Chapter 4
Dynamics

Part 2

4.3 Constrained Kinematics and Dynamics

Outline

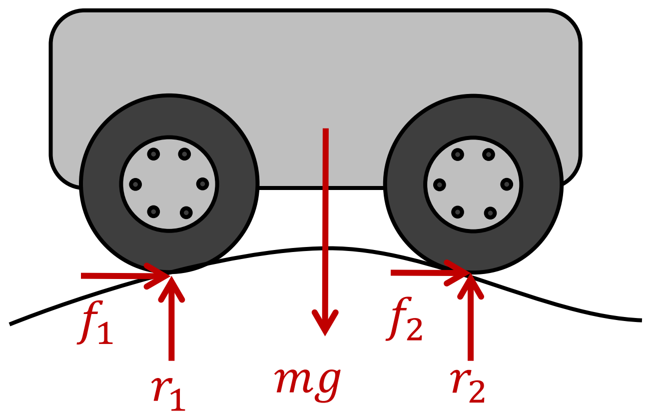
- 4.3 Constrained Kinematics and Dynamics
  - 4.3.1 Constraints of Disallowed Direction
  - 4.3.2 Constraints of Rolling without Slipping
  - 4.3.3 Lagrangian Dynamics
  - 4.3.4 Terrain Contact
  - 4.3.5 Trajectory Estimation and Prediction
  - Summary

Classes of ODE Models

- Unconstrained Velocity Driven
  - Assume you know all the velocities.
  - Constraints are automatically satisfied …
  - … or satisfy them as a separate process
- Constrained Velocity Driven
  - Assume you know unconstrained velocities.
  - Enforce constraints to determine disallowed directions and then integrate net velocities.
- Constrained Force Driven
  - Assume you know applied forces
  - Enforce constraints to determine constraint and then net forces.

What's the Big Deal with Dynamics?

- "Its all just F = ma integrated twice" Right?
- Well, what is f_1, f_2, r_1,and r_2?



- What about T = I α ? What are all the torques caused by the wheels?

Dynamics of WMR

- Two constraints:
  - Rolling without sliding
  - Terrain contact
- Two Formulations
  - **Second Kind:** Lagrangian formulation of dynamics computes the constraint forces automatically.
    - Few (generalized) coordinates. Complex nonlinear equations.
  - **First Kind:** Augmented formulation leaves them explicit.
    - All coordinates. Simple linear equations.
    - Lets you determine if the terrain can provide the forces.
    - We will do this kind.

4.3 Constrained WMR Models

- The "constraints" were not explicit in the last section.
  - Actually, they were, wheel equations are constraints too.
- Constrained models occur in both control and estimation contexts.
- Non slip constraints are nonholonomic, of abstract form: $\underline{c}(\underline{x}, \dot{\underline{x}}) = \underline{0}$
- Terrain contact constraints are holonomic, of abstract form: $\underline{c}(\underline{x}) = 0$

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Differentiating Holonomic Constraints

- If we differentiate a holonomic constraint:

$$\underline{c}(\underline{x}) = 0$$

- We get a form that resembles Pfaffian form:

$$\underline{c}_{\underline{x}}\,\dot{\underline{x}} = 0$$

- Where the constraint gradient $c_{\underline{x}}$ functions just like a disallowed direction of motion.
- This is convenient but this differentiated constraint can still be integrated so the constraint is still holonomic.

Expressing Wheel Constraints

- It is trivial to write an expression that says a wheel cannot move sideways.
- The implementation issue is how does a set of such statements constrain the motion of the body frame of the vehicle.
- We solve this problem by substituting the kinematics into the constraint:
  - Replace wheel velocity with body velocity.

4.3.1.1 Velocity Constraints

- We can describe the condition of rolling without slipping using a disallowed direction $\vec{w}_c$:

$$\vec{w}_c \cdot \vec{v}_c^{\,w} = 0$$

- Choose any coordinates to get a matrix form:

$$\underline{w}_c^T \underline{v}_c^w = 0 \qquad \textbf{(4.70)}$$

- Substitute the articulated wheel equation for $\vec{v}_c^{\,w}$

$$\underline{w}_c^T \underline{v}_c^w = \underline{w}_c^T H(\underline{r}_c^v)\underline{\dot{x}}_v^w + \underline{w}_c^T H(\underline{r}_c^s)\underline{\dot{x}}_s^v + \underline{w}_c^T \underline{v}_c^s = 0$$

- This is the general case in 3D for an articulated wheel.

4.3.1.1 Velocity Constraints (c fixed)

- In the special case where the contact point is fixed, the velocity reduces to the (nonoffset) wheel equation:

Wheel Equation $$\underline{v}_c^w = \underline{v}_v^w - [\underline{r}_c^v]^\times \underline{\omega}_v^w$$ **(4.40)**

- Substituting this into the wheel constraint gives:

$$\underline{w}_c^T(\underline{v}_v^w - [\underline{r}_c^v]^\times \underline{\omega}_v^w) = 0$$

- Define a "Pfaffian radius" $\underline{\rho}_c^T = \underline{w}_c^T[\underline{r}_c^v]^\times$ then:

$$\underline{w}_c^T \underline{v}_v^w - \underline{\rho}_c^T \underline{\omega}_v^w = 0$$ **(4.75)**

Wheel Constraint for c Fixed

- It says translational and rotational components must cancel in the disallowed direction.

4.3.1.2 Differentiated Velocity Constraints

- Differentiate Eqn 4.70 wrt time: $\underline{w}_c^T \underline{v}_c^w = 0$ **(4.70)**

$$\underline{w}_c^T \underline{a}_c^w + \dot{\underline{w}}_c^T \underline{v}_c^w = 0$$

- Substitute the articulated wheel equation for $\vec{v}_c^w$ and $\vec{a}_c^w$.

$$\underline{w}_c^T(\underline{a}_c^w = H(\underline{r}_c^v)\ \ddot{\underline{x}}_v^w + \Omega(\underline{\omega}_v^w)\underline{\rho}_c^v + H(\underline{r}_c^s)\ \ddot{\underline{x}}_s^v + \Omega(\underline{\omega}_s^v)\underline{\rho}_c^s + \underline{a}_c^s)$$
$$+ \dot{\underline{w}}_c^T(H(\underline{r}_c^v)\dot{\underline{x}}_v^w + H(\underline{r}_c^s)\dot{\underline{x}}_s^v + \underline{v}_c^s) = 0$$

- This is the general case in 3D for an articulated wheel.

4.3.1.1 Differentiated Velocity Constraints

(c fixed)

- In the special case where the contact point is fixed, the (nonoffset) wheel equations are substituted to generate:

$$\underline{w}_c^T \underline{a}_v^w - \underline{\rho}_c^T \underline{\alpha}_v^w + \underline{w}_c^T [\underline{\omega}_v^w]^{\times\times} \underline{r}_c^v + \dot{\underline{w}}_c^T \underline{v}_v^w - \dot{\underline{\rho}}_c^T \underline{\omega}_v^w = 0 \qquad \textbf{(4.83)}$$

Diff Wheel Constraint for c Fixed

- We defined a "Pfaffian radius rate" $\dot{\underline{\rho}}_c^T = \dot{\underline{w}}_c^T [\underline{r}_c^v]^{\times}$:

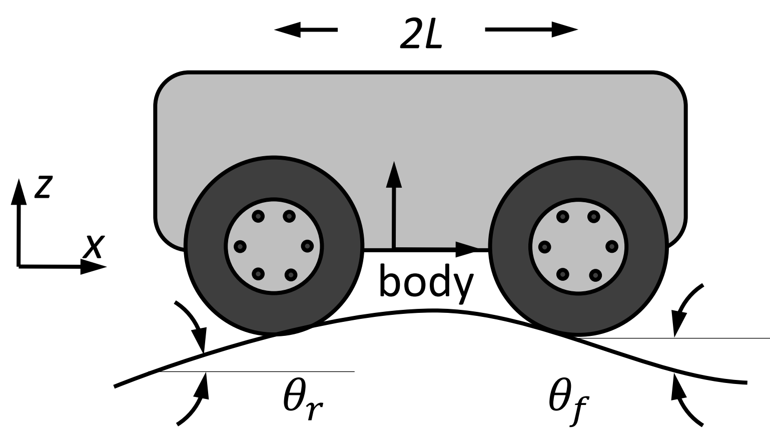
- This is valid in 3D for an articulated wheel with a fixed contact point.

