Analyzing training error

Analysis reveals:

- What α_t to choose for hypothesis h_t?
  What h_t to choose?

$$\alpha_t = \frac{1}{2} \ln \left(\frac{1 - \epsilon_t}{\epsilon_t} \right)$$

ε_t - weighted training error

- If each weak learner h_t is slightly better than random guessing ($\varepsilon_t < 0.5$), then training error of AdaBoost decays exponentially fast in number of rounds T.

$$\frac{1}{m} \sum_{i=1}^{m} \delta(H(x_i) \neq y_i) \leq \exp \left(-2 \sum_{t=1}^{T} (1/2 - \epsilon_t)^2 \right)$$

Training Error

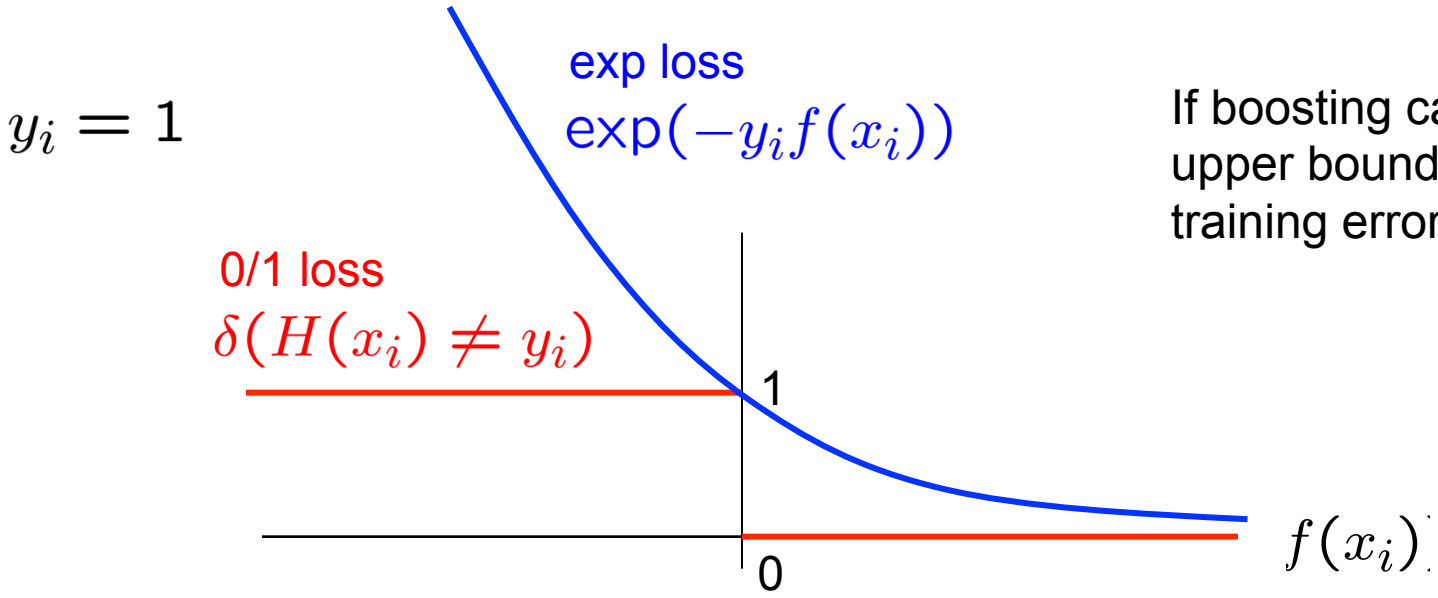
Analyzing training error

Training error of final classifier is bounded by:

$$\frac{1}{m}\sum_{i=1}^{m}\delta(H(x_i)\neq y_i) \leq \frac{1}{m}\sum_{i=1}^{m}\exp(-y_i f(x_i))$$

Convex upper bound

Where $f(x)=\sum_t \alpha_t h_t(x); H(x)=sign(f(x))$



If boosting can make upper bound → 0, then training error → 0

Analyzing training error

Training error of final classifier is bounded by:

$$\frac{1}{m}\sum_{i=1}^{m}\delta(H(x_i)\neq y_i) \leq \frac{1}{m}\sum_{i=1}^{m}\exp(-y_i f(x_i)) = \prod_t Z_t$$

