of 7 proof rules, **2 of which are novel**
- **Subclasses** of algebraic sets **characterized** by each proof rule

Assess the deductive power versus efficiency trade-off

- **Deductive power** increase $\rightsquigarrow$ **computational cost** increase ?
- What is the **practical efficiency** of those proof rules ?

Outline

Definitions

Gradient $\nabla h := (\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n})$

Lie Derivation $\mathfrak{D}_{\mathbf{p}}(h) := \frac{dh(\mathbf{x}(t))}{dt} = \langle \nabla h, \mathbf{p} \rangle \qquad (\dot{\mathbf{x}} = \mathbf{p})$

Singular Locus

$$\mathrm{SL}(h) := \{\mathbf{x} \in \mathbb{R}^n \mid \nabla h = \mathbf{0}\} = \left\{\mathbf{x} \in \mathbb{R}^n \mid \frac{\partial h}{\partial x_1} = 0 \wedge \cdots \wedge \frac{\partial h}{\partial x_n} = 0\right\}$$

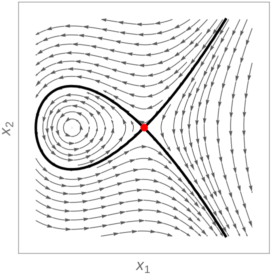
$h(\mathbf{x}) = 0$ ($\mathbf{x} \in V_{\mathbb{R}}(h)$) is **singular** if $\mathbf{x} \in \mathrm{SL}(h)$, **regular** otherwise.

Lie's Criterion

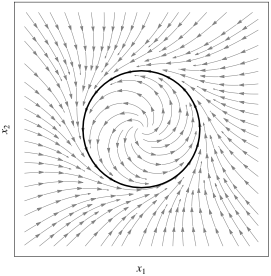
[Platzer, ITP 2012]

Necessary and sufficient for smooth invariant manifolds (Lie, 1893).

$$(\text{Lie})\frac{h = 0 \rightarrow (\mathfrak{D}_{\mathbf{p}}(h) = 0 \wedge \nabla h \neq \mathbf{0})}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}]\,(h = 0)}$$



$h = 0$ non-smooth ✗



$h = 0$ smooth ✓

Extensions of Lie

Contribution

Handling certain **singularities** (points where $\nabla h = \mathbf{0}$)

No flow in the problem variables at singularities on the variety

$$(\text{Lie}^{\circ})\frac{h = 0 \rightarrow \left(\mathfrak{D}_{\mathbf{p}}(h) = 0 \wedge \left(\nabla h = \mathbf{0} \rightarrow \mathbf{p} = \mathbf{0}\right)\right)}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}]\ (h = 0)}$$

Flow at singularities on the variety is directed into the variety

$$(\text{Lie}^{*})\frac{h = 0 \rightarrow \left(\mathfrak{D}_{\mathbf{p}}(h) = 0 \wedge \left(\nabla h = \mathbf{0} \rightarrow h(\mathbf{x} + \lambda \mathbf{p}) = 0\right)\right)}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}]\ (h = 0)} \ .$$

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Extensions of Lie: Lie$^\circ$

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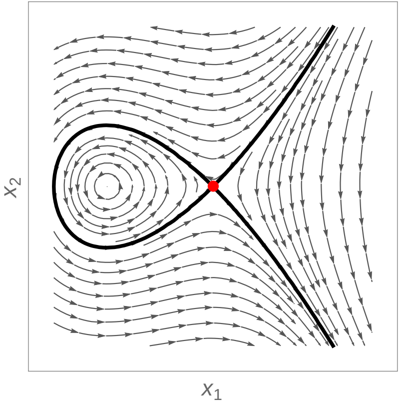

