Hidden Markov Models

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Slides courtesy: Eric Xing

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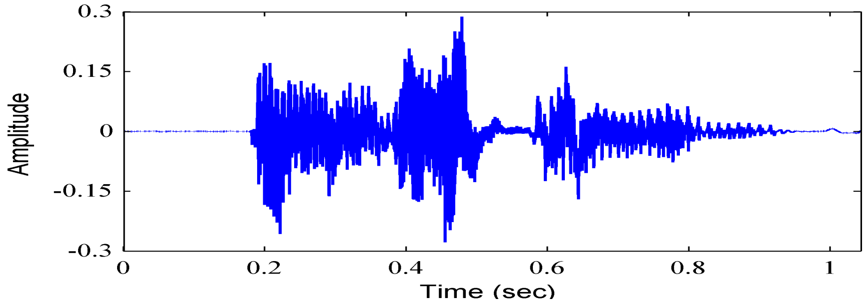
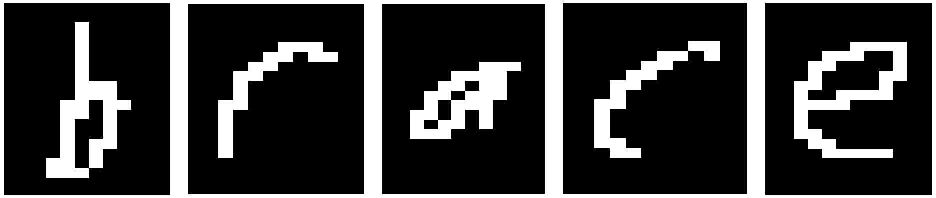
i.i.d to sequential data

- So far we assumed independent, identically distributed data $\{X_i\}_{i=1}^n \overset{iid}{\sim} p(\mathbf{X})$
- Sequential data
  - Time-series data

    E.g. Speech



  - Characters in a sentence



  - Base pairs along a DNA strand

Markov Models

$X_1 \to X_2 \to X_3 \to \ldots \to X_n$

- Joint Distribution

$$
\begin{aligned}
p(\mathbf{X}) &= p(X_1, X_2, \ldots, X_n) \\
&= p(X_1)p(X_2|X_1)p(X_3|X_2, X_1)\ldots p(X_i|X_{i-1}, \ldots, X_1) \\
&= \prod_{i=1}^{n} p(X_i|X_{i-1}, \ldots, X_1)
\end{aligned}
$$

Chain rule

- Markov Assumption (m^th^ order)

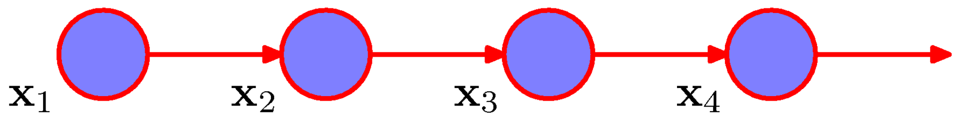
$$
p(\mathbf{X}) = \prod_{i=1}^{n} p(X_i|X_{i-1}, \ldots, X_{i-m})
$$

Current observation only depends on past m observations

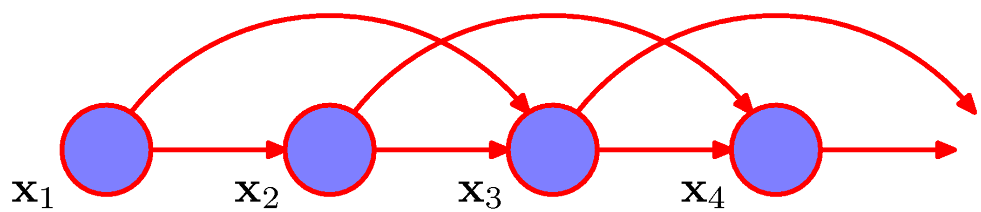
Markov Models

- Markov Assumption

1st order $$p(\mathbf{X}) = \prod_{i=1}^{n} p(X_i|X_{i-1})$$



2nd order $$p(\mathbf{X}) = \prod_{i=1}^{n} p(X_i|\underline{X_{i-1}, X_{i-2}})$$