Where $f(x) = \sum_t \alpha_t h_t(x); H(x) = sign(f(x))$

Proof: **Using Weight Update Rule**

$$D_1(i) = 1/m.$$

$$D_2(i) = \frac{1}{m}\frac{e^{-\alpha_1 y_i h_1(x_i)}}{Z_1}$$

$$D_3(i) = \frac{1}{m}\frac{e^{-\alpha_1 y_i h_1(x_i)} e^{-\alpha_2 y_i h_2(x_i)}}{Z_1 Z_2}$$

...

$$D_{T+1}(i) = \frac{1}{m}\frac{\exp(-y_i f(x_i))}{\prod_t Z_t}$$

Wts of all pts add to 1

$$\sum_{i=1}^{m} D_{T+1}(i) = 1$$

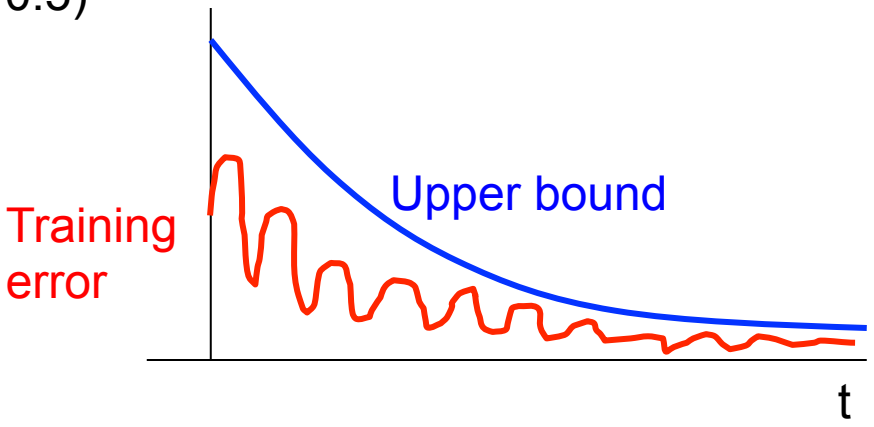
Analyzing training error

Training error of final classifier is bounded by:

$$\frac{1}{m}\sum_{i=1}^{m}\delta(H(x_i)\neq y_i) \leq \frac{1}{m}\sum_{i=1}^{m}\exp(-y_i f(x_i)) = \prod_t Z_t$$

Where $f(x) = \sum_t \alpha_t h_t(x); H(x) = sign(f(x))$

If $Z_t < 1$, training error decreases exponentially (even though weak learners may not be good ε_t~0.5)



What α_t to choose for hypothesis h_t?

Training error of final classifier is bounded by:

$$\frac{1}{m}\sum_{i=1}^{m}\delta(H(x_i)\neq y_i) \leq \frac{1}{m}\sum_{i=1}^{m}\exp(-y_i f(x_i)) = \prod_t Z_t$$

Where $f(x) = \sum_t \alpha_t h_t(x); H(x) = sign(f(x))$

If we minimize $\prod_t Z_t$, we minimize our training error

We can tighten this bound greedily, by choosing α_t and h_t on each iteration to minimize Z_t.

$$Z_t = \sum_{i=1}^{m} D_t(i)\exp(-\alpha_t y_i h_t(x_i))$$

What α_t to choose for hypothesis h_t?

We can minimize this bound by choosing α_t on each iteration to minimize Z_t.

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i))$$

For boolean target function, this is accomplished by [Freund & Schapire '97]:

$$\boxed{\alpha_t = \frac{1}{2} \ln \left(\frac{1 - \epsilon_t}{\epsilon_t} \right)}$$

Proof:

$$\begin{aligned} Z_t &= \sum_{i:y_i \neq h_t(x_i)} D_t(i) e^{\alpha_t} + \sum_{i:y_i = h_t(x_i)} D_t(i) e^{-\alpha_t} \\ &= \epsilon_t e^{\alpha_t} + (1 - \epsilon_t) e^{-\alpha_t} \end{aligned}$$

$$\frac{\partial Z_t}{\alpha_t} = \epsilon_t e^{\alpha_t} - (1 - \epsilon_t) e^{-\alpha_t} = 0 \qquad \Rightarrow e^{2\alpha_t} = \frac{1 - \epsilon_t}{\epsilon_t}$$

Bounding the error with choice of α_t

We can minimize this bound by choosing α_t on each iteration to minimize Z_t.

$$Z_t = \sum_{i=1}^{m} D_t(i) \exp(-\alpha_t y_i h_t(x_i))$$

For boolean target function, this is accomplished by [Freund & Schapire '97]:

$$\boxed{\alpha_t = \frac{1}{2} \ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right)}$$