4.3.1.3 Example: Terrain Contact

(Using a Disallowed Direction by Differentiation)

- Assume no suspensions and fixed contact points.
- Then, terrain normal is disallowed direction, and Equation 4.75 becomes …

$$\underline{n}_i^T \underline{v}_v^w + \underline{\rho}_i^T \underline{\omega}_v^w = 0$$



- Coordinates of contact points

$$x_f = x + Lc\theta - rs\theta_f \quad x_r = x - Lc\theta - rs\theta_r$$

$$z_f = z - Ls\theta - rc\theta_f \quad z_r = z + Ls\theta - rc\theta_r$$

- In matrix form, constraints are:

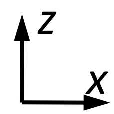
$$\underline{c}(\underline{x}) = \begin{bmatrix} \zeta(x_f) - z_f \\ \zeta(x_r) - z_r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ζ(x) is Terrain Elevation

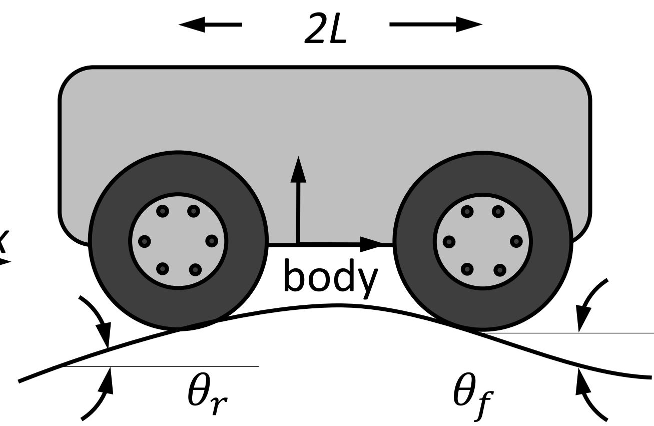
4.3.1.3 Example: Terrain Contact

- Constraint Jacobian:

$$\underline{c}_{\underline{x}} = \begin{bmatrix} \frac{\partial\zeta(x_f)}{\partial x_f}\frac{\partial x_f}{\partial x} & -1 & \frac{\partial\zeta(x_f)}{\partial x_f}\frac{\partial x_f}{\partial\theta} + Lc\theta \\ \frac{\partial\zeta(x_r)}{\partial x_r}\frac{\partial x_r}{\partial x} & -1 & \frac{\partial\zeta(x_r)}{\partial x_r}\frac{\partial x_r}{\partial\theta} - Lc\theta \end{bmatrix} = \begin{bmatrix} -t\theta_f & -1 & (-t\theta_f(-Ls\theta) + Lc\theta) \\ -t\theta_r & -1 & (-t\theta_r(Ls\theta) - Lc\theta) \end{bmatrix}$$



Terrain Gradient



- Suppose L=1; θ=0 ; slopes as shown:

$$\underline{c}_{\underline{x}} = \begin{bmatrix} 0.1 & -1 & 1 \\ -0.2 & -1 & -1 \end{bmatrix}$$

$$\dot{\underline{x}} = [I - \underline{c}_{\underline{x}}^T(\underline{c}_{\underline{x}}\underline{c}_{\underline{x}}^T)^{-1}\underline{c}_{\underline{x}}]f(\underline{x}, \underline{u}) = \begin{bmatrix} 0.9268 & -0.0463 & -0.1390 \end{bmatrix}^T$$

- This can be verified by a more conventional technique.

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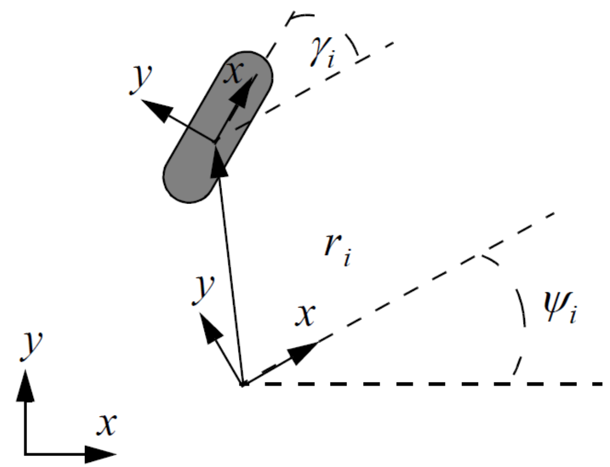
4.3.2.1.1 Pfaffian Constraint in Body Frame

$$\underline{w}_c^T \underline{v}_v^w - \underline{\rho}_c^T \underline{\omega}_v^w = 0 \quad (4.75)$$

- Position vector to wheel is:

$$\underline{r}_i^v = \begin{bmatrix} x_i^v & y_i^v \end{bmatrix}^T$$

- Unit vectors in body frame: $\hat{x}_i = \begin{bmatrix} c\gamma_i & s\gamma_i \end{bmatrix}^T \quad \hat{y}_i = \begin{bmatrix} -s\gamma_i & c\gamma_i \end{bmatrix}^T$
- Pfaffian radius $\underline{\rho}_i^T = [\hat{y}_i^T [\underline{r}_i^v]^\times]$ is:
- Or, as a matrix equation:



$$\underline{\rho}_i^T = \left[\begin{bmatrix} -s\gamma & c\gamma & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & y_i^v \\ 0 & 0 & -x_i^v \\ -y_i^v & x_i^v & 0 \end{bmatrix} \right]$$

$$\underline{\rho}_i^T = - \begin{bmatrix} 0 & 0 & (c\gamma x_i^v + s\gamma y_i^v) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -(\vec{r}_i^v \cdot \hat{x}_i) \end{bmatrix}$$

4.3.2.1.1 Pfaffian Constraint in Body Frame

- So equation 4.75 is:

$$\underline{w}_c^T \underline{v}_v^w - \underline{\rho}_c^T \underline{\omega}_v^w = 0 \quad (4.75)$$

$$\underline{w}_c^T \underline{v}_v^w - \underline{\rho}_c^T \underline{\omega}_v^w = 0$$

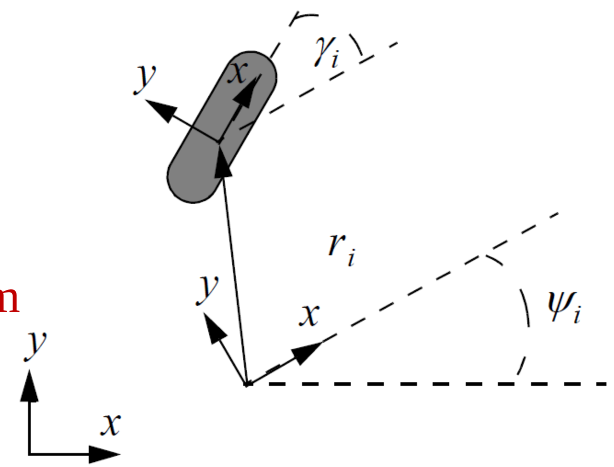
$$\begin{bmatrix} -s\gamma_i & c\gamma_i \end{bmatrix} \begin{bmatrix} v_x & v_y \end{bmatrix}^T - \begin{bmatrix} 0 & 0 & -(\vec{r}_i^{\,v} \cdot \hat{x}_i) \end{bmatrix} \omega_v^w = 0$$

- Writing it in terms of a state vector:

$$\begin{bmatrix} -s\gamma_i & c\gamma_i & (\underline{r}_i \cdot \hat{x}_i) \end{bmatrix} \begin{bmatrix} v_x & v_y & \omega \end{bmatrix}^T = 0$$

Pfaffian Form

$\underline{w}(\underline{x})*\dot{x}$



- This is the equation written in the body frame.

