General Solution for Linearized Error Propagation in Vehicle Odometry

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Abstract

Although odometry is nonlinear, it yields sufficiently to linearized analysis to produce a closed-form transition matrix and a symbolic general solution for both deterministic and stochastic error propagation. Accordingly, error propagation in vehicle odometry can be understood at a level of theoretical rigor equivalent to the well-known Schuler dynamics of inertial navigation. While response to initial conditions is path-independent, response to input errors can be related to path functionals. These trajectory moments are integral transforms which function like the moment of inertia or the Laplace transform - enabling many error propagation calculations to be performed by hand in closed-form.

1. Introduction

The problem of analytically computing the navigational error expected in odometry from a given set of sensor errors on a given trajectory seems to be both a fundamental and an unsolved problem. While a numerical solution to the problem of computing the resultant error is trivial, symbolic solutions yield dividends in the form of understanding the general case that numerical ones cannot. In this paper, the general solution is derived and validated.

1.1 Motivation

This work is motivated by a recurrent set of questions which arise for position estimation systems in mobile robots. Historically they have been answered numerically or in an ad hoc manner. How good do the sensors need to be? What kind of localization error can be expected if this particular sensor is used? Under what conditions do errors cancel out? What is the best way to calibrate the systematic or stochastic error model of this sensor? While the essentially dynamic nature of odometry error propagation is unavoidable, the integral transforms derived here enable rapid analytical algebraic solutions to many such design questions.

1.2 Prior work

The aerospace guidance community has enjoyed the benefits of a theoretical understanding of error propagation for at least five decades [1]. In inertial guidance, the governing differential equations and their solutions explain the stabilizing influence of gravity - horizontal errors exhibit oscillation with the characteristic Schuler period of 84 minutes.

Likewise, the essentially geometric nature of satellite navigation system error relationships was known before the GPS satellites were in operation [2]. Yet, the guidance community seems not to have provided the relevant analytical results for the land navigation systems which are typical of mobile robots - assemblies of wheel encoders, compasses, gyros, etc.

Analytical study of error propagation in mobile robot odometry appears only rarely in the literature. Early work in [3] concentrates on improving estimates for a single iteration of the estimation algorithm by incorporating knowledge of the geometry of the path followed between odometry updates. In [4], a geometric method is presented for the calibration of certain systematic errors on rectangular closed trajectories.

In [5], a recurrence equation solution is obtained for non systematic error on constant curvature trajectories. In [6], a one dimensional closed-form solution for a broader optimal estimation problem is presented. This paper presents the general solution for linearized systematic and random error propagation in odometry in the plane for any trajectory or error model.

1.3 Problem description

Odometry is a form of dead reckoning because the available measurements must be integrated to provide a navigation solution. In the common "forced dynamics" formulation of odometry, the measurements, normally denoted $\underline{z}(t)$, are identified with the usual control inputs $\underline{u}(t)$.

The state vector $\underline{x}(t)$ and input vector $\underline{u}(t)$ are:

$$\underline{x}(t) = \begin{bmatrix} x(t) & y(t) & \theta(t) \end{bmatrix}^T \quad \underline{u}(t) = \begin{bmatrix} V(t) & \omega(t) \end{bmatrix}^T$$

where the state is the vehicle pose and the inputs are the linear and angular velocity. The associated odometry equations in this case are those of the "integrated heading" case:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \\ \omega(t) \end{bmatrix}$$

The x axis has been implicitly chosen as the heading datum as illustrated below:

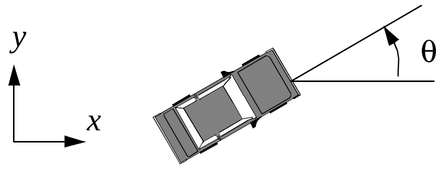

Figure 1: Coordinates for odometry.

Most of the notation used in subsequent sections follows that used in the classical texts such as [7]. Generally, an observer equation can be introduced to model how sensor indications project onto the inputs and states and the general situation is described by:

$$\begin{aligned} \dot{\underline{x}}(t) &= f(\underline{x}(t), \underline{u}(t)) \\ \underline{z}(t) &= h(\underline{x}(t), \underline{u}(t)) \end{aligned} \tag{1}$$

Many alternative formulations of odometry are possible. This one and many others have the key properties of being homogeneous in the inputs, nonlinear in the states, and reduced to echelon form. Due to the latter property, the general solution is immediate:

$$\begin{aligned} \theta(t) &= \theta(0) + \int_0^t \omega(t)dt \\ x(t) &= x(0) + \int_0^t V(t)\cos\theta(t)dt \\ y(t) &= y(0) + \int_0^t V(t)\sin\theta(t)dt \end{aligned} \tag{2}$$

Closed form solutions to these equations are nevertheless available only for the simplest of inputs and nonlinear error propagation is equally intractable.

This paper addresses the following problem. Let the inputs (or equivalently, the sensors) to the system be corrupted by additive errors $\delta\underline{u}(t)$ as follows:

$$\underline{u}'(t) = \underline{u}(t) + \delta\underline{u}(t) \tag{3}$$

Using these input errors and the system dynamics, determine the behavior of the associated errors $\delta\underline{x}(t)$ in the computed vehicle pose:

$$\underline{x}'(t) = \underline{x}(t) + \delta\underline{x}(t) \tag{4}$$

The errors can be systematic or random in nature, and solutions for either case are sought.

2. Nonlinear Error Dynamics

It is easy to see that substituting equation (3) into equation (2) generates equations with little hope of solution - although the simple cases of a zero curvature and constant curvature trajectory are solvable. Nonetheless, some important properties of odometry, and in many cases of odometry error propagation, can be discerned without solving the nonlinear equations. Based on the following discussion, it can be shown that odometry is memoryless, reversible, and motion dependent.