Proof:

$$\begin{aligned} Z_t &= \sum_{i:y_i \neq h_t(x_i)} D_t(i) e^{\alpha_t} + \sum_{i:y_i = h_t(x_i)} D_t(i) e^{-\alpha_t} \\ &= \epsilon_t e^{\alpha_t} + (1-\epsilon_t) e^{-\alpha_t} \\ &= 2\sqrt{\epsilon_t(1-\epsilon_t)} = \sqrt{1-(1-2\epsilon_t)^2} \end{aligned}$$

Dumb classifiers made Smart

Training error of final classifier is bounded by:

$$\frac{1}{m}\sum_{i=1}^{m}\delta(H(x_i)\neq y_i) \leq \prod_t Z_t = \prod_t \sqrt{1-(1-2\epsilon_t)^2}$$

$$\leq \exp\left(-2\sum_{t=1}^{T}(1/2-\epsilon_t)^2\right)$$

grows as ε_t moves away from 1/2

If each classifier is (at least slightly) better than random $\varepsilon_t < 0.5$

AdaBoost will achieve zero *training error* exponentially fast (in number of rounds T) !!

What about test error?

Boosting and Logistic Regression

Logistic regression assumes:

$$P(Y = 1|X) = \frac{1}{1 + \exp(f(x))} \qquad f(x) = w_0 + \sum_j w_j x_j$$

And tries to maximize data likelihood:

$$P(\mathcal{D}|f) \overset{\text{iid}}{=} \prod_{i=1}^{m} \frac{1}{1 + \exp(-y_i f(x_i))}$$

Equivalent to minimizing log loss

$$-\log P(\mathcal{D}|f) = \sum_{i=1}^{m} \ln(1 + \exp(-y_i f(x_i)))$$

Boosting and Logistic Regression

Logistic regression equivalent to minimizing log loss

$$\sum_{i=1}^{m} \ln(1 + \exp(-y_i f(x_i)))$$

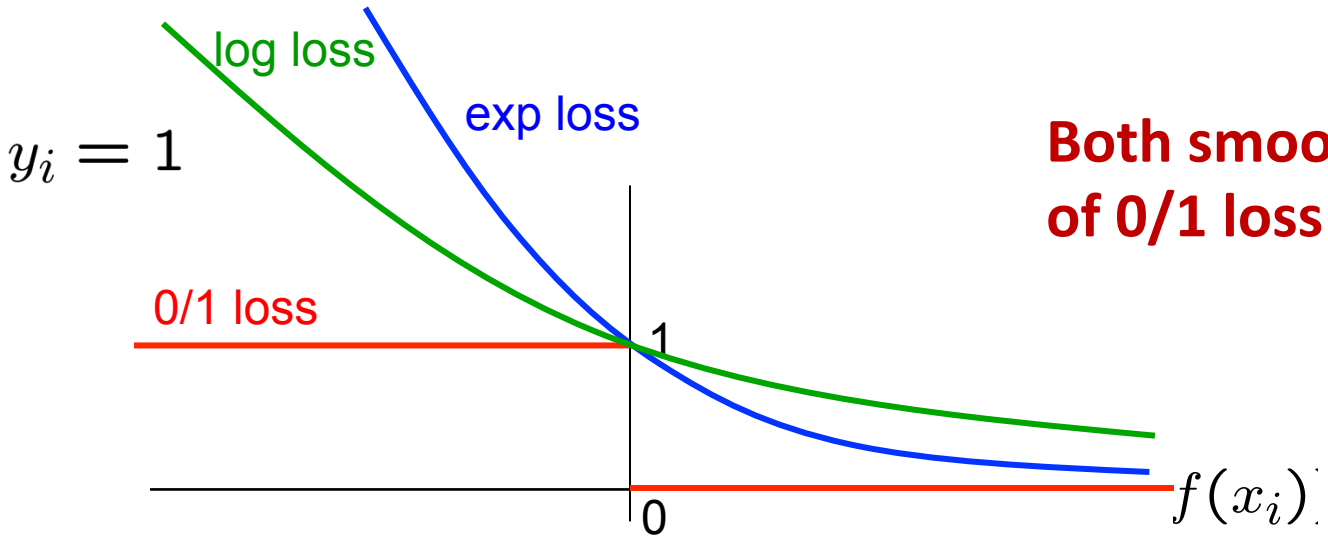
$$f(x) = w_0 + \sum_j w_j x_j$$

Boosting minimizes similar loss function!!

$$\frac{1}{m} \sum_{i=1}^{m} \exp(-y_i f(x_i))$$

$$f(x) = \sum_t \alpha_t h_t(x)$$

Weighted average of weak learners



Both smooth approximations of 0/1 loss!