Lie ✗   Lie$^\circ$ ✓   Lie* ✓

Extensions of Lie: Lie*

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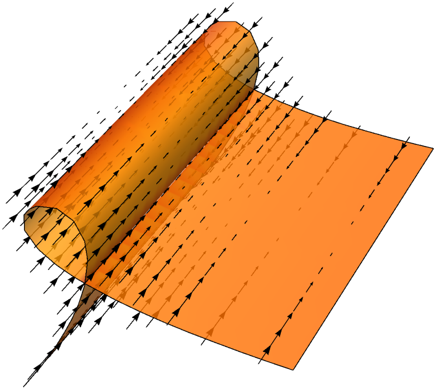


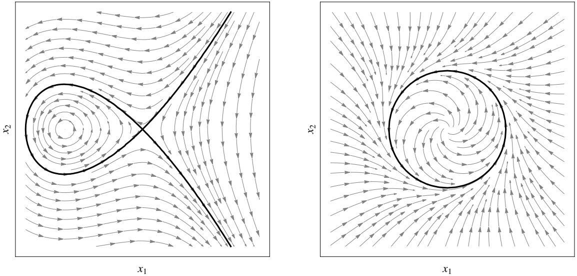
Lie ✗ Lie° ✗ Lie* ✓

Differential Invariant ($DI_=$)

[Platzer, J. Log. Comput. 2010]

Necessary and sufficient for conserved quantities (integrals of motion).

$$(DI_=)\frac{\mathfrak{D}_{\mathbf{p}}(h) = 0}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}]\ (h = 0)}$$



h conserved ✓ h not conserved ✗

Extensions of $\text{DI}_=$

[Sankaranarayanan et al., FMSD 2008]

Continuous consecutions (C-c) and **polynomial consecutions** (P-c) are Darboux polynomials (Darboux, 1878).

$$\text{(C-c)}\frac{\exists \lambda \in \mathbb{R},\ \mathfrak{D}_{\mathbf{p}}(h) = \lambda h}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}]\ (h = 0)},$$

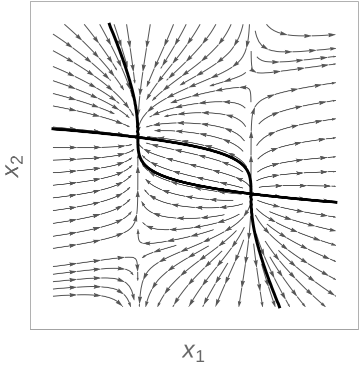
$$\text{(P-c)}\frac{\exists \lambda \in \mathbb{R}[\mathbf{x}],\ \mathfrak{D}_{\mathbf{p}}(h) = \lambda h}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}]\ (h = 0)}\ .$$

Extensions of $\mathrm{DI}_=$

[Sankaranarayanan et al., FMSD 2008]

$$\mathbf{p} = (3\left(x_1^2 - 4\right), -x_2^2 + x_1 x_2 + 3), \qquad h = x_2^4 + 2x_1 x_2^3 + 6x_2^2 + 2x_1 x_2 + x_1^2 + 3,$$

$$\mathfrak{D}_{\mathbf{p}}(h) = \underbrace{(6x_1 - 4x_2)}_{\lambda} h$$



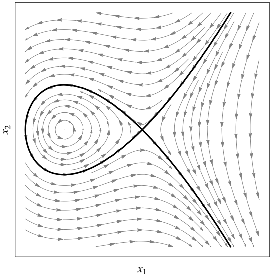
$\mathrm{DI}_=$ ✗ C-c ✗ P-c ✓

Differential Radical Invariants (DRI)

[G. et al., TACAS 2014, SAS 2014]

Necessary and sufficient for invariant varieties.

$$(\mathrm{DRI})\frac{h = 0 \rightarrow \bigwedge_{i=0}^{N-1} \mathfrak{D}_{\mathbf{p}}^{(i)}(h) = 0}{(h = 0) \rightarrow [\dot{\mathbf{x}} = \mathbf{p}]\ (h = 0)}$$



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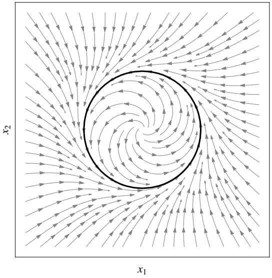

