# 1 Constrained Optimization

The following example is taken from http://people.brunel.ac.uk/mastjjb/jeb/or/morelp.html.

**Example** Suppose a company makes two products X and Y using two machines (A and B). Each X that is produced requires 50 minutes processing time on machine A and 30 minutes processing time on machine B. Each Y that is produced requires 24 minutes processing time on machine A and 33 minutes processing time on machine B.

Available processing time on machine A is forecast to be 40 hours and on machine B is forecast to be 35 hours. The demand for X in the current week is forecast to be 45 units and for Y is forecast to be 5 units. Company policy is to maximise the combined sum of the number of X and Y at the end of the week. Formulate the problem of deciding how much of each product to make in the current week.

## 2 Duality

Examples in this section are borrowed from 10725 lectures by Ryan Tibshirani.

**Example 1** Let's consider a simpler optimization problem

$$\begin{aligned} \min_{x,y} \quad & x + y \\ \text{subject to} \quad & x + y \geq 2 \\ & x, y \geq 0 \end{aligned}$$

What is the lower bound?

**Example 2** What is the lower bound of the following problem?

$$\begin{aligned} \min_{x,y} \quad & x + 3y \\ \text{subject to} \quad & x + y \geq 2 \\ & x, y \geq 0 \end{aligned}$$

**Example 3** Can you solve the general form of this problem?

$$\begin{aligned} \min_{x,y} \quad & px + qy \\ \text{subject to} \quad & x + y \geq 2 \\ & x, y \geq 0 \end{aligned}$$

How can you get the exact lower bound?

We call this solution the dual form of the original problem.

**Primal**

$$\begin{aligned} \min_{x,y} \quad & px + qy \\ \text{subject to} \quad & x + y \geq 2 \\ & x, y \geq 0 \end{aligned}$$

**Dual**

$$\begin{aligned} \max_{a,b,c} \quad & 2a \\ \text{subject to} \quad & a + b = p \\ & a + c = q \\ & a, b, c \geq 0 \end{aligned}$$

**Duality for general form LP** More generally, let $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{r \times n}$, $h \in \mathbb{R}^r$.

**Primal**

$$\begin{aligned} \min_{x} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & Gx \leq h \end{aligned}$$

**Dual**

$$\begin{aligned} \max_{u,v} \quad & b^T u - h^T v \\ \text{subject to} \quad & -A^T u - G^T v = c \\ & v \geq 0 \end{aligned}$$

## 3 Lagrange Dual problem

Given a primal problem

$$\begin{aligned} \min_x \quad & f(x) \\ \text{subject to} \quad & h_i(x) \leq 0, \quad i = 1, \ldots, m \\ & l_i(x) = 0, \quad j = 1, \ldots, r \end{aligned}$$

We define the Lagrangian as

$$L(x, u, v) = f(x) + \sum_{i=1}^{m} u_i h_i(x) + \sum_{j=1}^{r} v_j l_j(x), \qquad u_i \geq 0$$

Note that for any $u_i \geq 0$ and $v_j$, $f(x) \geq L(x, u, v)$.

**Lagrange dual problem** We let $g(u, v) = \min_x L(x, u, v)$ (x is not constrained). The best lower bound for $f^*$ is obtained by maximizing $g(u, v)$ over all feasible $u$ and $v$, i.e.

$$\begin{aligned} \max_x \quad & g(u, v) \\ \text{subject to} \quad & u_i \geq 0 \end{aligned}$$

**Weak duality** For any feasible $x$ and feasible $u, v$,

$$f^* \geq g^*$$

**Strong duality** We say strong duality holds if

$$f^* = g^*$$

**Slater's condition** If the primal is a convex problem, i.e. $f$ and $h_i$ are convex, and $l_i$ are affine, and there exists at least one strictly feasible point $x$, meaning

$$h_i(x) < 0 \ \ \forall i \ \ \textbf{and} \ \ l_j(x) = 0 \ \ \forall j$$

then strong duality holds.

**Convexity** The dual problem is always convex (or concave), even though the original primal problem is non-convex.

# 4 KKT conditions

For a problem with strong duality, $x^*$ and $u^*$, $v^*$ are primal and dual solutions if and only if $x^*$ and $u^*$, $v^*$ satisfy the KKT conditions.

- $0 \in \partial(L(x, u, v))$
- $u_i \cdot h_i(x) = 0$ for all $i$
- $h_i(x) \leq 0$ and $l_j(x) = 0$ for all $i, j$
- $u_i \geq 0$ for all $i$

**Example** Use KKT conditions to solve the following constrained optimization problem.

$$\begin{aligned} \min_{a,b,c} \quad & x^2 + y^2 \\ \text{subject to} \quad & x + y - 1 = 0 \end{aligned}$$