Boosting and Logistic Regression

Logistic regression:

- Minimize log loss

$$\sum_{i=1}^{m} \ln(1 + \exp(-y_i f(x_i)))$$

- Define

$$f(x) = \sum_j w_j x_j$$

where x_j predefined features

(linear classifier)

- Jointly optimize over all weights w_0, w_1, w_2…

Boosting:

- Minimize exp loss

$$\sum_{i=1}^{m} \exp(-y_i f(x_i))$$

- Define

$$f(x) = \sum_t \alpha_t h_t(x)$$

where $h_t(x)$ defined dynamically to fit data

(not a linear classifier)

- Weights α_t learned per iteration t incrementally

Hard & Soft Decision

Weighted average of weak learners $f(x) = \sum_t \alpha_t h_t(x)$

Hard Decision/Predicted label: $H(x) = sign(f(x))$

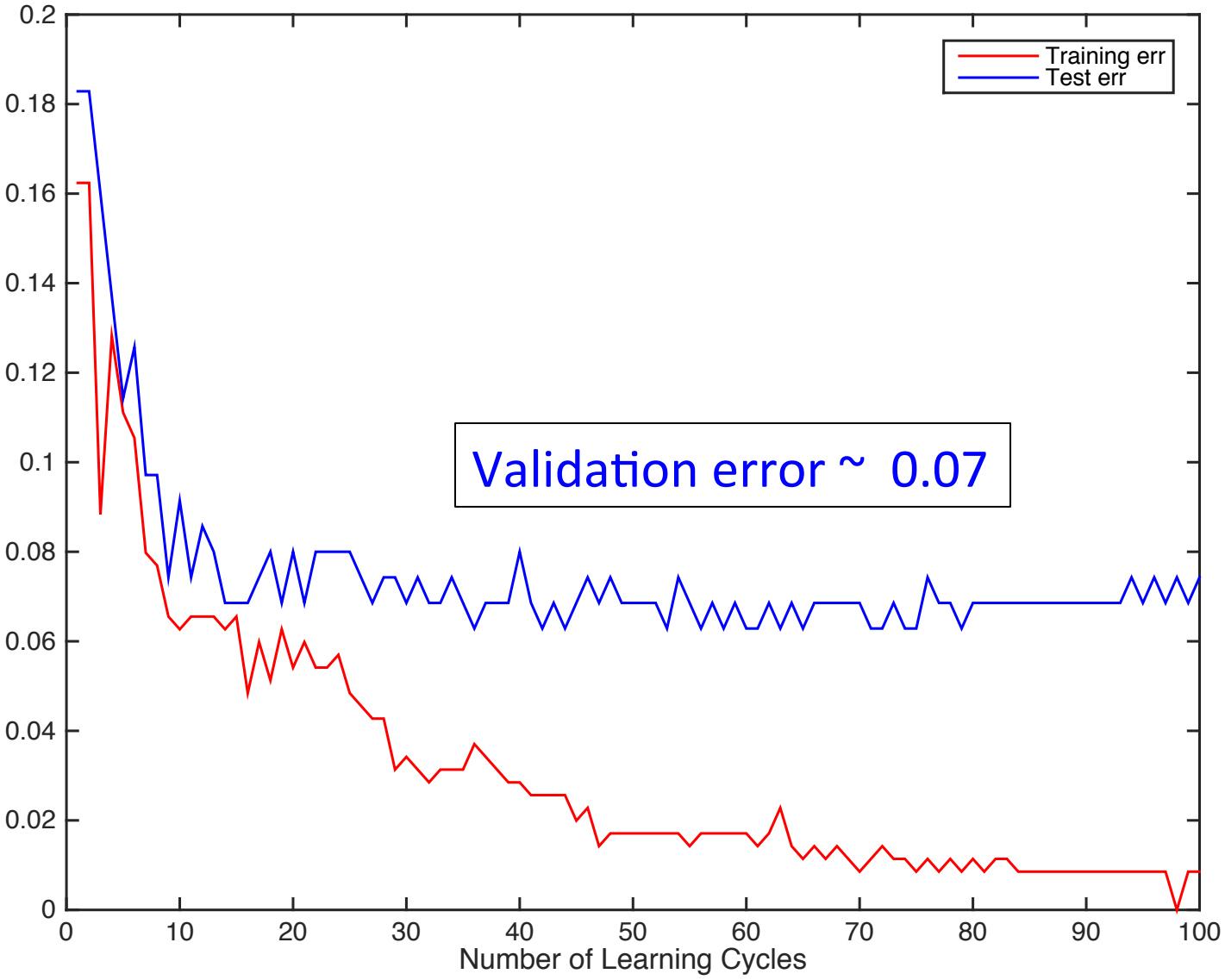
Soft Decision:
(based on analogy with logistic regression)