4.3.2.1.2 Pfaffian Constraint in Inertial Frame

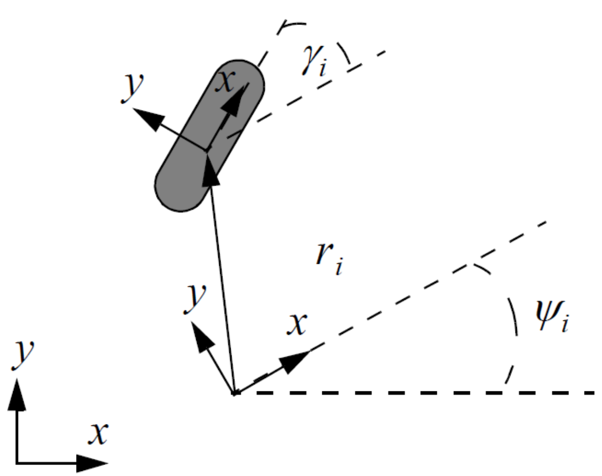
$$\underline{w}_c^T \, {}^w\underline{v}_v - \underline{\rho}_c^T \, {}^w\underline{\omega}_v = 0 \quad (4.75)$$

- Now the unit vectors are:

$$\hat{x}_i = \begin{bmatrix} c\psi\gamma_i & s\psi\gamma_i \end{bmatrix}^T \qquad \hat{y}_c = \begin{bmatrix} -s\psi\gamma_i & c\psi\gamma_i \end{bmatrix}^T$$

- Writing it in terms of a state vector:

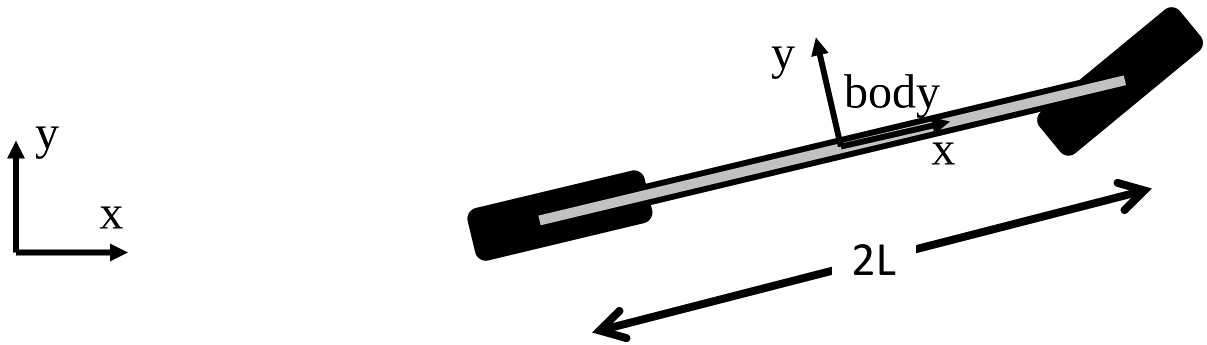
$$\begin{bmatrix} -s\psi\gamma_i & c\psi\gamma_i & (\underline{r}_i \cdot \hat{x}_i) \end{bmatrix} \begin{bmatrix} v_x & v_y & \omega \end{bmatrix}^T = 0 \quad (4.87)$$



- This is the equation written in the inertial frame.

4.3.2.2 Velocity Driven Bicycle

- Simplest, sufficiently complex case to illustrate most issues.



- 3 dof in total (x,y,θ)
- 2 nonholonomic constraints (2 constraints)
- 1 dof left in the tangent plane

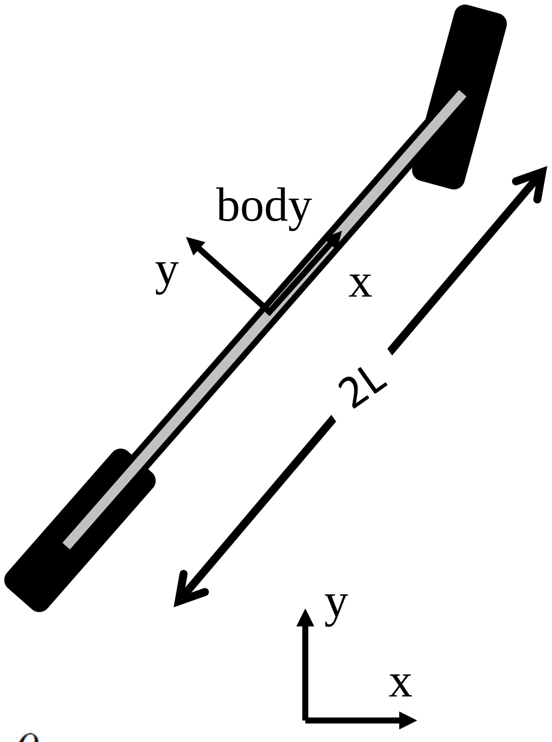
4.3.2.2 Velocity Driven Bicycle

- The first order model is:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) = \underline{u}$$

$$\underline{c}_r(\underline{x}, \dot{\underline{x}}) = \underline{w}_r(\underline{x})\,\dot{\underline{x}} = 0$$

$$\underline{c}_f(\underline{x}, \dot{\underline{x}}) = \underline{w}_f(\underline{x})\,\dot{\underline{x}} = 0$$



- Position vectors:

$$\underline{r}_f = \begin{bmatrix} L & 0 \end{bmatrix}^T \qquad \underline{r}_r = \begin{bmatrix} -L & 0 \end{bmatrix}^T$$

- By Equation 4.87:

$$\underline{c}_r(\underline{x}, \dot{\underline{x}}) = \underline{w}_r(\underline{x})\,\dot{\underline{x}} = \begin{bmatrix} -s\psi & c\psi & -L \end{bmatrix} \begin{bmatrix} v_x & v_y & \omega \end{bmatrix}^T = 0$$

$$\underline{c}_f(\underline{x}, \dot{\underline{x}}) = \underline{w}_f(\underline{x})\,\dot{\underline{x}} = \begin{bmatrix} -s\psi\gamma & c\psi\gamma & Lc\gamma \end{bmatrix} \begin{bmatrix} v_x & v_y & \omega \end{bmatrix}^T = 0$$

4.3.2.2 Velocity Driven Bicycle

algorithm *VelocityDrivenBicycle* ( )

$L \leftarrow 1 \;\; ; \;\; \gamma \leftarrow 0.1 \; ; \;\; \psi_0 \leftarrow -atan[(tan(\gamma))/2]$

$x \leftarrow \begin{bmatrix} 0 & 0 & \psi_0 \end{bmatrix} ; \; v_x \leftarrow 1 \; ; \; \dot{x} \leftarrow \begin{bmatrix} v_x & 0 & -v_x sin \frac{(\psi_0)}{L} \end{bmatrix}$

$\Delta t \leftarrow 0.01 \; ; \;\; t_{max} \leftarrow 2\pi / \dot{x}(3)$

for$(t \leftarrow 0 \; ; t < t_{max} \; ; t \leftarrow t + \Delta t)$

$\quad \psi \leftarrow x(3) \; ; \;\; C \leftarrow C_{\dot{x}} \leftarrow \begin{bmatrix} -s\psi & c\psi & -L \\ -s\psi\gamma & c\psi\gamma & Lc\gamma \end{bmatrix} ;$

$\quad \hat{v} \leftarrow \begin{bmatrix} \dot{x}(1) & \dot{x}(2) \end{bmatrix}^T ; \qquad \hat{v} \leftarrow \hat{v} / |\hat{v}|$

$\quad f_x \leftarrow f_x(t)$;// or just 1; $u \leftarrow f_x \hat{v}$;

$\quad \lambda \leftarrow (CC^T)^{-1}(Cu)$ Drift Trim Step

$\quad \dot{x} \leftarrow (u - C^T\lambda)$

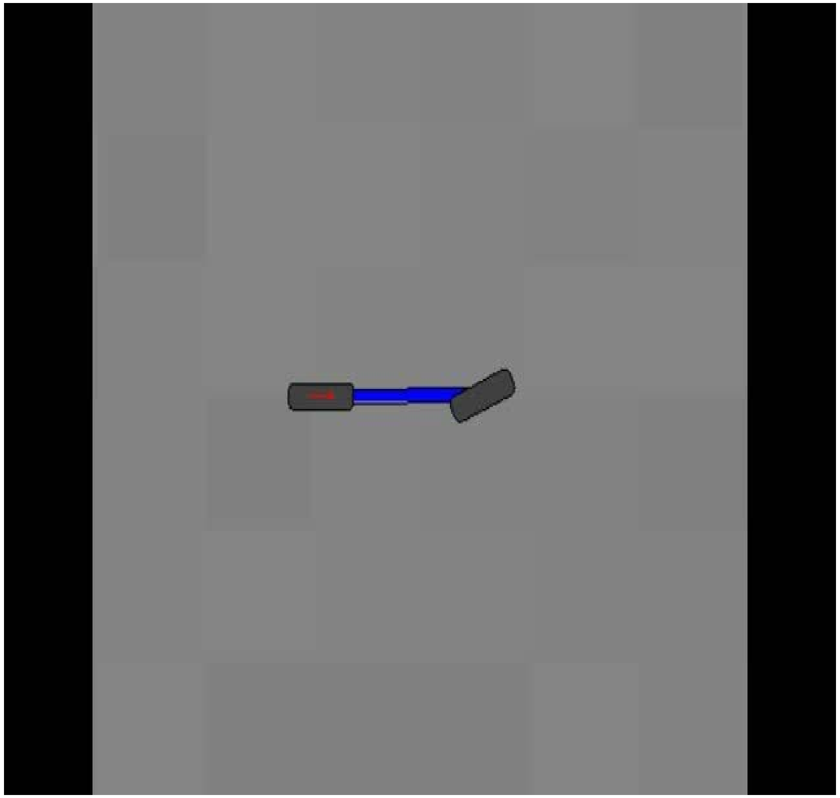
$\quad x \leftarrow x + \dot{x}\Delta t$

`endfor`

return

Algorithm 4.1: First-Order Model of a Single Body Bicycle. This model can be coded quickly and once debugged will drive the vehicle in a circle. Try steering it by setting $\gamma = cos(2\pi t / t_{max})$.