2.1 Homogeneity and reversibility

When the nonlinear system dynamics satisfy:

$$f[\underline{x}(t), k \times \underline{u}(t)] = k^n \times f[\underline{x}(t), \underline{u}(t)] \tag{5}$$

for some constant k, the system is said to be homogeneous of degree n with respect to $\underline{u}(t)$. Such systems are memoryless - their zero input response is zero.

Systems of odd order homogeneity are odd with respect to $\underline{u}(t)$ and hence reversible because they can be driven precisely back over their original trajectory with little effort by simply reversing the input signal $\underline{u}(t)$ in time. Systems of even order homogeneity are even with respect to $\underline{u}(t)$ and monotone because the sign of the state derivative is invariant under changes in the inputs.

2.2 Rate inputs and motion dependence

If a particular input $u_i(t)$ to a homogeneous system can be written as the time rate of some other parameter such as s, it can be divided by the input without creating a singularity:

$$\frac{f[\underline{x}(t), \underline{u}(t)]}{u_i(t)} = \frac{f[\underline{x}(t), \underline{u}(t)]}{(ds)/(dt)} = finite \tag{6}$$

Hence, a change of variable becomes possible and the system can be written in the form:

$$\frac{d}{ds}\underline{x}(s) = \tilde{f}[\underline{x}(s), \underline{u}(s)] \tag{7}$$

Such a system is motion dependent - the state advances only under nonzero velocity conditions. The distinction is important because inertially derived sensor errors continue to grow when motion stops, whereas sensor errors for terrain relative indications do not.

2.3 Closure and path independence

State space trajectories which close on themselves are of special interest in error analysis. In general, a trajectory closes on the interval $[0, T]$ when the closure condition is satisfied by the state space trajectory:

$$\underline{x}(T) - \underline{x}(0) = \int_0^T F[\underline{u}(t)]dt = 0 \tag{8}$$

In a general solution, the integrand $F(\ )$ will depend on the inputs but not the state. A few special cases are important for error analysis.

The first is symmetry. If it is possible to partition the interval of integration such that every time t_1 can be paired with another time t_2 such that:

$$F[\underline{u}(t_1)]dt = -F[\underline{u}(t_2)]dt \tag{9}$$

then the closure condition will be satisfied.

The second important special case is path independence. When the integrand can be written in the form:

$$\underline{F}[\underline{u}(t)]dt = \frac{\partial}{\partial \underline{u}}\underline{p}(\underline{u}) \bullet d\underline{u} = d\underline{p}(\underline{u}) \tag{10}$$

the resulting total differential can be integrated in closed form:

$$\underline{x}(t) = p[\underline{u}(t)] \tag{11}$$

Such integrals are relevant here because they provide the conditions for which odometry error must cancel on closed trajectories of any shape.

2.4 Path independence of velocity scale error

That velocity (e.g encoder) scale errors are path independent follows from odometry's homogeneity in velocity. If linear velocity is corrupted by additive errors are follows:

$$V'(t) = V(t) + \delta V(t)$$

while angular velocity is unchanged, then the perturbed position vector must vanish on a closed path:

$$\begin{gathered}\int_0^t \begin{bmatrix} c\theta \\ s\theta \end{bmatrix} V'(t)dt = \int_0^t \begin{bmatrix} c\theta \\ s\theta \end{bmatrix} V(t)dt + \int_0^t \begin{bmatrix} c\theta \\ s\theta \end{bmatrix} \delta V(t)dt \\ \oint \begin{bmatrix} dx' \\ dy' \end{bmatrix} = \oint \begin{bmatrix} dx \\ dy \end{bmatrix} + \oint \begin{bmatrix} d(\delta x) \\ d(\delta y) \end{bmatrix}\end{gathered} \tag{12}$$

Both the left side and the first term on the right vanish identically by definition of position coordinates. Hence, the last term must vanish as well and the path independent response of odometry (in its full nonlinear form) to velocity scale errors is established.

3. Linearized Error Dynamics

Here, the governing equations of odometry error dynamics are developed and described.

3.1 Linearization

Perturbative techniques linearize nonlinear dynamical systems in order to study their first order behavior. With reference to equation (1), several Jacobian matrices are defined which may depend on the state and the input, and are evaluated on some reference trajectory:

$$\begin{aligned} F(t) &= \frac{\partial}{\partial \underline{x}}\underline{f}\Big|_{\underline{x},\,\underline{u}} & G(t) = L(t) &= \frac{\partial}{\partial \underline{u}}\underline{f}\Big|_{\underline{x},\,\underline{u}} \\ H(t) &= \frac{\partial}{\partial \underline{x}}\underline{h}\Big|_{\underline{x},\,\underline{u}} & M(t) = N(t) &= \frac{\partial}{\partial \underline{u}}\underline{h}\Big|_{\underline{x},\,\underline{u}} \end{aligned} \tag{13}$$

To be consistent with common notation, $G(t)$ will be used when discussing systematic error and $L(t)$ when discussing random error. $M(t)$ and $N(t)$ will be used similarly.

Equation (1) is linearized as follows:

$$\begin{aligned} \delta\dot{\underline{x}}(t) &= F\underline{x}(t), \underline{u}(t)\delta\underline{x}(t) \\ &+ G\{\underline{x}(t), \underline{u}(t)\}\delta\underline{u}(t) + L\{\underline{x}(t), \underline{u}(t)\}\delta\underline{w}(t) \\ \delta\underline{z}(t) &= H\{\underline{x}(t), \underline{u}(t)\}\delta\underline{x}(t) \\ &+ M\{\underline{x}(t), \underline{u}(t)\}\delta\underline{u}(t) + N\{\underline{x}(t), \underline{u}(t)\}\delta\underline{w}(t) \end{aligned} \tag{14}$$

Parameters can be considered to be constant inputs. Subsequently, notational dependence on the state and the input will be suppressed for brevity but all of these matrices generally depend on both. These provide a linear approximation to the propagation of systematic error as well as a description for the propagation of the mean of random error.

A second input vector $\delta\underline{w}(t)$ has been introduced to differentiate systematic from random error sources. $\delta\underline{w}(t)$ is simply the component of the original $\delta\underline{u}(t)$ which is random. By superposition, systematic and random error sources can be treated independently.

