10/36-702 Homework 2
Additional hints for problem 4

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4 Let $\mathcal{H}$ be a Hilbert space of functions. Suppose that the evaluation functionals $\delta_x f = f(x)$ are continuous. Show that $\mathcal{H}$ is a reproducing kernel Hilbert space and find the kernel.

h1 **Dual space**

Suppose the original space is $\mathcal{H} = \{f : \mathcal{X} \to \mathbb{R}\}$, where $\mathcal{X}$ could be $[0,1]^d$. A functional is $\alpha : \mathcal{H} \to \mathbb{R}$, i.e. $\alpha(f)$ returns a real number for every $f \in \mathcal{H}$, e.g. the evaluation functional defined above. The dual space $\mathcal{H}^*$ consists of all continuous linear functionals. Continuity of α means that if $f_1, \ldots, f_n, f \in \mathcal{H}$ satisfies $\lim_n \|f_n - f\|_{\mathcal{H}} = 0$, then $\alpha(f_n)$, which is a number, converges to $\alpha(f)$. Linearity of α is defined after 49.1 at the beginning of the Function Space chapter.

When $\mathcal{H}$ is a L^2 space on $[0,1]$, δ_x needs not be continuous. A counter example is $f_1(x) = I_{\{x<0.5\}}(x)$ and $f_2(x) = I_{\{x\leq 0.5\}}(x)$. By definition,

$$\|f_1 - f_2\|_{\mathcal{H}} = \Big(\int_0^1 |f_1(x) - f_2(x)|^2 dx\Big)^{\frac{1}{2}} = 0.$$

On the other hand, $\delta_{0.5}(f_1) \neq \delta_{0.5}(f_2)$.

h2 **Riesz representation theorem**

`http://en.wikipedia.org/wiki/Riesz_representation_theorem`

For any $\alpha \in \mathcal{H}^*$, there exists a unique $f_\alpha \in \mathcal{H}$ s.t. $\forall g \in \mathcal{H}, \alpha(g) = \langle f_a, g\rangle_{\mathcal{H}}$. Notice $\langle \alpha, g\rangle$, which spans two spaces, is not allowed.

h3 **RKHS only needs** (according to Larry)

to be a Hilbert space

have a symmetric kernel

with the reproducing property