Demos : Unibody Bike



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4.3.3 Lagrangian Dynamics

- Recall, the equations are of the form

$$\begin{bmatrix} M & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \ddot{\underline{x}} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{F}^{ext} \\ \underline{F}_d \end{bmatrix}$$

- Using elimination, the solution is:

$$\underline{F}_d = -C \dot{\underline{x}}$$ From definition of Fd

$$\underline{\lambda} = (CM^{-1}C^T)^{-1}(CM^{-1}\underline{F}^{ext} - \underline{F}_d)$$

$$\ddot{\underline{x}} = M^{-1}(\underline{F}^{ext} - C^T\underline{\lambda})$$

- Do this every time step and integrate acceleration twice.

4.3.31 Differentiated Pfaffian Constraints

(For a Wheel)

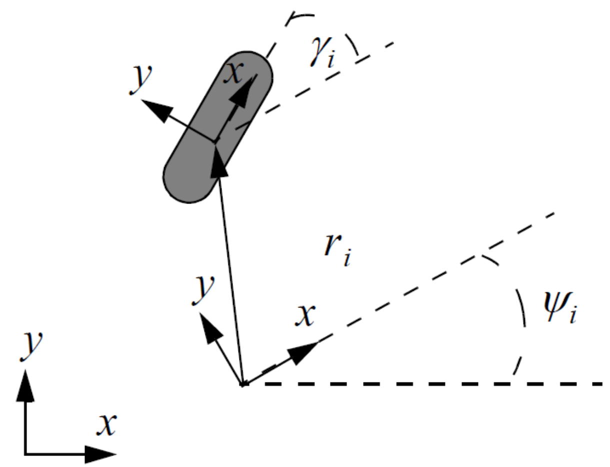
$$\underline{w}_c^T \underline{a}_v^w - \underline{\rho}_c^T \underline{\alpha}_v^w + \underline{w}_c^T [\underline{\omega}_v^w]^{\times\times} \underline{r}_c^v + \dot{\underline{w}}_c^T \underline{v}_v^w - \dot{\underline{\rho}}_c^T \underline{\omega}_v^w = 0 \quad (4.83)$$

- Two key components of equation 4.83 are:

$$\underline{w}_c^T = \hat{y}_i = \begin{bmatrix} -s\psi\gamma_i & c\psi\gamma_i \end{bmatrix}^T \qquad \underline{\rho}_i^T = \begin{bmatrix} 0 & 0 & -(\vec{r}_i^v \cdot \hat{x}_i) \end{bmatrix}$$

- After much manipulation:

$$\begin{bmatrix} -s\psi\gamma_i \\ c\psi\gamma_i \\ (\underline{r}_i \cdot \hat{x}_i) \end{bmatrix}^T \begin{bmatrix} a_x \\ a_y \\ \alpha \end{bmatrix} = \begin{bmatrix} (\omega + \dot{\gamma}_i) c\psi\gamma_i \\ (\omega + \dot{\gamma}_i) s\psi\gamma_i \\ -\dot{\gamma}_i (\underline{r}_i \cdot \hat{y}_i) \end{bmatrix}^T \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$



- This is, by necessity, written in an **inertial frame**.

4.3.3.2 (Unibody) Force Driven Bicycle

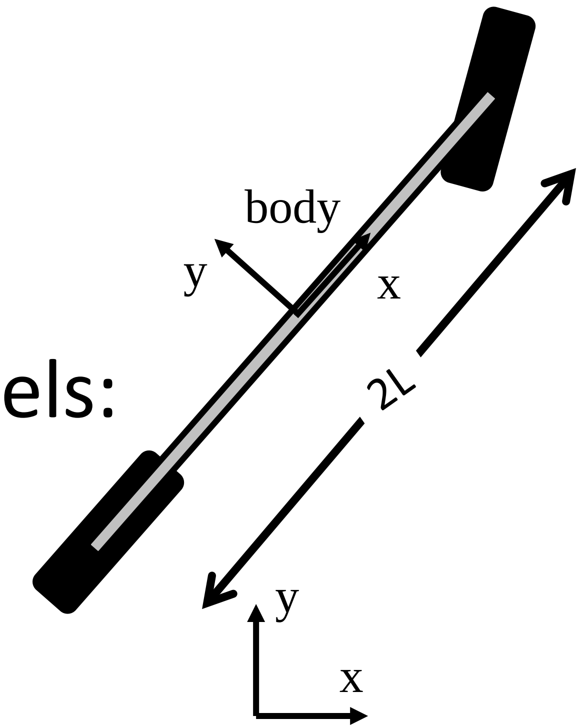
- Assume wheels are massless.
- Equations of motion for body:

$$M\underline{\ddot{x}} = \underline{F}^{ext} + \underline{F}^{con}$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ \alpha \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ \tau \end{bmatrix}^{ext} + \begin{bmatrix} F_x \\ F_y \\ \tau \end{bmatrix}^{con}$$

- Differentiated constraints for wheels:

$$\begin{bmatrix} -s\psi & c\psi & -L \\ -s\psi\gamma & c\psi\gamma & Lc\gamma \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ \alpha \end{bmatrix} = \begin{bmatrix} \omega c\psi & \omega s\psi & 0 \\ (\omega+\dot{\gamma})c\psi\gamma & (\omega+\dot{\gamma})s\psi\gamma & \dot{\gamma}Ls\gamma \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$



- Relevant Jacobians:

$$\underline{c}_{\underline{x}} = -\begin{bmatrix} \omega c\psi & \omega s\psi & 0 \\ (\omega+\dot{\gamma})c\psi\gamma & (\omega+\dot{\gamma})s\psi\gamma & \dot{\gamma}Ls\gamma \end{bmatrix} \qquad \underline{c}_{\underline{\dot{x}}} = \begin{bmatrix} -s\psi & c\psi & -L \\ -s\psi\gamma & c\psi\gamma & Lc\gamma \end{bmatrix}$$

4.3.3.3 Force Driven Bicycle

algorithm $LagrangeBicycle()$

$m \leftarrow I \leftarrow L \leftarrow 1\ ;\ \ \gamma \leftarrow 0.1\ ;\ \ \psi_0 \leftarrow -atan[(tan(\gamma))/2]$

$x \leftarrow \begin{bmatrix} 0 & 0 & \psi_0 \end{bmatrix}\ ;\ v_x \leftarrow 1\ ;\ \dot{x} \leftarrow \begin{bmatrix} v_x & 0 & -v_x sin\dfrac{(\psi_0)}{L} \end{bmatrix}$

$\Delta t \leftarrow 0.01\ ;\ \ M \leftarrow diag(\begin{bmatrix} m & m & I \end{bmatrix})\ ;\ \ t_{max} \leftarrow 2\pi/\dot{x}(3)$

$M^{-1} \leftarrow inverse(M))$

for$(t \leftarrow 0\ ; t < t_{max}\ ; t \leftarrow t + \Delta t)$

$\quad \psi \leftarrow x(3)\ ;\ \ C \leftarrow C_{\dot{x}} \leftarrow \begin{bmatrix} -s\psi & c\psi & -L \\ -s\psi\gamma & c\psi\gamma & Lc\gamma \end{bmatrix}\ ;\ \ \omega \leftarrow \dot{x}(3)$

$\quad C_x \leftarrow -\begin{bmatrix} \omega c\psi & \omega s\psi & 0 \\ (\omega + \dot{\gamma})c\psi\gamma & (\omega + \dot{\gamma})s\psi\gamma & \dot{\gamma}Ls\gamma \end{bmatrix}$

$\quad \hat{v} \leftarrow \begin{bmatrix} \dot{x}(1) & \dot{x}(2) \end{bmatrix}^T;\ \ \ \hat{v} \leftarrow \hat{v}/|\hat{v}|$

$\quad f_x \leftarrow f_x(t)$; // or just 0; $F^{ext} \leftarrow f_x\hat{v}\ ;\ F_d \leftarrow -C_x\dot{x}$

$\quad \lambda \leftarrow (CM^{-1}C^T)^{-1}(CM^{-1}F^{ext} - F_d)$

$\quad \ddot{x} \leftarrow M^{-1}(F^{ext} - C^T\lambda)$

$\quad \dot{x} \leftarrow \dot{x} + \ddot{x}\Delta t\ ;\ \ \dot{x} \leftarrow [I - C^T(CC^T)^{-1}C]\,\dot{x}$ Drift Trim Step

$\quad x \leftarrow x + \dot{x}\Delta t$

endfor

return

Algorithm 4.2: Lagrangian Model of a Single Body Bicycle. This model can be coded quickly and once debugged will drive the vehicle in a circle.