Markov Models

parameters in stationary model d-ary variables

- Markov Assumption

1st order $$p(\mathbf{X}) = \prod_{i=1}^{n} p(X_i | X_{i-1})$$ $O(d^2)$

b = X_i, a = X_{i-1}

mth order $$p(\mathbf{X}) = \prod_{i=1}^{n} p(X_i | X_{i-1}, \ldots, X_{i-m})$$ $O(d^{m+1})$

d×1 d×1 d×1

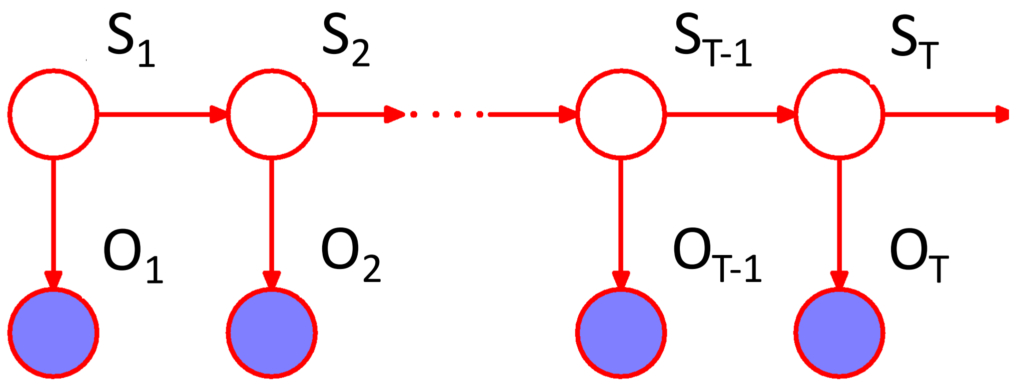
n-1th order $$p(\mathbf{X}) = \prod_{i=1}^{n} p(X_i | X_{i-1}, \ldots, X_1)$$ $O(d^n)$

≡ no assumptions – complete (but directed) graph

Homogeneous/stationary Markov model (probabilities of transitioning from a particular state value to another value doesn't depend on i)

Hidden Markov Models

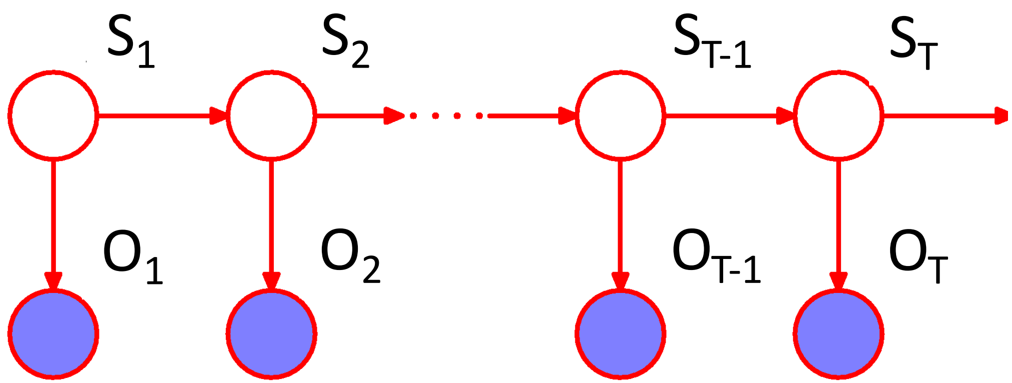
- Distributions that characterize sequential data with few parameters but are not limited by strong Markov assumptions.



Observation space $O_t \in \{y_1, y_2, \ldots, y_K\}$

Hidden states $S_t \in \{1, \ldots, S\}$

Hidden Markov Models



$$p(S_1, \ldots, S_T, O_1, \ldots, O_T) \quad = \quad \prod_{t=1}^{T} p(O_t|S_t) \prod_{t=1}^{T} p(S_t|S_{t-1})$$

1st order Markov assumption on hidden states $\{S_t\}$ t = 1, …, T

(can be extended to higher order).