$$P(Y = 1|X) = \frac{1}{1 + \exp(f(x))}$$

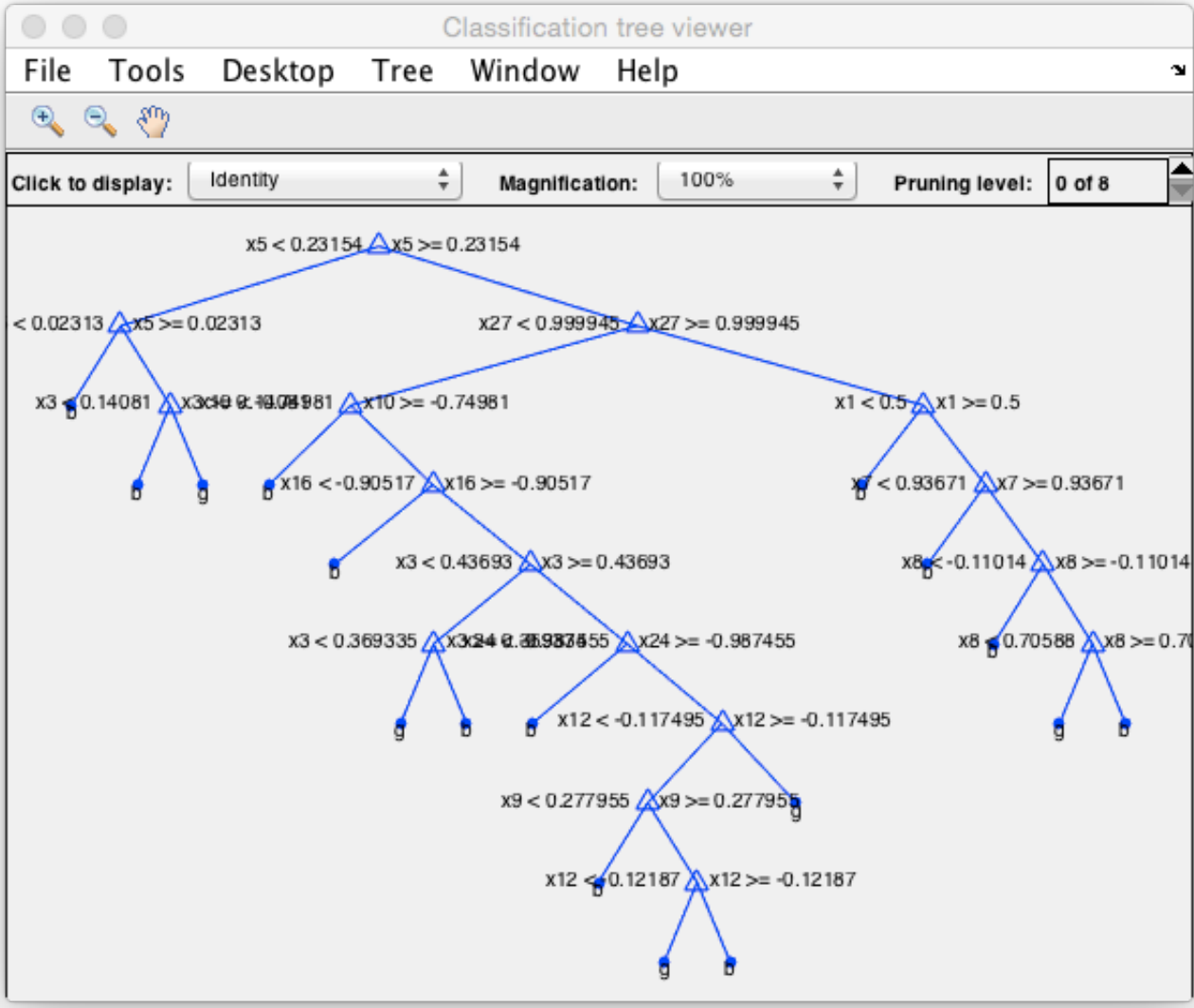
Matlab example - boosting

```
% UCI dataset
% 34 features, 351 samples
% binary classification
load ionosphere;

rng(2); % For reproducibility

ClassTreeEns = fitensemble(X,Y,'AdaBoostM1',100,'Tree');
rsLoss = resubLoss(ClassTreeEns,'Mode','Cumulative');
plot(rsLoss,'r');
hold on

ClassTreeEns = fitensemble(X,Y,'AdaBoostM1',100,'Tree',...
    'Holdout',0.5);
genError = kfoldLoss(ClassTreeEns,'Mode','Cumulative');
plot(genError,'b');
xlabel('Number of Learning Cycles');
legend('Training err', 'Test err')
```