Simplified Wheel Constraints

- When the system state vector uses coordinates of the wheel contact point $\underline{r}_i = 0$ and $\gamma_i = 0$, so:

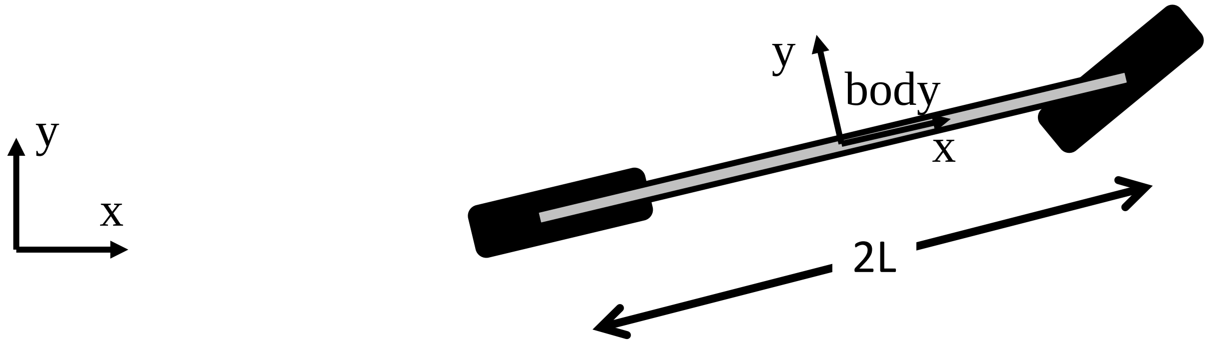
$$\underline{w}_c^T(\underline{x})\dot{\underline{x}} = \begin{bmatrix} -s\psi & c\psi & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = 0$$

- … and the derivative is:

$$\underline{w}_c^T(\underline{x})\ddot{\underline{x}} = \begin{bmatrix} -s\psi \\ c\psi \\ 0 \end{bmatrix}^T \begin{bmatrix} a_x \\ a_y \\ \alpha \end{bmatrix} = -\dot{\underline{w}}_c^T(\underline{x})\dot{\underline{x}} = \begin{bmatrix} (\omega + \dot{\gamma}_i)c\psi\gamma_i \\ (\omega + \dot{\gamma}_i)s\psi\gamma_i \\ -\dot{\gamma}_i(\underline{r}_i \cdot \hat{y}_i) \end{bmatrix}^T \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = \begin{bmatrix} \omega c\psi \\ \omega s\psi \\ 0 \end{bmatrix}^T \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$

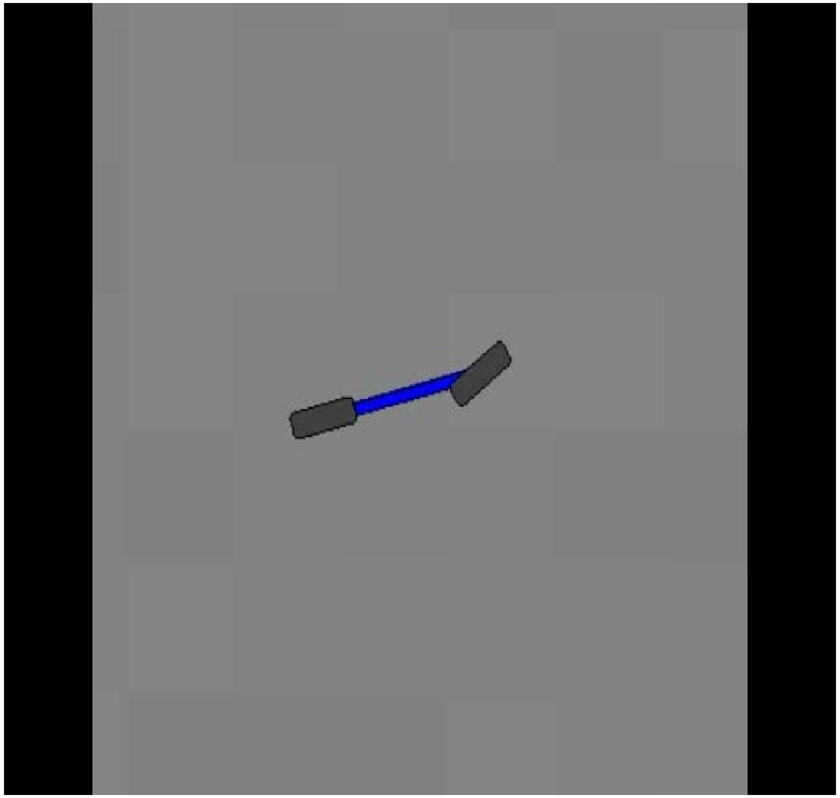
(Multibody) Bicycle

- Now have three bodies with mass.



- 9 dof in total: 3 X (x,y,q)
- 1 rigidity constraint (3 constraints)
- 1 rotary (steer) joint (2 constraints)
- 2 nonholonomic constraints (2 constraints)
- 2 dof left in tangent plane (steer, V)

Demos : 3 Body Bike



Other Issues

- Allowing wheel slip according to a specific model.
- Computing explicit constraint forces.
- Inconsistent constraints.
- Redundant constraints.

Outline

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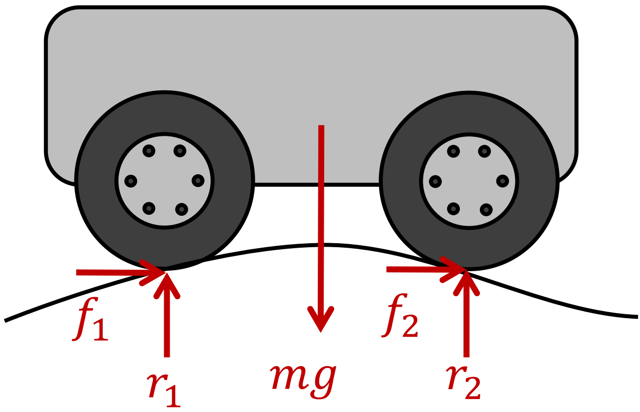
  - 4.3.1 Constraints of Disallowed Direction
  - 4.3.2 Constraints of Rolling without Slipping
  - 4.3.3 Lagrangian Dynamics
  - <u>4.3.4 Terrain Contact</u>
  - 4.3.5 Trajectory Estimation and Prediction
  - Summary

WMR Constraints and DOFs

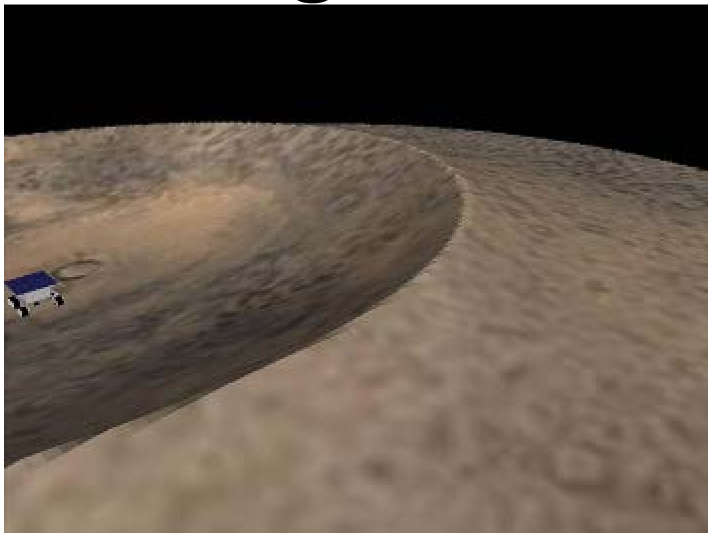
- Assuming:
  - the robot will stay in contact with the ground.
  - 1 dof of suspension to fix 4 wheels on ground.
- Terrain following: 3 dof
  - Attitude (pitch, roll) and altitude determined from terrain.
- Inputs: 2 dof
  - Usually, there are only 2 very distinct dof actuated.
- Wheel No Slip Constraints: 1 dof.