Note: O_t depends on all previous observations $\{O_{t-1},\ldots O_1\}$

Hidden Markov Models

- Parameters – stationary/homogeneous markov model (independent of time t)

Initial probabilities

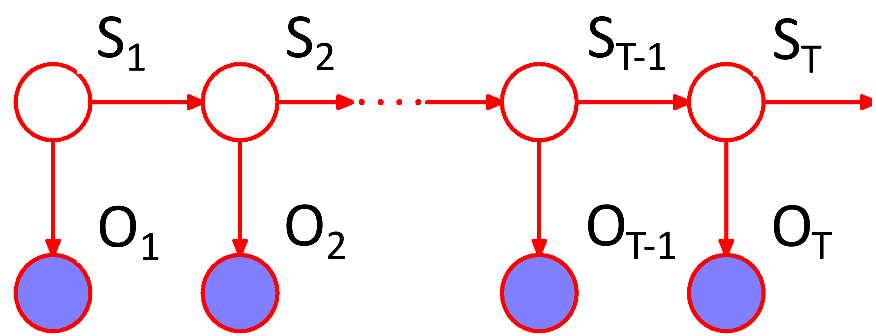
$p(S_1 = i) = \pi_i$ $\quad$ $i=1..S$

Transition probabilities

$p(S_t = j | S_{t-1} = i) = p_{ij}$ $\quad$ S^2

Emission probabilities

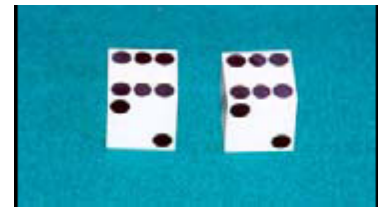
$p(O_t = y | S_t = i) = q_i^y$ $\quad$ KS



$$p(\{S_t\}_{t=1}^T, \{O_t\}_{t=1}^T) = p(S_1) \prod_{t=2}^T p(S_t|S_{t-1}) \prod_{t=1}^T p(O_t|S_t)$$

HMM Example

- The Dishonest Casino



A casino has two die: $P(O=5 \mid S=F)$

Fair dice

$P(1) = P(2) = P(3) = P(5) = P(6) = 1/6$

Loaded dice

$P(1) = P(2) = P(3) = P(5) = 1/10$

$P(6) = ½$ $P(O=5 \mid S=L)$

Casino player switches back-&-forth between fair and loaded die once every 20 turns

HMM Problems

GIVEN: A sequence of rolls by the casino player

1245526462146146136136661664661636616366163616515615115146123562344

S = F/L?

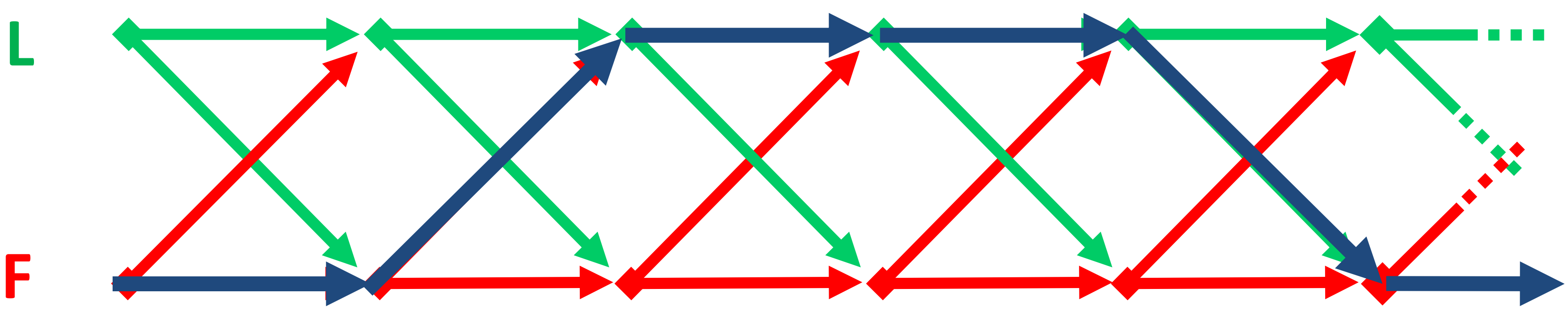
QUESTION

- How likely is this sequence, given our model of how the casino works?
  - This is the **EVALUATION** problem in HMMs
- What portion of the sequence was generated with the fair die, and what portion with the loaded die?
  - This is the **DECODING** question in HMMs
- How “loaded” is the loaded die? How “fair” is the fair die? How often does the casino player change from fair to loaded, and back?
  - This is the **LEARNING** question in HMMs