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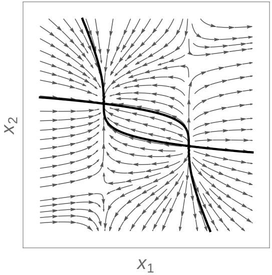

✓

How to compare these different proof rules ?

$$\{\mathrm{DI}_=, \text{C-c}, \text{P-c}, \mathrm{Lie}, \mathrm{Lie}^\circ, \mathrm{Lie}^*, \mathrm{DRI}\}$$

For some classes of problems, the premises of the proof rules lead to *decision procedures*.

Natural questions:

- Given two decision procedures, which is more practical?
- Are any of these proof rules redundant?

To answer these, we perform

- Theoretical comparison
- Empirical performance analysis

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Outline

Order Relation

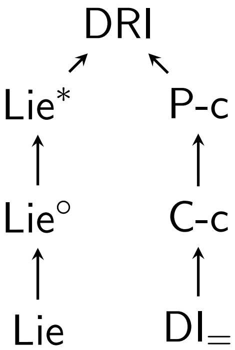
$$(\mathrm{R}_A)\frac{A}{(h=0) \longrightarrow [\dot{\mathbf{x}} = \mathbf{p}](h=0)} \qquad (\mathrm{R}_B)\frac{B}{(h=0) \longrightarrow [\dot{\mathbf{x}} = \mathbf{p}](h=0)}$$

Partial Order

$\mathrm{R}_A \preccurlyeq \mathrm{R}_B$ if and only if $A \implies B$.

- $\mathrm{R}_A \sim \mathrm{R}_B$ ($\mathrm{R}_A \preccurlyeq \mathrm{R}_B$ and $\mathrm{R}_A \succcurlyeq \mathrm{R}_B$) **Equivalence**.
- $\mathrm{R}_A \prec \mathrm{R}_B$ ($\mathrm{R}_A \preccurlyeq \mathrm{R}_B$ and $\mathrm{R}_A \not\succcurlyeq \mathrm{R}_B$) **Strict increase** of deductive power

Hasse Diagram: Deductive Hierarchy



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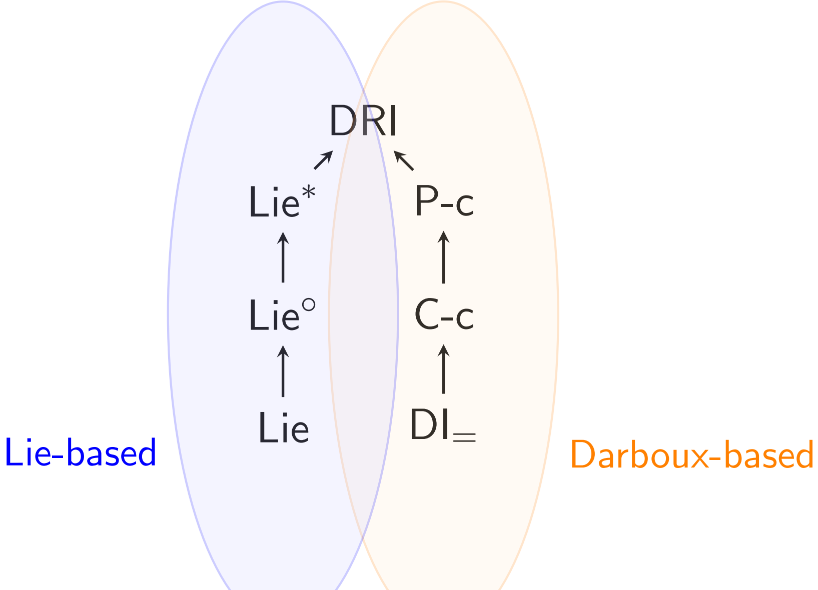
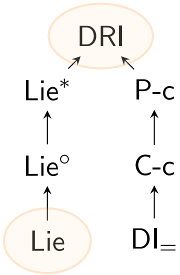
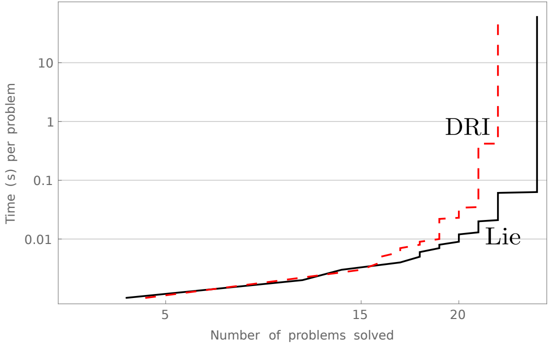