Constraints include no wheel slip and terrain following

Effect of Terrain Following

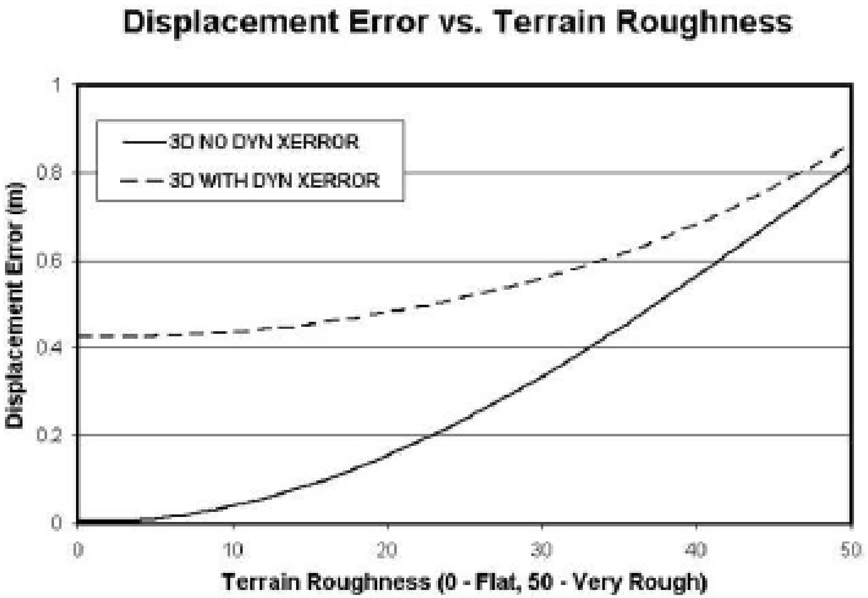
- Where the robot will go depends on the terrain.
- For motion along the body x axis and rotation around body z:



$$x(s) = x_0 + \int_0^s \cos(\psi(s))\cos(\theta(x(s), y(s)))ds$$

$$y(s) = y_0 + \int_0^s \sin(\psi(s))\cos(\theta(x(s), y(s)))ds$$

$$\psi(s) = \psi_0 + \int_0^s \frac{\cos(\phi(x(s), y(s)))}{\cos(\theta(x(s), y(s)))} k(s)ds$$



Basic Terrain Following

- Start with known (x,y,z) from DE:

$$z_i = \text{terrain}(x_i, y_i) \quad \forall i$$ wheels

$$\theta_{left} = (z_1 - z_3)/L$$

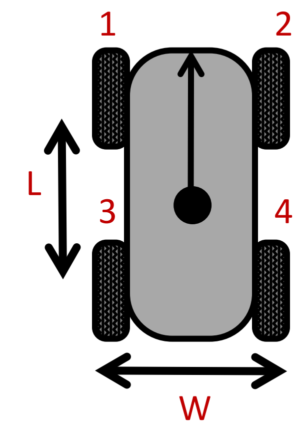
$$\theta_{right} = (z_2 - z_4)/L$$

$$\theta = (\theta_{right} + \theta_{left})/2$$

$$\phi_{front} = (z_1 - z_2)/W$$

$$\phi_{back} = (z_3 - z_4)/W$$

$$\phi = (\phi_{front} + \phi_{back})/2$$



- Simple but not so accurate.

4.3.4.1 Least Residual Terrain Following

- Assumes no suspension.
- Minimize total residual of wheel heights and terrain elevations.

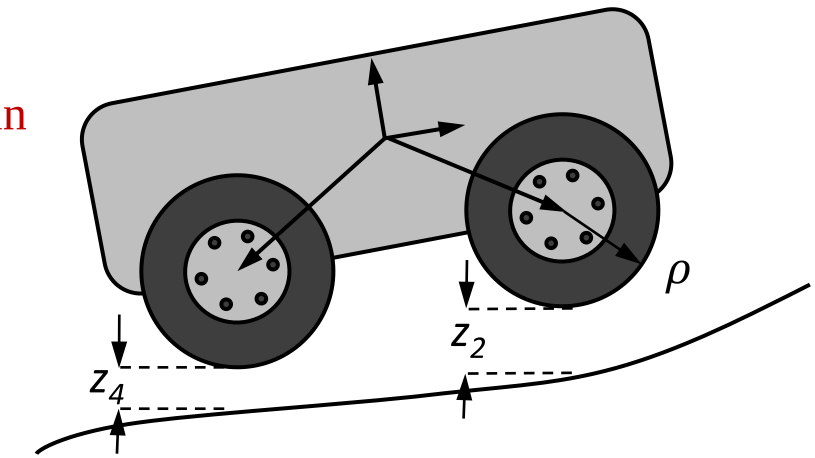
$$\text{minimize:}_{\underline{x}} \quad f(\underline{x}) = \frac{1}{2}\underline{r}^T\underline{r}$$

$$\text{where:} \quad \underline{r}(\underline{x}) = \underline{z} - h(\underline{x}_f, \underline{x})$$

$$z_i = (R_b^w \underline{r}_i^b)_z - \rho$$

Terrain

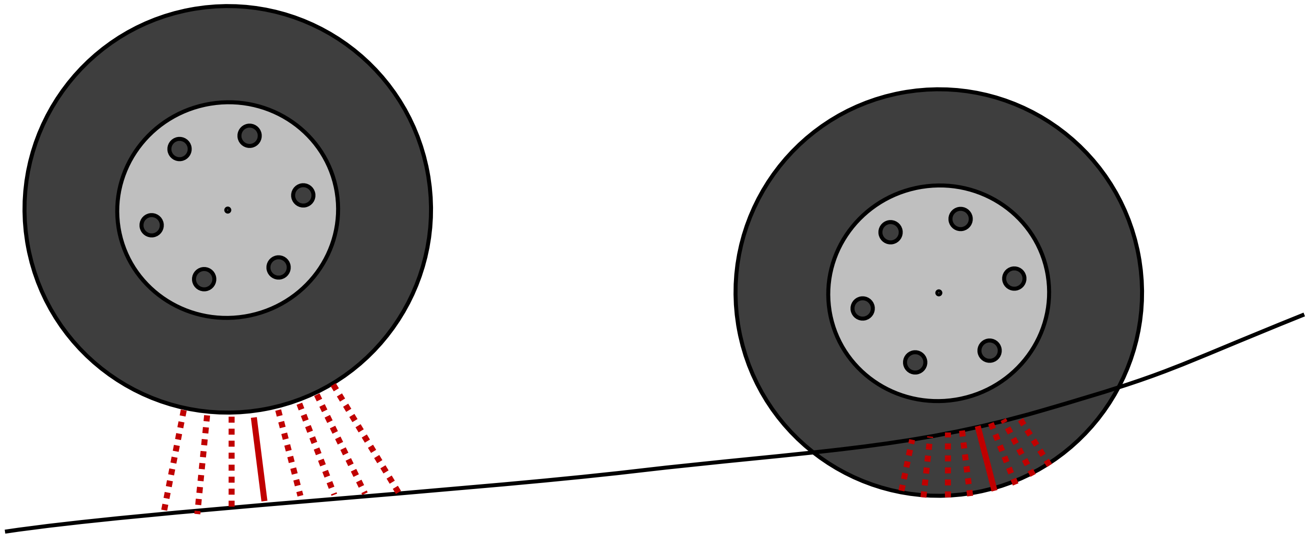
Wheel position



- Unconstrained optimization: Solve using nonlinear least squares.

Computing Wheel Contact Points

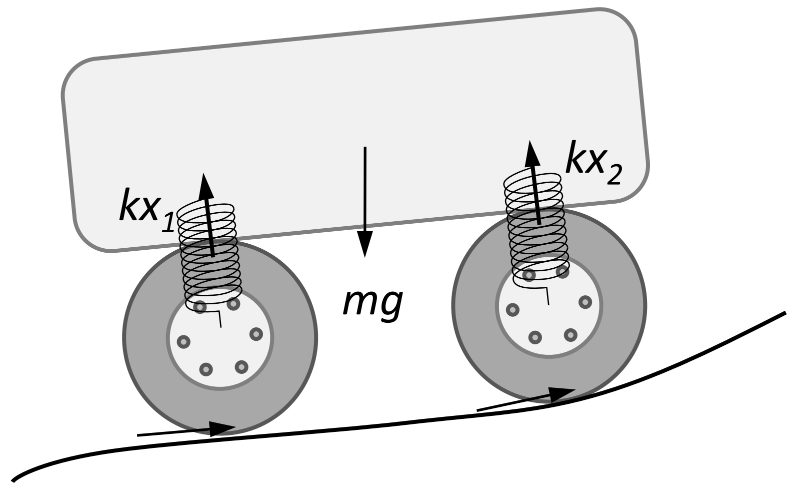
- These are not necessarily on the bottoms of the wheels.
- Contact points occur at local minima or maxima of (perpendicular) distance from wheel surface.