HMM Example

- Observed sequence: $\{O_t\}_{t=1}^{T}$

1 → 4 → 3 → 6 → 6 → 4 →

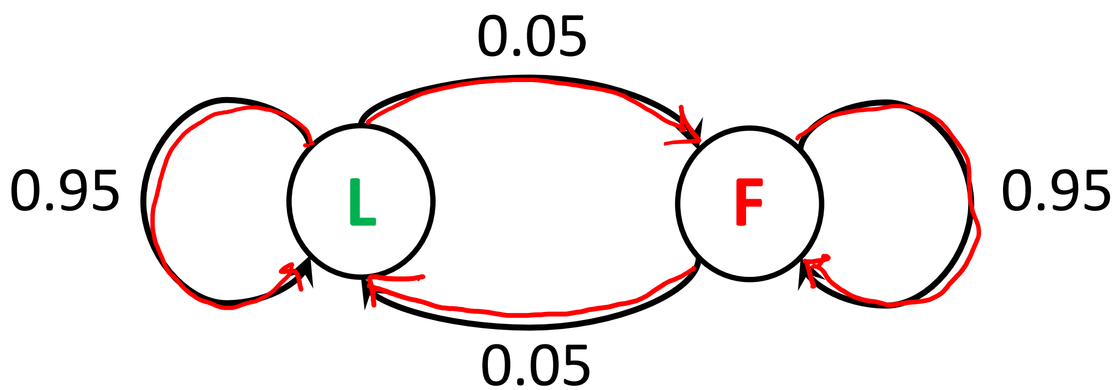


- Hidden sequence $\{S_t\}_{t=1}^{T}$ or segmentation):

F → F → L → L → L → F →

State Space Representation

- Switch between **F** and **L** once every 20 turns (1/20 = 0.05)



- HMM Parameters

Initial probs	$P(S_1 = L) = 0.5 = P(S_1 = F)$
Transition probs	$P(S_t = L/F \mid S_{t-1} = L/F) = 0.95$
	$P(S_t = F/L \mid S_{t-1} = L/F) = 0.05$
Emission probabilities	$P(O_t = y \mid S_t = F) = 1/6 \qquad y = 1,2,3,4,5,6$
	$P(O_t = y \mid S_t = L) = 1/10 \qquad y = 1,2,3,4,5$
	$= 1/2 \qquad y = 6$

Three main problems in HMMs

- Evaluation – Given HMM parameters & observation seqn $\{O_t\}_{t=1}^T$

  find $\underline{p(\{O_t\}_{t=1}^T)}$ prob of observed sequence

- Decoding – Given HMM parameters & observation seqn $\{O_t\}_{t=1}^T$

  find $\arg\max_{s_1,\ldots,s_T} \underline{p(\{S_t\}_{t=1}^T | \{O_t\}_{t=1}^T)}$ most probable sequence of hidden states

  $\max_{s_1,\ldots s_T} p(S_T = s_T, \ldots, S_t = s_t, \ldots, S_1 = s_1 | O_1, \ldots O_T)$

- Learning – Given HMM with unknown parameters and $\{O_t\}_{t=1}^T$ observation sequence

  find $\arg\max_\theta p(\{O_t\}_{t=1}^T | \theta)$ parameters that maximize likelihood of observed data

HMM Algorithms

- Evaluation – What is the probability of the observed sequence? Forward Algorithm

$p(S_3 = L \mid O_1 \ldots O_T)$

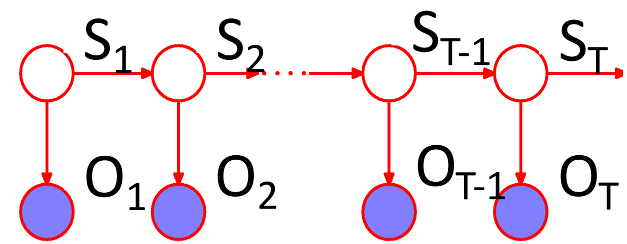
- Decoding – What is the probability that the third roll was loaded given the observed sequence? Forward-Backward Algorithm