For the random errors $\delta\underline{w}(t)$, equation (14) is used only in a heuristic sense in stochastic calculus because direct integration of random signals is beyond the scope of traditional calculus.

3.2 Deterministic case

For deterministic error, the linearized dynamics take the form of:

$$\begin{aligned} \delta\dot{\underline{x}}(t) &= F(t)\delta\underline{x}(t) + G(t)\delta\underline{u}(t) \\ \delta\underline{z}(t) &= H(t)\delta\underline{x}(t) + M(t)\delta\underline{u}(t) \end{aligned} \tag{15}$$

The first is the linear perturbation equation. The second, the linearized observer, is not a differential equation. It can be solved and substituted into the state equations to produce an equation of the same form as the original perturbation equation. It can be dispensed with for the balance of the paper and the deterministic case can be considered to be defined by the linear perturbation equation:

$$\delta\dot{\underline{x}}(t) = F(t)\delta\underline{x}(t) + G(t)\delta\underline{u}(t) \tag{16}$$

Once the matrices are filled in, and error models are assumed, it is instructive to evaluate equation (16) for motion dependence, reversibility, etc.

3.3 Stochastic case

For practical treatment of random errors, the second moment or "covariance" of the error is considered and the state covariance and input spectral density matrices are defined:

$$\begin{aligned} P &= Exp(\delta \underline{x}(t)\delta \underline{x}(t)^T) \\ Q &= Exp(\delta \underline{w}(t)\delta \underline{w}(\tau)^T)\delta(t-\tau) \end{aligned} \tag{17}$$

In the second case, the Dirac delta signifies that the random sequence $\delta \underline{w}(t)$ is white and that the units of Q are a time rate of covariance. The solution for linear variance propagation is derived in several texts including [7]:

$$\begin{aligned} \dot{P}(t) &= F(t)P(t) + P(t)F(t)^T + L(t)Q(t)L(t)^T \\ R(t) &= H(t)P(t)H(t)^T + M(t)Q(t)M(t)^T \end{aligned} \tag{18}$$

The first is the linear variance equation. The second, the linear stochastic observer, is not a differential equation. It can be substituted into the state equations to produce an equation of the same form as the original variance equation. It can be dispensed with for the balance of the paper and the stochastic case can be considered to be defined by the linear variance equation:

$$\dot{P}(t) = F(t)P(t) + P(t)F(t)^T + L(t)Q(t)L(t)^T \tag{19}$$

4. Solution to Linearized Systems

Here, the solution equations are developed and described.

4.1 General solutions

The equations to be solved are equations (16) and (19). It is well known that the solution to these equations rests on the knowledge of a very important matrix called the transition matrix, denoted $\Phi(t, \tau)$. One definition for this matrix is that it is the solution to:

$$\frac{d}{dt}\Phi(t,\tau) = F(t)\Phi(t,\tau) \tag{20}$$

The general solutions for the propagation of systematic and random error are respectively of the form of the vector and matrix convolution integrals:

$$\begin{aligned} \delta \underline{x}(t) &= \Phi(t, t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)\delta \underline{u}(\tau)d\tau \\ P(t) &= \Phi(t, t_0)P(t_0)\Phi^T(t, t_0) \\ &+ \int_{t_0}^{t} \Phi(t,\tau)L(\tau)Q(\tau)L^T(\tau)\Phi^T(t,\tau)d\tau \end{aligned} \tag{21}$$

Once a trajectory and an error model are assumed, the only unknown in both of these equations is the transition matrix.

4.1.1 *Error propagation behavior*

The important behaviors of error propagation are evident from the structure of these solutions. Both solutions consist of a state (initial conditions $\underline{x}(t_0)$ or $P(t_0)$) response and an input ($\delta \underline{u}(\tau)$ or $Q(\tau)$) response. The state response is always path independent and hence it vanishes on any closed trajectory. Perhaps surprisingly, this means both general solutions can exhibit extrema and even zeros.

Although the input response may be path independent or otherwise vanish under conditions of symmetry, it is generally path dependent. In other words, error propagation is a functional defined on the inputs and it can, in certain cases, be expressed as a functional on the reference state space trajectory. The integrand in the systematic case is signed and can exhibit zeros. In the stochastic case, the integrand is positive semidefinite and its contribution is nondecreasing.

4.1.2 *Input transition matrix*

It is very useful to define the (potentially nonsquare) input transition matrix as:

$$\tilde{\Phi}(t,\tau) = \Phi(t,\tau)L(\tau) = \Phi(t,\tau)G(\tau) \tag{22}$$

This matrix maps a given systematic or random error at time τ onto its net effect on the resultant state error occurring later at time t. Linearization for the purposes of studying error propagation amounts to treating errors occurring at different times independently of each other - their compounded effects being higher order. This matrix is the defining matrix for each form of odometry - capturing both the effects of system dynamics and of the input measurement errors.

4.1.3 *Influence matrices*

Let $\underline{\tilde{\Phi}}_i$ denote the ith column of the input transition matrix. Notice that for a given element δu_i of $\delta \underline{u}$, its contribution to the solution integrands in equation (21) is:

$$\begin{aligned} d[\delta \underline{x}(t,\tau)] &= \underline{\tilde{\Phi}}_i \delta u_i d\tau \\ dP(t,\tau) &= q_{ij}(\underline{\tilde{\Phi}}_i \underline{\tilde{\Phi}}_j^T)d\tau \end{aligned} \tag{23}$$

Hence, the influence vectors $\underline{\tilde{\Phi}}_i$ define the projection of each individual element of the input (measurement) error vector onto the entire output (state) error vector. Similarly, the outer product influence matrices $\underline{\tilde{\Phi}}_i \underline{\tilde{\Phi}}_j^T$ define the projection of each element of the input (measurement) covariance matrix onto the entire output (state) covariance matrix.