Matlab example - boosting



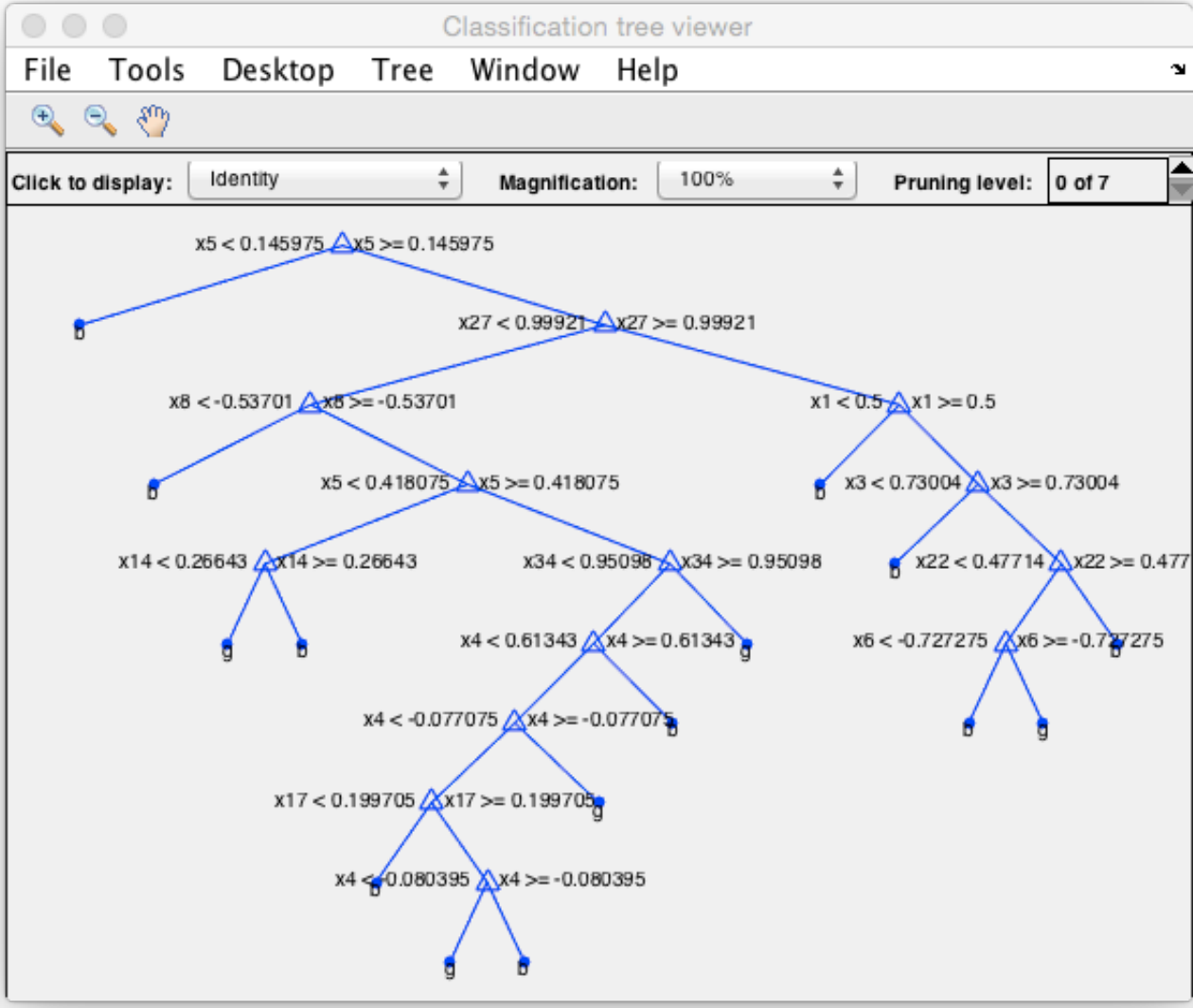
Matlab example – decision tree




```
load ionosphere
% UCI dataset
% 34 features, 351 samples
% binary classification
rng(100)

%Default MinLeafSize = 1
tc = fitctree(X,Y);
cvmodel = crossval(tc);
view(cvmodel.Trained{1},'Mode','graph')
kfoldLoss(cvmodel)
```