  – What is the most likely die sequence given the observed sequence? Viterbi Algorithm $\max_{S_1 \ldots S_T} p(S_1 \ldots S_T \mid O_1 \ldots O_T)$

- Learning – Under what parameterization is the observed sequence most probable? Baum-Welch Algorithm (EM)

Evaluation Problem

- Given HMM parameters $p(S_1), p(S_t|S_{t-1}), p(O_t|S_t)$ & observation sequence $\{O_t\}_{t=1}^T$



find probability of observed sequence

$$p(\{O_t\}_{t=1}^T) = \sum_{S_1,\dots,S_T} p(\{O_t\}_{t=1}^T, \{S_t\}_{t=1}^T)$$

$$= \sum_{S_1,\dots,S_T} p(S_1) \prod_{t=2}^T p(S_t|S_{t-1}) \prod_{t=1}^T p(O_t|S_t)$$

requires summing over all possible hidden state values at all times – S^T exponential # terms!

$S \times S \times S \times \dots S$ (T times)

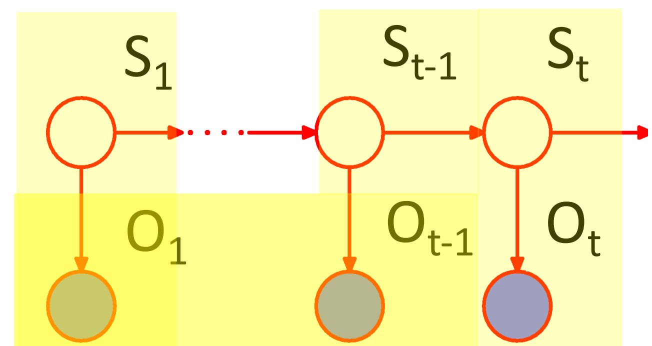
Instead: $p(\{O_t\}_{t=1}^T) = \sum_k \underbrace{p(\{O_t\}_{t=1}^T, S_T = k)}_{\alpha_T^k}$

Compute recursively

Forward Probability

$$p(\{O_t\}_{t=1}^T) = \sum_k p(\{O_t\}_{t=1}^T, S_T = k) = \sum_k \alpha_T^k$$

Compute forward probability α_t^k recursively over t



$$\alpha_t^k \quad := \quad p(O_1, \ldots, O_t, S_t = k)$$

$= \sum_i p(O_1 \ldots O_t, S_t = k, S_{t-1} = i)$

Introduce S_{t-1}

$\vdots$

Chain rule

$= \sum_i p(O_t | S_t = k)\, p(S_t = k | S_{t-1} = i)\, p(O_1 - O_{t-1}, S_{t-1} = i)$

Markov assumption

$$= \quad p(O_t|S_t = k) \sum_i \alpha_{t-1}^i p(S_t = k | S_{t-1} = i)$$

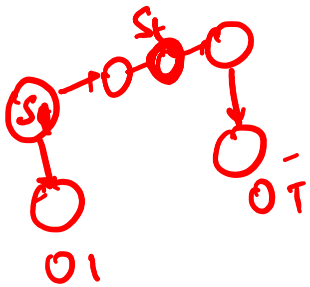
Forward Algorithm

Can compute α_t^k for all k, t using dynamic programming:

- Initialize: $\alpha_1^k = p(O_1 | S_1 = k)\, p(S_1 = k)$ for all k
- Iterate: for t = 2, …, T

$$\alpha_t^k = p(O_t | S_t = k) \sum_i \alpha_{t-1}^i \, p(S_t = k | S_{t-1} = i) \quad \text{for all } k$$