Clearly:

$$\underline{\tilde{\Phi}}_i \underline{\tilde{\Phi}}_i^T = symmetric \qquad \underline{\tilde{\Phi}}_i \underline{\tilde{\Phi}}_j^T = (\underline{\tilde{\Phi}}_j \underline{\tilde{\Phi}}_i^T)^T$$

In terms of influence matrices, equation (21) can be rewritten as:

$$\begin{aligned} \delta \underline{x}(t) &= \Phi(t, t_0)\delta \underline{x}(t_0) + \int_{t_0}^{t} \Big(\sum_i \underline{\tilde{\Phi}}_i \delta u_i\Big) d\tau \\ P(t) &= \Phi(t, t_0) P(t_0) \Phi^T(t, t_0) \\ &+ \int_{t_0}^{t} \Big(\sum_i \sum_j q_{ij} (\underline{\tilde{\Phi}}_i \underline{\tilde{\Phi}}_j^T)\Big) d\tau \end{aligned} \tag{24}$$

The columns of the input transition matrix constitute a basis for the time derivative of error because the result for any error source must reside in the column space of $\tilde{\Phi}(t, \tau)$.

4.1.4 Trajectory moment matrices

The order of integration and summation can be reversed when convenient. Also, when the errors are constant or can be rendered so under a change of variable, they can be moved outside the integrals to produce:

$$\begin{aligned} \delta \underline{x}(t) &= \Phi(t, t_0)\delta \underline{x}(t_0) + \sum_i \left[\int_{t_0}^{t} \underline{\tilde{\Phi}}_i d\tau\right] \delta u_i \\ P(t) &= \Phi(t, t_0) P(t_0) \Phi^T(t, t_0) \\ &+ \sum_i \sum_j \left[\int_{t_0}^{t} \underline{\tilde{\Phi}}_i \underline{\tilde{\Phi}}_j^T d\tau\right] q_{ij} \end{aligned} \tag{25}$$

This is the trajectory moment form of the error propagation equations. The two expressions in square brackets are respectively the trajectory moment vector and the trajectory moment matrix. Both are intrinsic properties of the trajectory - independent of the form of the errors. Their role in estimation is similar to the role of the Laplace Transform in control, the Fourier transform in circuits, the moment of inertia in mechanics and the moments of probability distributions (the mean is the first probabilistic moment, variance the second).

When errors have simple forms, the effects of errors and trajectories can be decoupled, and the latter can be tabulated for specific trajectories and used like any other integral transform to convert problems in differential equations into equivalent algebraic ones.

Trajectory moment matrices constitute a basis for covariance in dynamical systems. Indeed, by setting any one error source to unity and the rest to zero, it is clear that they are numerically equal to the covariance - and so they must behave exactly like covariance matrices. In this sense, the structure of covariance itself is exposed in the trajectory moment matrix - because it is one.

4.2 Solution for commutative dynamics

Given that the system Jacobian of odometry is a time varying matrix, linear systems theory shows that the transition matrix exists, but generally provides little guidance in finding it. Luckily, odometry is a special case.

Consider the following particular matrix exponential of a definite integral of the system Jacobian:

$$\Psi(t, \tau) = \exp[R(t, \tau)] = \exp\Big(\int_{\tau}^{t} F(\zeta) d\zeta\Big) \tag{26}$$

It is easy to show that when this matrix commutes with the system dynamics matrix:

$$\Psi(t, \tau) F(t) = F(t) \Psi(t, \tau) \tag{27}$$

it satisfies equation (20) and therefore is the transition matrix which solves the associated time-varying linear system. This property of commutative dynamics is the key to generating a general solution.

5. Application To Odometry

This section derives the error propagation equations for a few common forms of odometry.

5.1 Direct heading odometry

Direct heading refers to the case where a direct measurement of heading is available rather than its derivative. For example, a compass could be used to measure heading directly and a transmission encoder could be used to measure the linear velocity of the vehicle.

The heading and error in heading are respectively equal at all times to the heading measurement and its error. Considering the heading to be an input, the state equations are:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} V(t)\cos\theta(t) \\ V(t)\sin\theta(t) \end{bmatrix} \tag{28}$$

Where:

$$\begin{aligned} \underline{x}(t) &= \begin{bmatrix} x(t) & y(t) \end{bmatrix}^T & Q(t) &= \begin{bmatrix} \sigma_{vv} & \sigma_{v\theta} \\ \sigma_{v\theta} & \sigma_{\theta\theta} \end{bmatrix} \\ \underline{u}(t) &= \begin{bmatrix} V(t) & \theta(t) \end{bmatrix}^T \\ \delta\underline{x}(t) &= \begin{bmatrix} \delta x(t) & \delta y(t) \end{bmatrix}^T & P(t) &= \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \\ \delta\underline{u}(t) &= \begin{bmatrix} \delta V(t) & \delta\theta(t) \end{bmatrix}^T \end{aligned} \tag{29}$$

The Jacobians are:

$$F(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad G(t) = \begin{bmatrix} c\theta(t) & -V(t)s\theta(t) \\ s\theta(t) & V(t)c\theta(t) \end{bmatrix} \tag{30}$$

So the linearized error dynamics are:

$$\frac{d}{dt}\begin{bmatrix}\delta x(t)\\ \delta y(t)\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta x(t)\\ \delta y(t)\end{bmatrix} + \begin{bmatrix}c\theta & -Vs\theta\\ s\theta & Vc\theta\end{bmatrix}\begin{bmatrix}\delta V(t)\\ \delta\theta(t)\end{bmatrix}$$

The rest of the matrices leading to the input transition matrix are therefore as follows:

$$R(t,\tau) = \int_\tau^t \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} d\zeta = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \tag{31}$$

$$\Phi(t,\tau) = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \tilde{\Phi}(t,\tau) = \begin{bmatrix}c\theta(\tau) & -V(\tau)s\theta(\tau)\\ s\theta(\tau) & V(\tau)c\theta(\tau)\end{bmatrix}$$

Note that $V(t)c\theta(t) = \dot{x}(t)$ etc. could be substituted. The transition matrix clearly satisfies (27).