Outline

Smooth invariant manifolds (Lie vs DRI)

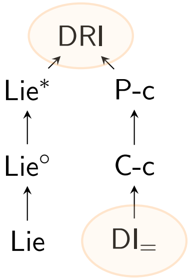
Lie and DRI **decide** invariance for **smooth invariant manifolds**.

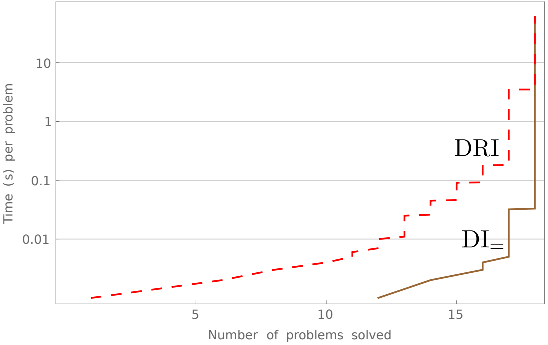




Functional invariants (DI vs DRI)

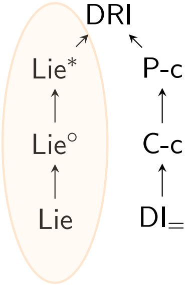
$DI_=$ and DRI **decide** invariance of varieties of **conserved quantities**.

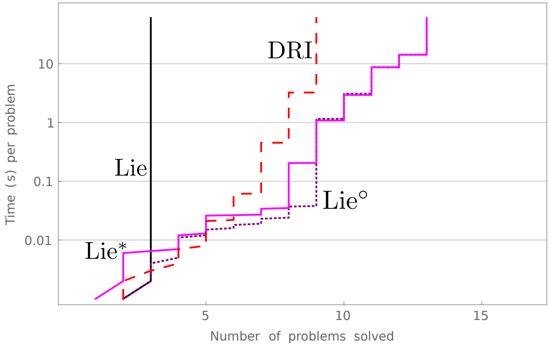




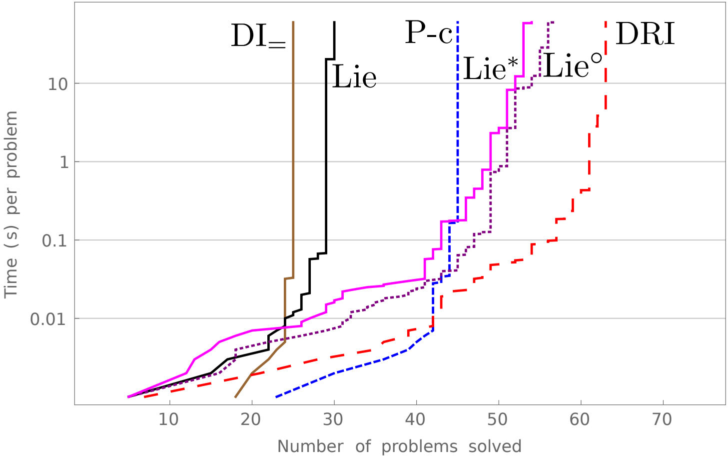
Singularities at Equilibria (Lie, Lie° & Lie* vs DRI)

Lie°, Lie* and DRI **decide** invariance for varieties of with **singularities that are equilibrium points**.