Validation error = 0.1254

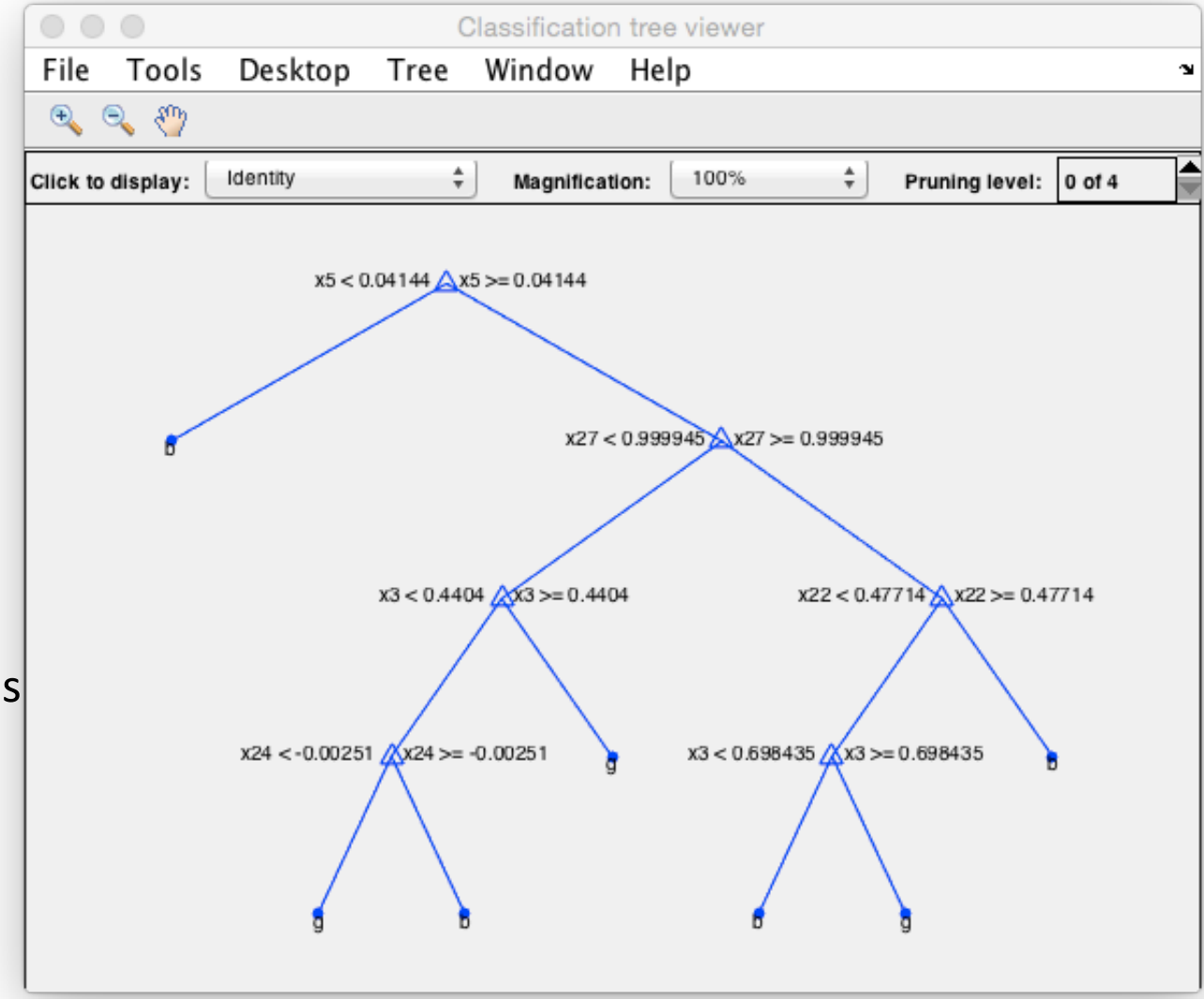
Matlab example – decision tree




```
load ionosphere
% UCI dataset
% 34 features, 351 samples
% binary classification
rng(100)

%Default MinLeafSize = 1
tc = fitctree(X,Y, 'MinLeafSize',2);
cvmodel = crossval(tc);
view(cvmodel.Trained{1},'Mode','graph')
kfoldLoss(cvmodel)
```