Substituting into the general solution in equation (21) gives:

$$\begin{aligned}\delta\underline{x}(t) &= \delta\underline{x}(0) + \int_0^t \begin{bmatrix}c\theta & -Vs\theta\\ s\theta & Vc\theta\end{bmatrix}\begin{bmatrix}\delta V(t)\\ \delta\theta(t)\end{bmatrix} d\tau\\ P(t) &= P(0)\\ &+ \int_{t_0}^t \begin{bmatrix}c\theta & -Vs\theta\\ s\theta & Vc\theta\end{bmatrix}\begin{bmatrix}\sigma_{vv} & \sigma_{v\theta}\\ \sigma_{v\theta} & \sigma_{\theta\theta}\end{bmatrix}\begin{bmatrix}c\theta & -Vs\theta\\ s\theta & Vc\theta\end{bmatrix}^T d\tau\end{aligned} \tag{32}$$

This result is the general linearized solution for the propagation of systematic and random error in 2D direct heading odometry for any trajectory or error model. When the velocity error is a scale error (i.e. $\delta V(t) \propto V(t)$), its effects are path independent.

5.2 Integrated heading odometry

In integrated heading odometry, an angular velocity indication is available and a heading state is necessary which is then integrated to get the heading. For example, a gyro could be used to measure heading rate and a transmission encoder, groundspeed radar, or fifth wheel encoder could be used to measure the linear velocity of the vehicle. This is the case given in equation (1):

$$\frac{d}{dt}\begin{bmatrix}x(t)\\ y(t)\\ \theta(t)\end{bmatrix} = \begin{bmatrix}V(t)\cos\theta(t)\\ V(t)\sin\theta(t)\\ \omega(t)\end{bmatrix} \tag{33}$$

Where:

$$\begin{aligned}\underline{x}(t) &= \begin{bmatrix}x(t) & y(t) & \theta(t)\end{bmatrix}^T \quad P(t) = \begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{x\theta}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{y\theta}\\ \sigma_{x\theta} & \sigma_{y\theta} & \sigma_{\theta\theta}\end{bmatrix}\\ \underline{u}(t) &= \begin{bmatrix}V(t) & \omega(t)\end{bmatrix}^T\\ \delta\underline{x}(t) &= \begin{bmatrix}\delta x(t) & \delta y(t) & \delta\theta(t)\end{bmatrix}^T\\ \delta\underline{u}(t) &= \begin{bmatrix}\delta V(t) & \delta\omega(t)\end{bmatrix}^T \quad Q(t) = \begin{bmatrix}\sigma_{vv} & \sigma_{v\omega}\\ \sigma_{v\omega} & \sigma_{\omega\omega}\end{bmatrix}\end{aligned} \tag{34}$$

The Jacobians are:

$$F(t) = \begin{bmatrix}0 & 0 & -Vs\theta\\ 0 & 0 & Vc\theta\\ 0 & 0 & 0\end{bmatrix} \quad G(t) = \begin{bmatrix}c\theta(t) & 0\\ s\theta(t) & 0\\ 0 & 1\end{bmatrix} \tag{35}$$

So the linearized error dynamics are:

$$\frac{d}{dt}\begin{bmatrix}\delta x(t)\\ \delta y(t)\\ \delta\theta(t)\end{bmatrix} = \begin{bmatrix}0 & 0 & -Vs\theta\\ 0 & 0 & Vc\theta\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\delta x(t)\\ \delta y(t)\\ \delta\theta(t)\end{bmatrix} + \begin{bmatrix}c\theta(t) & 0\\ s\theta(t) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\delta V(t)\\ \delta\omega(t)\end{bmatrix}$$

The rest of the matrices leading to the input transition matrix are therefore as follows:

$$\begin{aligned}R(t,\tau) &= \int_\tau^t \begin{bmatrix}0 & 0 & -Vs\theta\\ 0 & 0 & Vc\theta\\ 0 & 0 & 0\end{bmatrix} d\zeta = \begin{bmatrix}0 & 0 & -\Delta y(t,\tau)\\ 0 & 0 & \Delta x(t,\tau)\\ 0 & 0 & 0\end{bmatrix}\\ \Phi(t,\tau) &= \begin{bmatrix}1 & 0 & -\Delta y(t,\tau)\\ 0 & 1 & \Delta x(t,\tau)\\ 0 & 0 & 1\end{bmatrix}\\ \tilde{\Phi}(t,\tau) &= \begin{bmatrix}c\theta(\tau) & -\Delta y(t,\tau)\\ s\theta(\tau) & \Delta x(t,\tau)\\ 0 & 1\end{bmatrix}\end{aligned} \tag{36}$$

The transition matrix follows from the fact that $R^2(t,\tau) = 0$ in this case. The following shorthand expressions for the coordinates of the endpoint from the perspective of the point $[x(\tau), y(\tau)]$ have also been used:

$$\begin{aligned}\Delta x(t,\tau) &= [x(t) - x(\tau)]\\ \Delta y(t,\tau) &= [y(t) - y(\tau)]\end{aligned} \tag{37}$$

Integrals involving these quantities in particular must be manipulated with slightly more care to preserve the distinction between t (the endpoint of the trajectory) and τ (the variable of integration). Substituting into the general solution in equation (21) gives:

$$\begin{aligned}\delta\underline{x}(t) &= \underline{IC}_d + \int_0^t \begin{bmatrix}c\theta & -\Delta y(t,\tau)\\ s\theta & \Delta x(t,\tau)\\ 0 & 1\end{bmatrix}\begin{bmatrix}\delta V(\tau)\\ \delta\omega(\tau)\end{bmatrix} d\tau\\ P(t) &= IC_s\\ &+ \int_0^t \begin{bmatrix}c\theta & -\Delta y(t,\tau)\\ s\theta & \Delta x(t,\tau)\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sigma_{vv} & \sigma_{v\omega}\\ \sigma_{v\omega} & \sigma_{\omega\omega}\end{bmatrix}\begin{bmatrix}c\theta & -\Delta y(t,\tau)\\ s\theta & \Delta x(t,\tau)\\ 0 & 1\end{bmatrix}^T d\tau\end{aligned} \tag{38}$$

