The corresponding Bayes error is

$$L^* = L(g^*) = \mathbf{E}\{\min(\eta(X), 1 - \eta(X))\} = \mathbf{E}\left\{\frac{\min(c, X)}{c + X}\right\}.$$

While we could deduce the Bayes decision from η alone, the same cannot be said for the Bayes error L^*—it requires knowledge of the distribution of X. If $X = c$ with probability one (as in an army school, where all students are forced to study c hours per week), then $L^* = 1/2$. If we have a population that is nicely spread out, say, X is uniform on $[0, 4c]$, then the situation improves:

$$L^* = \frac{1}{4c}\int_0^{4c} \frac{\min(c, x)}{c + x} dx = \frac{1}{4}\log\frac{5e}{4} \approx 0.305785.$$

Far away from $x = c$, discrimination is really simple. In general, discrimination is much easier than estimation because of this phenomenon.

2.3 Another Simple Example

Let us work out a second simple example in which $Y = 0$ or $Y = 1$ according to whether a student fails or passes a course. X represents one or more observations regarding the student. The components of X in our example will be denoted by T, B, and E respectively, where T is the average number of hours the students watches TV, B is the average number of beers downed each day, and E is an intangible quantity measuring extra negative factors such as laziness and learning difficulties. In our cooked-up example, we have

$$Y = \begin{cases} 1 & \text{if } T + B + E < 7 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if T, B, and E are known, Y is known as well. The Bayes classifier decides 1 if $T + B + E < 7$ and 0 otherwise. The corresponding Bayes probability of error is zero. Unfortunately, E is intangible, and not available to the observer. We only have access to T and B. Given T and B, when should we guess that $Y = 1$? To answer this question, one must know the joint distribution of (T, B, E), or, equivalently, the joint distribution of (T, B, Y). So, let us assume that T, B, and E are i.i.d. exponential random variables (thus, they have density e^{-u} on $[0, \infty)$). The Bayes rule compares $\mathbf{P}\{Y = 1|T, B\}$ with $\mathbf{P}\{Y = 0|T, B\}$ and makes a decision consistent with the maximum of these two values. A simple calculation shows that

$$\begin{aligned} \mathbf{P}\{Y = 1|T, B\} &= \mathbf{P}\{T + B + E < 7|T, B\} \\ &= \mathbf{P}\{E < 7 - T - B|T, B\} \\ &= \max\left(0, 1 - e^{-(7-T-B)}\right). \end{aligned}$$

The crossover between two decisions occurs when this value equals 1/2. Thus, the Bayes classifier is as follows:

$$g^*(T, B) = \begin{cases} 1 & \text{if } T + B < 7 - \log 2 = 6.306852819\ldots \\ 0 & \text{otherwise.} \end{cases}$$