Least Energy Terrain Following

- Let $\underline{x} = [x_1 \quad x_2 \quad x_3 \quad x_4]^T$ …
  - represent spring deflections
- Spring forces are …

$$^w f_i = [R_b^w(\theta, \phi) k x_i]_z$$



- Problem formulation

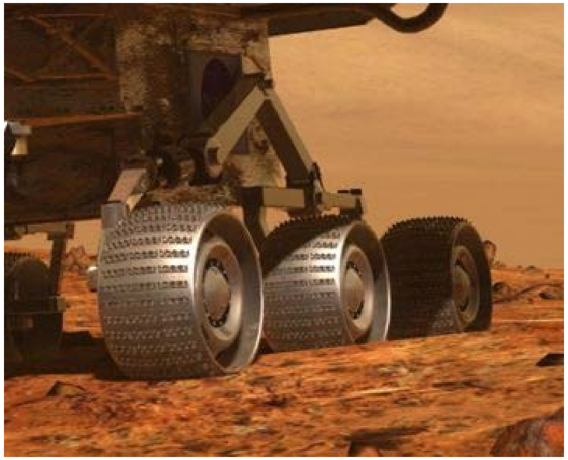
$$\text{minimize:}_{\underline{x}} \quad f(\underline{x}) = \frac{1}{2}\underline{x}^T\underline{x}$$

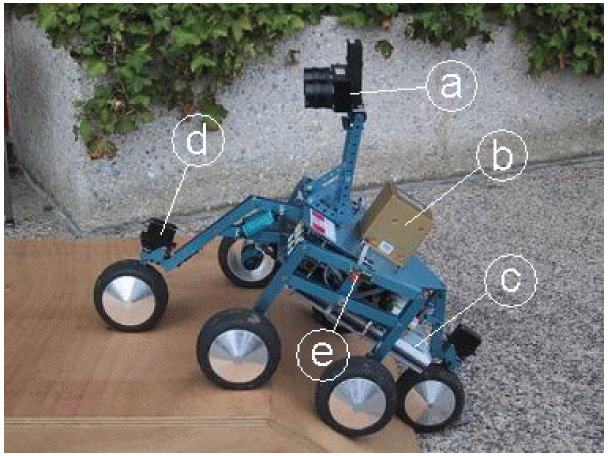
$$\text{subject to:} \quad ({}^w f_1 + {}^w f_2 + {}^w f_3 + {}^w f_4 = mg)$$ Springs carry weight

$$\underline{c}(\underline{x}) = \underline{z} - h(\underline{x}_f, \underline{x}) = \underline{0}$$ 4 wheel contact constraints

- Require suspension to be in minimum energy configuration.

Some Terrain Following Robots





The MER Rovers Spirit and Opportunity used a rocker-bogie design that was intended to keep the forces on all wheels roughly constant.

CMU Rover prototype called Scarab used an averaging suspension that kept the bogy at the average pitch of the left and right halves.

EPFL rover SHRIMP uses four-bar mechanisms to achieve extended climbing capabilities.

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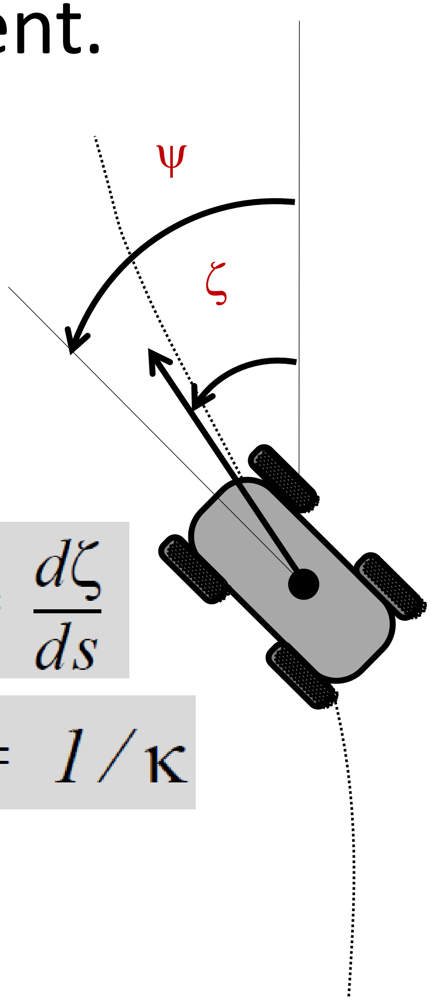
4.3.5 Trajectory Estimation and Prediction

- Up to now, we have concentrated on how you form the differential equation.
- Next issue is how do you integrate it.
- There are two purposes:
  - For state estimation, especially odometry (inputs are measurements)
  - For state prediction in predictive control (inputs are controls)
- Convenience of using body coordinates is now over. Must convert velocities to earth fixed frame to integrate.

4.3.5.1 Heading, Yaw and Curvature

- Heading z is the angle of the path tangent.
- Yaw y is the direction of the forward looking axis
- These may be related or unrelated on a given vehicle.
- Curvature is a property of the path. $\kappa = \frac{d\zeta}{ds}$
- Radius of curvature is its reciprocal. $R = 1/\kappa$
- By the chain rule: $\dot{\zeta} = \frac{d\zeta}{ds}\frac{ds}{dt} = \kappa v$

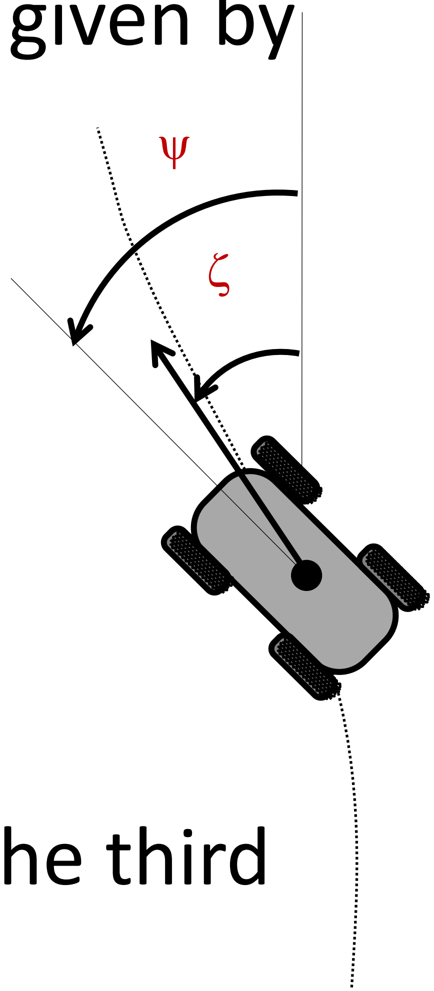


Heading and Yaw Rates

- The rotation rate of the path tangent is given by the chain rule:

$$\dot{\zeta} = \frac{d\zeta}{ds}\frac{ds}{dt} = \kappa v$$

Eqn 4.104



- Only when ζ = ψ can we write

$$\dot{\psi} = \omega = \kappa V$$

- Knowing any two of these determines the third one.

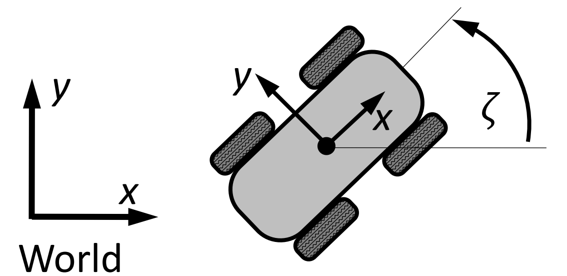
Rate Coordinates

- Hence there are two ways to specify the instantaneous motion…
- κ, v (Curvature-Speed)
  - (+) relates more directly to steering
  - (+) can be derived readily from path
  - (-) point turns are singular
- ω,v (Ang Velocity-Speed)
  - (-) curvature depends on two inputs
  - (-) need path and speed to derive
  - (+) can represent point turns
  - (+) general case.