Note, for example, that the translational error resulting from gyro bias is proportional to the product of excursion $\Delta x(t,\tau)$ and time. The initial state response for this case is:

$$
\underline{IC}_d = \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix} \delta\underline{x}(0) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ 0 \end{bmatrix} + \begin{bmatrix} -y(t)\delta\theta(0) \\ x(t)\delta\theta(0) \\ \delta\theta(0) \end{bmatrix}
$$

$$
IC_s = \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx}(0) & \sigma_{xy}(0) & \sigma_{x\theta}(0) \\ \sigma_{xy}(0) & \sigma_{yy}(0) & \sigma_{y\theta}(0) \\ \sigma_{x\theta}(0) & \sigma_{y\theta}(0) & \sigma_{\theta\theta}(0) \end{bmatrix} \begin{bmatrix} 1 & 0 & -y(t) \\ 0 & 1 & x(t) \\ 0 & 0 & 1 \end{bmatrix}^T \tag{39}
$$

This result is the general linearized solution for the propagation of systematic and random error in 2D integrated heading odometry for any trajectory and any error model. Again, the effects of velocity scale error are path independent.

5.2.1 Intuitive interpretation

It is clear now that the solution could have been written by inspection. The initial conditions affect the endpoint error in a predictable manner and the remaining terms amount to an addition of the effects felt at the endpoint at time t of the linear and angular errors occurring at each time τ between the start and end as illustrated in Figure 2.

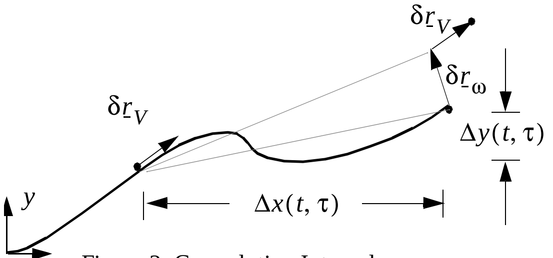

Figure 2: Convolution Integral.

The matrix relating input systematic errors occurring at time τ to their later effect at time t is:

$$
d\begin{bmatrix} \delta x(t) \\ \delta y(t) \\ \delta\theta(t) \end{bmatrix} = \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta V(\tau) \\ \delta\omega(\tau) \end{bmatrix} d\tau \tag{40}
$$

Therefore, the covariance relationship is:

$$
d\begin{bmatrix} \sigma_{xx}(t) & \sigma_{xy}(t) & \sigma_{x\theta}(t) \\ \sigma_{xy}(t) & \sigma_{yy}(t) & \sigma_{y\theta}(t) \\ \sigma_{x\theta}(t) & \sigma_{y\theta}(t) & \sigma_{\theta\theta}(t) \end{bmatrix}
= \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix} \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix}^T d\tau \tag{41}
$$

These expressions are exactly what equation (38) is integrating. Note that the reference trajectory is not updated to reflect the compounding effect of the input error histories over time. Linearization amounts to treating all errors as if they do not compound.

5.3 Differential heading odometry

Differential heading odometry is a special case of integrated heading odometry where angular velocity is derived from the differential indications of wheel linear velocities and the wheel tread W. Let there be a left wheel and a right wheel on either side of the vehicle reference point as shown below:

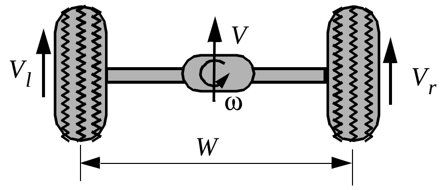

Figure 3: Differential Heading Odometry.

Let the measurement vector be the velocities of the two wheels:

$$
\underline{z}(t) = \begin{bmatrix} V_r(t) & V_l(t) \end{bmatrix}^T \tag{42}
$$

The relationship between these and the equivalent integrated heading inputs is:

$$
\underline{z}(t) = M\underline{u}(t) \quad \begin{bmatrix} V_r(t) \\ V_l(t) \end{bmatrix} = \begin{bmatrix} 1 & W/2 \\ 1 & -W/2 \end{bmatrix} \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} \tag{43}
$$

The inverse relationship is immediate:

$$
\underline{u}(t) = M^{-1}\underline{z}(t)
$$

$$
\begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/W & -1/W \end{bmatrix} \begin{bmatrix} V_r(t) \\ V_l(t) \end{bmatrix} \tag{44}
$$

Hence, known errors can be converted to an equivalent set of errors in integrated heading:

$$
\begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix}_{DH} = \begin{bmatrix} 1/2 & 1/2 \\ 1/W & -1/W \end{bmatrix} \begin{bmatrix} \delta V_r(t) \\ \delta V_l(t) \end{bmatrix} \tag{45}
$$

$$
\begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix}_{DH} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix} \begin{bmatrix} \sigma_{rr} & \sigma_{rl} \\ \sigma_{rl} & \sigma_{ll} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{W} & -\frac{1}{W} \end{bmatrix}^T
$$

This transformation will introduce appropriate correlations between the linear and angular velocity uncertainties due to their common dependence on two other variables. Under this substitution, the solution is identical to the integrated heading case:

$$
\delta\underline{x}(t) = \underline{IC}_d + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta V(t) \\ \delta\omega(t) \end{bmatrix}_{DH} d\tau
$$

$$
P(t) = IC_s + \int_0^t \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{vv} & \sigma_{v\omega} \\ \sigma_{v\omega} & \sigma_{\omega\omega} \end{bmatrix}_{DH} \begin{bmatrix} c\theta & -\Delta y(t,\tau) \\ s\theta & \Delta x(t,\tau) \\ 0 & 1 \end{bmatrix}^T d\tau \tag{46}
$$

This result is the general linearized solution for the propagation of systematic and random error in 2D differential heading odometry for any trajectory and any error model.