- Termination: $p(\{O_t\}_{t=1}^T) = \sum_k \alpha_T^k$



Decoding Problem 1

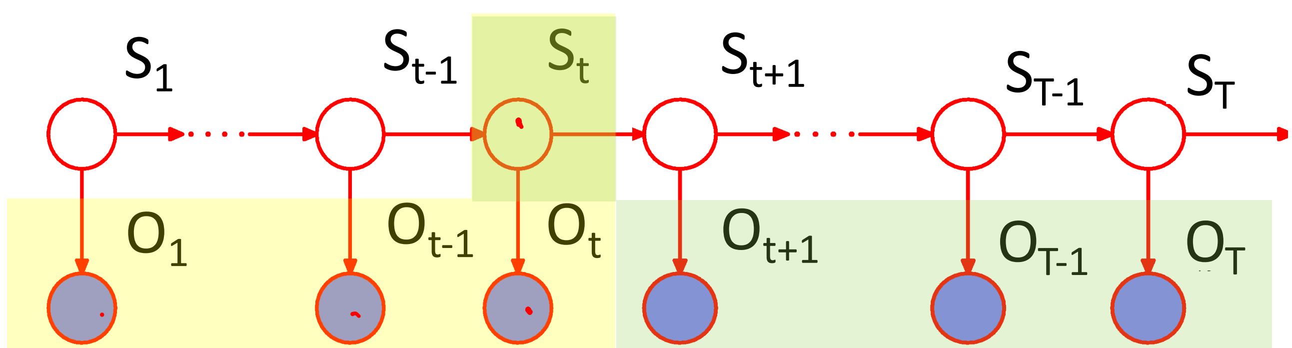
- Given HMM parameters $p(S_1), p(S_t|S_{t-1}), p(O_t|S_t)$ & observation sequence $\{O_t\}_{t=1}^T$

$p(S_3=2|S\ 3)$

find probability that hidden state at time t was k $p(S_t = k|\{O_t\}_{t=1}^T)$

$$\begin{aligned} p(S_t = k, \{O_t\}_{t=1}^T) &= p(O_1, \ldots, O_t, S_t = k, O_{t+1}, \ldots, O_T) \\ &= \underbrace{p(O_1, \ldots, O_t, S_t = k)}_{\boxed{\alpha_t^k}} \underbrace{p(O_{t+1}, \ldots, O_T|S_t = k)}_{\boxed{\beta_t^k}} \end{aligned}$$

Compute recursively



Backward Probability

$$p(S_t = k, \{O_t\}_{t=1}^T) = p(O_1, \ldots, O_t, S_t = k)p(O_{t+1}, \ldots, O_T | S_t = k) = \alpha_t^k \beta_t^k$$

Compute ~~forward~~ backward probability β_t^k recursively over t

β_T^k

$$\beta_t^k \quad := \quad p(O_{t+1}, \ldots, O_T | S_t = k)$$

$= \sum_i p(O_{t+1} \ldots O_T | S_t = k, S_{t+1} = i)$

Introduce S_{t+1}

$= \sum_i p(O_{t+1} | S_{t+1} = i)$ ✓

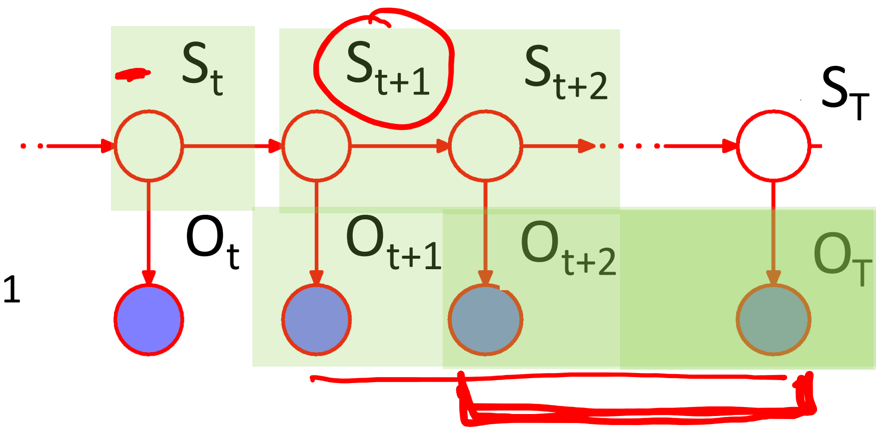
$p(S_{t+1} = i | S_t = k)$ ✓

Chain rule

$p(O_{t+2} \ldots O_T | S_{t+1} = i)$

Markov assumption

$$= \quad \sum_i p(S_{t+1} = i | S_t = k) p(O_{t+1} | S_{t+1} = i) \beta_{t+1}^i$$



Backward Algorithm

Can compute β_t^k for all k, t using dynamic programming:

- Initialize: $\beta_T^k = 1$ for all k
- Iterate: for t = T-1, …, 1

$$\underbrace{\beta_t^k}_{T-1} = \sum_i p(S_{t+1} = i | S_t = k) p(O_{t+1} | S_{t+1} = i) \underbrace{\beta_{t+1}^i}_{T} \quad \text{for all k}$$

- Termination: $p(S_t = k, \{O_t\}_{t=1}^T) = \alpha_t^k \beta_t^k$

$$p(S_t = k | \{O_t\}_{t=1}^T) = \frac{p(S_t = k, \{O_t\}_{t=1}^T)}{p(\{O_t\}_{t=1}^T)} = \frac{\alpha_t^k \beta_t^k}{\sum_i \alpha_t^i \beta_t^i}$$

$\max_{k \in L,F} p(S_3 = k | \{O\})$

Most likely state vs. Most likely sequence

- Most likely state assignment at time t

$$\arg\max_k p(S_t = k | \{O_t\}_{t=1}^T) = \arg\max_k \alpha_t^k \beta_t^k$$

E.g. Which die was most likely used by the casino in the third roll given the observed sequence?

- Most likely assignment of state sequence

$$\arg\max_{\{S_t\}_{t=1}^T} p(\{S_t\}_{t=1}^T | \{O_t\}_{t=1}^T)$$

E.g. What was the most likely sequence of die rolls used by the casino given the observed sequence?

Not the same solution !

MLA of x?
MLA of (x,y)?

x	y	P(x,y)
0	0	0.35
0	1	0.05
1	0	0.3
1	1	0.3

Decoding Problem 2

- Given HMM parameters $p(S_1), p(S_t|S_{t-1}), p(O_t|S_t)$ & observation sequence $\{O_t\}_{t=1}^T$

  find most likely assignment of state sequence

$$\arg\max_{\{S_t\}_{t=1}^T} p(\{S_t\}_{t=1}^T | \{O_t\}_{t=1}^T) = \arg\max_{\{S_t\}_{t=1}^T} p(\{S_t\}_{t=1}^T, \{O_t\}_{t=1}^T) \quad / p\{O\}$$

$$= \arg\max_k \underbrace{\max_{\{S_t\}_{t=1}^{T-1}} p(S_T = k, \{S_t\}_{t=1}^{T-1}, \{O_t\}_{t=1}^T)}_{V_T^k}$$

Viterbi prob

Compute recursively

V_T^k - probability of most likely sequence of states ending at state $S_T = k$

Viterbi Decoding

$$\max_{\{S_t\}_{t=1}^T} p(\{S_t\}_{t=1}^T, \{O_t\}_{t=1}^T) = \max_k V_T^k$$

Compute probability V_t^k recursively over t

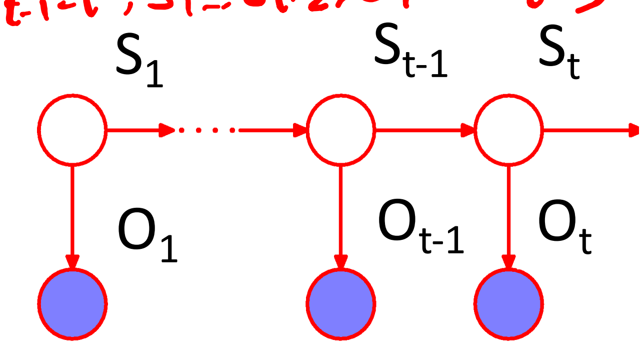
$$V_t^k \quad := \quad \max_{S_1,\ldots,S_{t-1}} p(S_t = k, S_1, \ldots, S_{t-1}, O_1, \ldots, O_t)$$

$$= \max_{S_1 .. S_{t-1}} p(O_t | S_t = k)\, p(S_t = k | S_{t-1} = i)\, \overbrace{p(S_{t-1} = i, S_1 .. S_{t-2}, O_1 .. O_{t-1})}^{V_{t-1}^i}$$

.