4.3.5.2 Fully Actuated WMR in Plane

- Velocity is often intrinsically known in the body frame:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \psi(t) \end{bmatrix} = \begin{bmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x(t) \\ v_y(t) \\ \dot{\psi}(t) \end{bmatrix}$$



- The matrix converts coordinates of velocity from body to terrain tangent plane.
- The is the generic 2D velocity kinematics of any vehicle.

4.3.5.2 Fully Actuated WMR in Plane

- If heading and yaw are the same ($\zeta = \psi$), lateral velocity vanishes by definition:

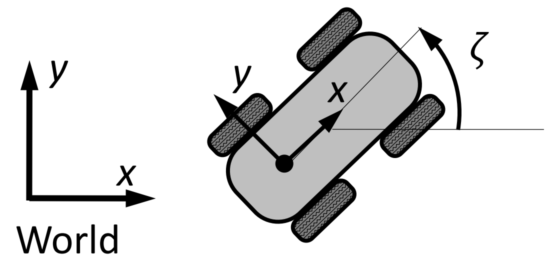
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \zeta(t) \end{bmatrix} = \begin{bmatrix} cos\,\zeta(t) & -sin\,\zeta(t) & 0 \\ sin\,\zeta(t) & cos\,\zeta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x(t) \\ 0 \\ \dot{\zeta}(t) \end{bmatrix}$$

- By assumption, the velocity vector is expressed in a frame aligned with the velocity vector.

4.3.5.3 UnderActuated WMR in Plane

- If the vehicle frame is at center of rear wheels of a car then $\zeta = \psi$. Substitute Eqn 1.104 into last result:

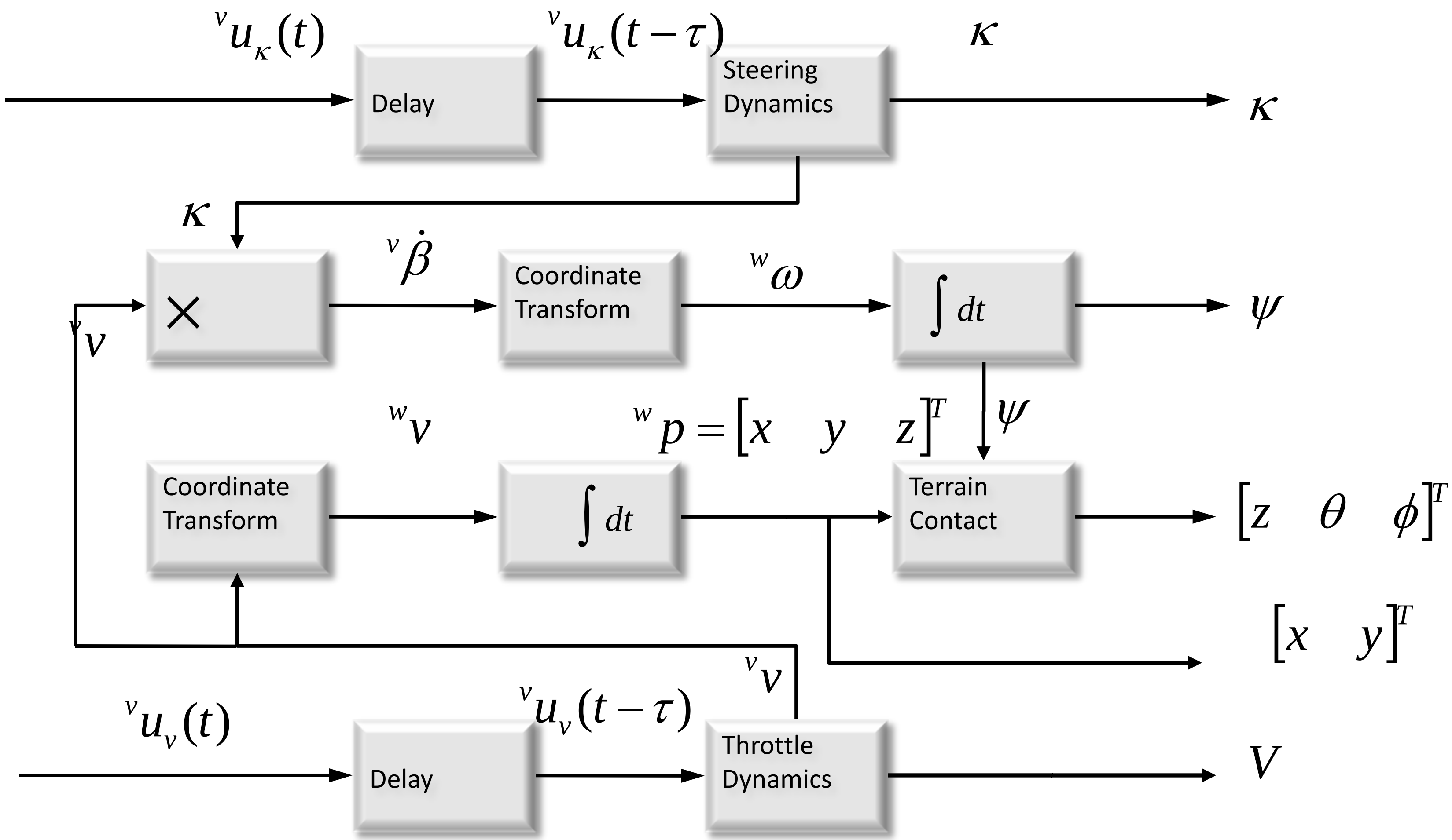
$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ \psi(t) \end{bmatrix} = \begin{bmatrix} \cos\psi(t) & 0 \\ \sin\psi(t) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v(t) \\ \kappa(t)v(t) \end{bmatrix}$$



- Its integral is simply:

$$\begin{bmatrix} x(t) \\ y(t) \\ \psi(t) \end{bmatrix} = \begin{bmatrix} x(0) \\ y(0) \\ \psi(0) \end{bmatrix} + \int_0^t \begin{bmatrix} \cos\psi(t) \\ \sin\psi(t) \\ \kappa(t) \end{bmatrix} v(t)dt$$

4.3.5.4 Fully Actuated WMR in 3D



Since any vehicle has a curvature and speed, this is quite general

4.3.5.4.1 Coordinate Transforms

(in last figure)

- Linear velocity:

$$ {}^{w}\underline{v}_{w}^{w} = Rotz(\psi)Roty(\theta)Rotx(\phi)\,{}^{v}\underline{v}_{v}^{w} $$

$$ {}^{w}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_{v}^{w} = \begin{bmatrix} c\psi c\theta & (c\psi s\theta s\phi - s\psi c\phi) & (c\psi s\theta c\phi + s\psi s\phi) \\ s\psi c\theta & (s\psi s\theta s\phi + c\psi c\phi) & (s\psi s\theta c\phi - c\psi s\phi) \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} {}^{v}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}_{v}^{w} $$

- Angular velocity:

$$ \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & \frac{s\phi}{c\theta} & \frac{c\phi}{c\theta} \end{bmatrix} {}^{v}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_{v}^{w} $$

4.3.5.4.1 Coordinate Transforms

- If the vehicle moves instantaneously in the terrain tangent plane, then:

$$\underline{c}(\underline{x}) = {}^{v}\begin{bmatrix} v_z \\ \omega_x \\ \omega_y \end{bmatrix}^{\mathrm{w}}_{\mathrm{v}} = \underline{0}$$

Typo in book here

- Substitute in last slide. This gives the result:

$$^{w}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}^{w}_{v} = \begin{bmatrix} c\psi c\theta & (c\psi s\theta s\phi - s\psi c\phi) \\ s\psi c\theta & (s\psi s\theta s\phi + c\psi c\phi) \\ -s\theta & c\theta s\phi \end{bmatrix} {}^{v}\begin{bmatrix} v_x \\ v_y \end{bmatrix}^{w}_{v}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} c\phi t\theta \\ -s\phi \\ \dfrac{c\phi}{c\theta} \end{bmatrix} {}^{v}\begin{bmatrix} \omega_z \end{bmatrix}^{w}_{v}$$

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Summary

- Disallowed Directions of Motion:
  - Terrain following is a holonomic constraint but it can be written in a form that uses a disallowed direction.
  - Wheel slip constraints are nonholonomic and use a disallowed direction
  - The difference is that the first direction is fixed in the world frame and the second in the vehicle frame.
- WMR Kinematics and WMR (Lagrangian) Dynamics can both be formulated as constrained differential equations.
- Terrain contract can also be formulated as an energy minimization problem.
- Trajectory estimation and prediction require a conversion of body velocities to world coordinates and integration wrt time.