6. Properties of Trajectory Moments

Individual elements of the trajectory moment matrices will be called trajectory moments. Trajectory moments are responsible for many of the interesting behaviors of error propagation. Many properties can be defined on these integrals which are related to the properties of the original differential equations, and there are additional important properties. A few of the most important are discussed here.

6.1 Interrelationships

Derivatives (integrals) of trajectory moments are often equal to lower (higher) order moments. For example, the following moments arise in equation (46) for velocity scale errors. They will be referred to respectively as first and second order due to the order of the integrand. They tend to apply respectively to systematic and random error propagation:

$$S_x = \int_0^t \Delta x(t,\tau) V d\tau = \int_0^s [x(s) - x(\xi)] d\xi$$

$$S_{xx} = \int_0^t \Delta x^2(t,\tau) V d\tau = \int_0^s [x(s) - x(\xi)]^2 d\xi$$

Applying Liebnitz rule to the second yields:

$$\frac{dS_{xx}}{ds} = \int_0^s 2[x(s) - x(\xi)]x'(s) d\xi = 2\cos\theta S_x(s) \tag{47}$$

Thus, systematic and random error propagation are inextricably related. Indeed, zeros of the first order moments can coincide with extrema of the second order moments, etc.

6.2 Path independence

Certain moments are path independent. For example, the following moment arises in equation (32) for velocity scale errors. It is just one of the coordinates of the endpoint:

$$S_c = \int_0^t \cos\theta V d\tau = \int_0^s \cos\theta ds = x(s) \tag{48}$$

This result explains why odometry based on a good gyroscope exhibits excellent performance at the point of closure of an arbitrarily shaped trajectory.

6.3 Reversibility and irreversibility

First order motion dependent moments which are even in the associated input are reversible because reversing the input reverses the moment:

$$\begin{aligned} &\int_0^s F[u_i(s)] ds + \int_s^0 F[-u_i(s)] ds \\ &= \int_0^s F[u_i(s)] ds + \int_s^0 F[u_i(s)] ds = 0 \end{aligned} \tag{49}$$

Many systematic error moments are independent of (hence even in) linear or angular velocity and the associated systematic errors cancel when the vehicle is driven back over its original path.

In the case of second order moments, it is usually appropriate to interpret the differentials as unsigned quantities in order to ensure that the associated variance is irreversible - increasing in both directions of motion:

$$ds = |V| dt \qquad d\theta = |\omega| dt$$

6.4 Symmetry, centroids and zeros

Some moments may vanish at specific places whether or not the trajectory closes there. For example, consider the moment S_x defined earlier. It can be written in terms of the current position, the distance travelled and the instantaneous centroid location:

$$S_x = s x(s) - \int_0^s x(\xi) d\xi = s[x(s) - \bar{x}(s)] \tag{50}$$

Moments of this type vanish at the centroid of the associated coordinate. This observation subsumes earlier comments on symmetry as a special case. This result explains why errors tend to be reduced or even cancel on symmetric trajectories. Many errors will cancel on a figure-8 trajectory for instance.

6.5 Monotonicity and conservation

Consider the general symmetric influence matrix formed from any column of the input transition matrix:

$$\tilde{\Phi} = \underline{\tilde{\Phi}}\underline{\tilde{\Phi}}^T \tag{51}$$

Let its elements be denoted as follows:

$$\tilde{\Phi} = \begin{bmatrix} \phi_{xx} & \phi_{xy} & \phi_{x\theta} \\ \phi_{xy} & \phi_{yy} & \phi_{y\theta} \\ \phi_{x\theta} & \phi_{y\theta} & \phi_{\theta\theta} \end{bmatrix} \tag{52}$$

This matrix must be positive semidefinite by construction. Its integral is the associated trajectory moment matrix M:

$$M = \int_0^t \tilde{\Phi} dt \tag{53}$$

This matrix must also be positive semidefinite. Every eigenvalue of M is a nondecreasing function of time. The trace of M, also equal to the sum of its eigenvalues, and the "total variance", is also a nondecreasing function of time:

$$\frac{d}{dt}[\phi_{xx} + \phi_{yy} + \phi_{\theta\theta}] \geq 0$$

This implies that any decrease in one diagonal element of a moment matrix must be accompanied by at least as great an increase in the others (conservation).

For example, consider the moments:

$$S_{cc} + S_{ss} = \int_0^s [\cos\theta^2 + \sin\theta^2] ds = s$$

These two moments are vertically symmetric about a line whose slope is unity. On any trajectory, decreases in one relative to the line must be offset by equivalent increases in the other. Uncertainty ellipses may rotate in space but they must grow monotonically when expressed in principal coordinates.

7. Validation

Error propagation results were verified on the integrated heading case by comparing the linearized solutions of the paper with an exact nonlinear numerical solution for both systematic and random errors. The input error characteristics were as follows:

Table 1: Error Sources for Theory Validation

Error Source	Deterministic	Random
Linear Velocity	$\delta V = \alpha V$ $\alpha = 0.05$	$\sigma_v = \delta V$
Angular Velocity	$\delta\omega = b$ $b = 30° / \mathrm{hr}$	$\sigma_\omega = \delta\omega$

These models represent a systematic scale error of 5% on velocity and a motion dependent random walk stochastic velocity error of equal standard deviation. A systematic gyro bias of 30 degrees/hr is used as well as a bias stability of equivalent standard deviation.

Such error magnitudes are considerably larger than might be expected in a practical situation. The intention here is to stress the linearity assumption and provide a common error magnitude for both systematic and random sources which is large enough to be noticeable in the following figures.

In the systematic case, straightforward numerical quadrature can be used to integrate the nonlinear dynamics in both the perturbed and unperturbed case and the difference between the two is obtained as the exact nonlinear solution within the limits of time discretization.