Validation error = 0.1168

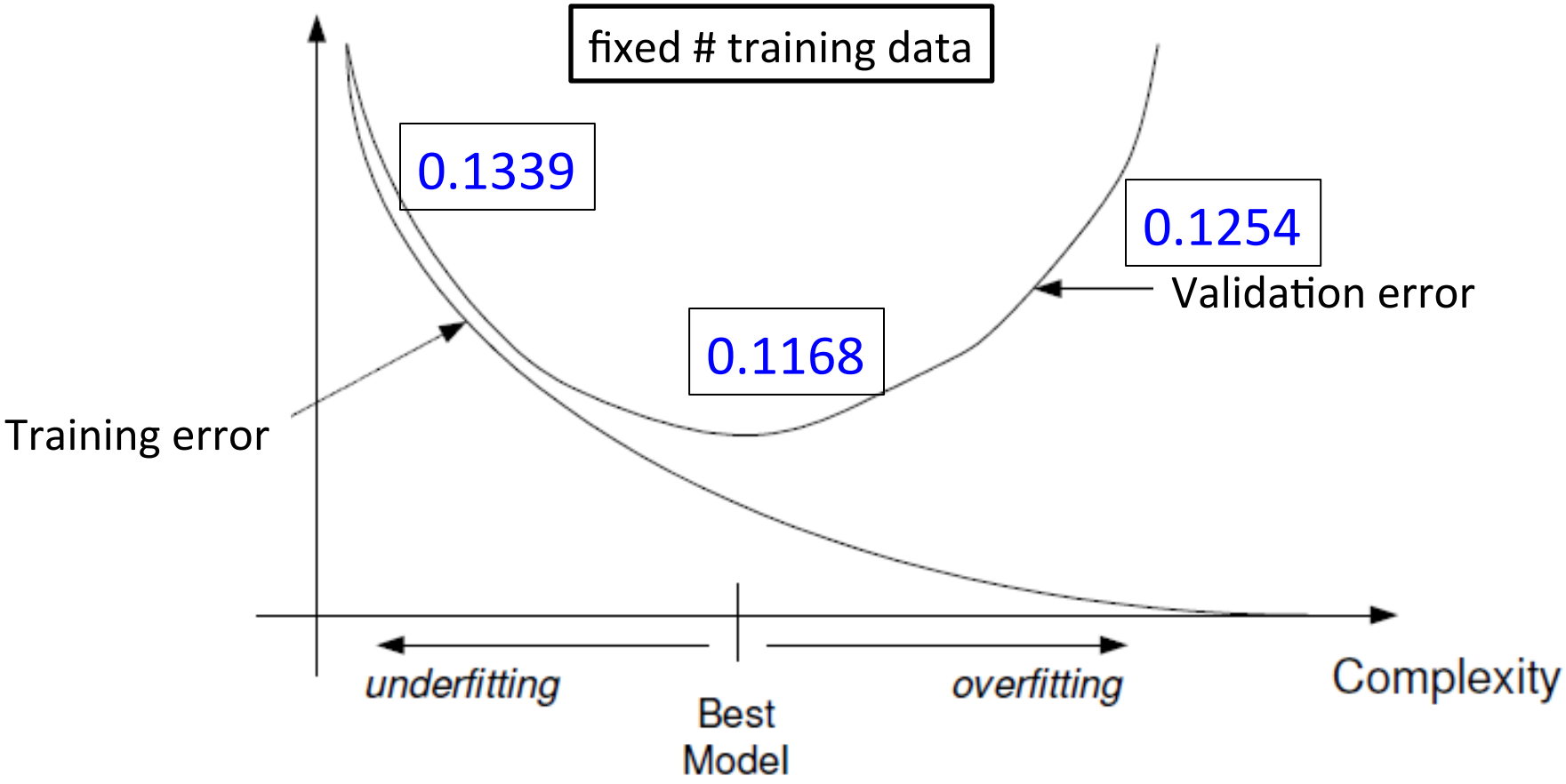
Matlab example – decision tree




```
load ionosphere
% UCI dataset
% 34 features, 351 samples
% binary classification
rng(100)

%Default MinLeafSize = 1
tc = fitctree(X,Y, 'MinLeafSize',10);
cvmodel = crossval(tc);
view(cvmodel.Trained{1},'Mode','graph')
kfoldLoss(cvmodel)
```

Validation error = 0.1339

Matlab example – decision trees



MinLeafSize 10 MinLeafSize 2 MinLeafSize 1