Experimental Performance of All Proof Rules



Outline

Square-free Reduction

Square-free reduction of a polynomial

$$h = h_1^{\alpha_1} h_2^{\alpha_2} \cdots h_k^{\alpha_k}$$

Geometrically $V_{\mathbb{R}}(h) \equiv_{\mathbb{R}} V_{\mathbb{R}}(\mathrm{SF}(h))$.

- SF **automated pre-processing** step in **computer algebra** systems
- Is it a "good idea" to apply SF for invariance checking ?

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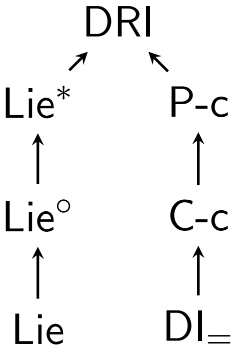

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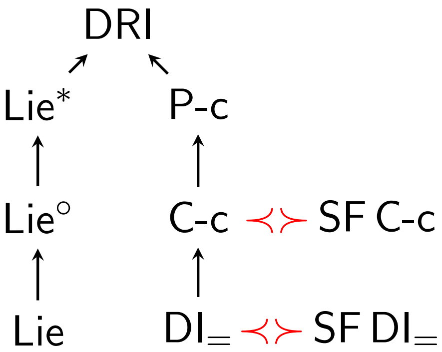

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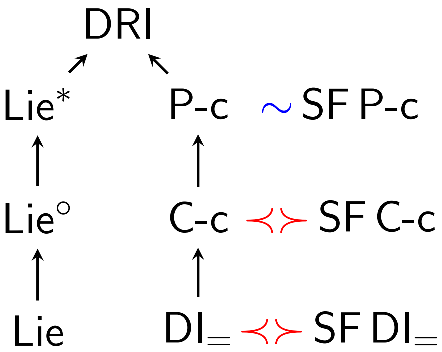

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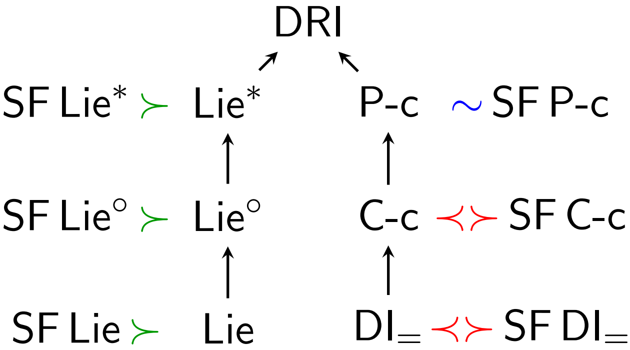

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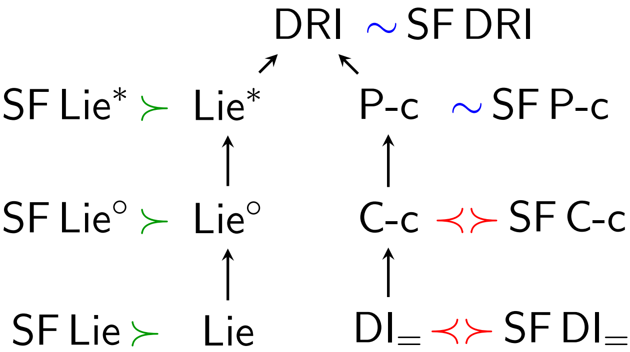

Summary

- The **most deductively powerful** rule DRI **performs very well**.
- C-c and P-c are made **redundant** by DRI.
- SF reduction is always of **benefit** to the Lie-**based** proof rules
- SF with $DI_=$ and C-c yields new **incomparable** proof rules.
- SF with P-c is **as powerful as** P-c alone.
- SF may introduce a **performance penalty** for DRI.

Conclusions

DRI has good performance on average

Apply DRI first with a time-out.

Sufficient proof rules are useful

Exploit the computational sweet spots of sufficient conditions

Thank you for attending !