For stochastic error, Monte Carlo simulation was used. Using 500,000 independent, unbiased, Gaussian random variables, 250 discrete time random signals for linear and angular velocity errors were generated and used to corrupt the nominal inputs.

Figure 4 illustrates the result of nonlinear simulation of an arbitrary trajectory chosen to contain a loop but exhibit no particular symmetry. The velocity for the test is 0.25 m/s, the total time is 210 secs and the time step is 0.5 secs.

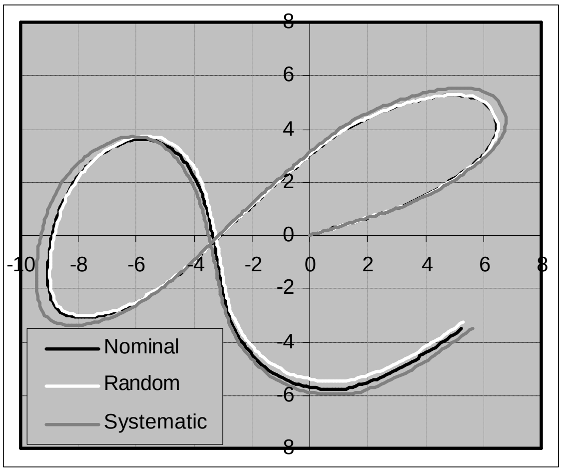

Figure 4: Test Trajectory: Effects of both systematic and random error are shown. Systematic error is initially larger but mostly cancels on the closed loop. Random error is more subdued but more persistent.

While there is a single systematic result to plot, only one representative of the 250 randomly perturbed trajectories is shown. Due to the tendency of random error to cancel, the effect of systematic error is more dramatic when compared to a random error.

The effect of the systematic velocity scale error is evident in the increase in error magnitude with the radius from the origin. As predicted by theory, substantial accumulated systematic errors in the far left vanish when the loop closes. For random error, the overall growth rate is more subdued but it nonetheless accumulates to nontrivial levels over time.

The difference between linearized and nonlinear deterministic error is shown in Figure 5.

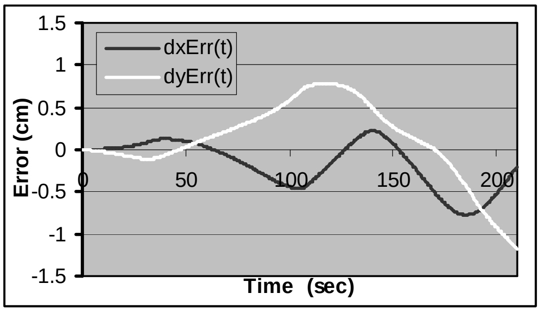

Figure 5: Deterministic Linearization Error: Linearized error dynamics are accurate to 2.5% even in the presence of unrealistically large encoder and gyro errors.

Clearly, the linearized solution is an excellent approximation even for errors of this magnitude. The difference between exact and linearized solutions barely exceeds 1 cm or 2.5% of the actual maximum error magnitude of 40 cm. This result validates the systematic part of equation (38).

The results of the Monte Carlo simulation are provided in Figure 6 for the translational variances and co-variances. Rotational variance is linear by construction and translational-rotational covariances agree similarly with theory.

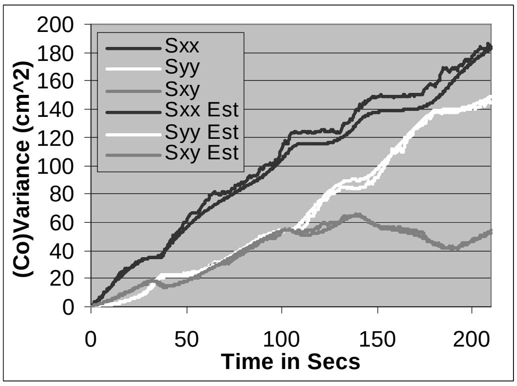

Figure 6: Monte Carlo Simulation Compared to Theory: Agreement with theory is excellent. Note how the two translational variances vary symmetrically about a steady growth curve.

The agreement between theory and simulation is excellent. Note how the two translational variances exhibit conservation behavior by varying symmetrically about an average steady growth curve.

Overall, three classes of error can be expected in the stochastic case: linearization, discretization, and sampling error. The first two classes have been demonstrated to be relatively small by the systematic error results. These curves show that for a sample size of 250 pairs of input random processes, the sample variance of the state covariance matrix tracks the linearized theoretical population variance quite well. This result validates the stochastic part of equation (38).

8. Conclusions

The commutative dynamics condition holds for commonly encountered forms of odometry when the equations are expressed in a forced dynamics manner and linearized. Given that the system Jacobian is also upper triangular, a symbolic expression for the transition matrix is obtained easily. Subsequent application of the theory of linear systems provides the integrals for error propagation in closed form.

Resultant state estimation error is always a combination of the state response and the input response. The former is always path independent and vanishes on any closed trajectory. The latter can often be reduced to expressions involving path functionals or moments which are analogous to the moments of mechanics and the Laplace transforms of control theory.

Trajectory moments are responsible for the path dependent behavior of odometry error and a one-to-one correspondence can be established between elements of the input error covariance matrix and the trajectory moment which projects its value onto the state covariance. Over time, error evolves as does the trajectory moment, so it represents the driving mechanism behind propagation.

Analytic expressions are important tools in the development of theory. The present results enable the symbolic application of linear systems theory, optimal control, calculus of variations, etc. to any application which attempts to account in some way for error propagation.

In addition to their pedagogic value, these results can be used in design to determine acceptable levels of sensor error. They can be used in development to accentuate response to individual error sources for on-line or off-line calibration or evaluative purposes. They can be used in operation to plan trajectories in order to minimize exposure to specific error sources, or determine optimal approaches to error compensation. The availability of an analytical theory of error propagation enables many problems to be solved rapidly by referring to a table of trajectory moments.

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