. Chain rule

. Markov assumption



$$= \quad p(O_t | S_t = k) \max_i p(S_t = k | S_{t-1} = i) V_{t-1}^i$$

Viterbi Algorithm

Can compute V_t^k for all k, t using dynamic programming:

- Initialize: $V_1^k = p(O_1|S_1=k)p(S_1 = k)$ for all k
- Iterate: for t = 2, …, T

$$V_t^k = p(O_t|S_t = k) \max_i p(S_t = k|S_{t-1} = i)V_{t-1}^i \quad \text{for all k}$$

- Termination: $\max_{\{S_t\}_{t=1}^T} p(\{S_t\}_{t=1}^T, \{O_t\}_{t=1}^T) = \max_k V_T^k$

Traceback:

$$S_T^* = \arg\max_k V_T^k$$

$$S_{t-1}^* = \arg\max_i p(S_t^*|S_{t-1} = i)V_{t-1}^i$$

Computational complexity

- What is the running time for Forward, Forward-Backward, Viterbi?

$$\alpha_t^k = \underline{q_k^{O_t}} \sum_i \alpha_{t-1}^i \, \underline{p_{i,k}}$$

$$\beta_t^k = \sum_i p_{k,i} \, q_i^{O_{t+1}} \, \beta_{t+1}^i$$

$$V_t^k = q_k^{O_t} \, \max_i p_{i,k} \, V_{t-1}^i$$

$O(S^2T)$ linear in T instead of $O(S^T)$ exponential in T!

Learning Problem

- Given HMM with unknown parameters $\theta = \{\{\pi_i\}, \{p_{ij}\}, \{q_i^k\}\}$ and observation sequence $\mathbf{O} = \{O_t\}_{t=1}^{T}$

  find parameters that maximize likelihood of observed data

  $$\arg\max_{\theta} p(\{O_t\}_{t=1}^{T}|\theta)$$

  But likelihood doesn't factorize since observations not i.i.d.

  hidden variables – state sequence $\{S_t\}_{t=1}^{T}$

  EM (Baum-Welch) Algorithm:

  E-step – Fix parameters, find expected state assignments

  M-step – Fix expected state assignments, update parameters

Baum-Welch (EM) Algorithm

- Start with random initialization of parameters
- **E-step** – Fix parameters, find expected state assignments

$$\gamma_i(t) = p(S_t = i|O,\theta) = \frac{\alpha_t^i \beta_t^i}{\sum_j \alpha_t^j \beta_t^j}$$

Forward-Backward algorithm

$$\xi_{ij}(t) = p(S_{t-1} = i, S_t = j|O,\theta)$$

$$= \frac{p(S_{t-1} = i|O,\theta)p(S_t = j, O_t, \ldots, O_T|S_{t-1} = i,\theta)}{p(O_t, \ldots, O_T|S_{t-1} = i,\theta)}$$

$$= \frac{\gamma_i(t-1)\ p_{ij}\ q_j^{O_t}\ \beta_t^j}{\beta_{t-1}^i}$$

Baum-Welch (EM) Algorithm

- Start with random initialization of parameters
- **E-step**

$$\gamma_i(t) = \underline{p(S_t = i | O, \theta)}$$

$$\xi_{ij}(t) = \underline{p(S_{t-1} = i, S_t = j | O, \theta)}$$

$\sum_{t=1}^{T} \gamma_i(t)$ = expected # times in state i

$\sum_{t=1}^{T\text{-}1} \gamma_i(t)$ = expected # transitions from state i

$\sum_{t=1}^{T-1} \xi_{ij}(t)$ = expected # transitions from state i to j

- **M-step**

$$\pi_i = \gamma_i(1)$$

$$p_{ij} = \frac{\sum_{t=1}^{T-1} \xi_{ij}(t)}{\sum_{t=1}^{T-1} \gamma_i(t)}$$

$$q_i^k = \frac{\sum_{t=1}^{T} \delta_{O_t=k} \gamma_i(t)}{\sum_{t=1}^{T} \gamma_i(t)}$$

HMMs.. What you should know

- Useful for modeling sequential data with few parameters using discrete hidden states that satisfy Markov assumption
- Representation - initial prob, transition prob, emission prob,

  State space representation
- Algorithms for inference and learning in HMMs
  - Computing marginal likelihood of the observed sequence: forward algorithm
  - Predicting a single hidden state: forward-backward
  - Predicting an entire sequence of hidden states: viterbi
  - Learning HMM parameters: an EM algorithm known as Baum-Welch