Analytical Forward Projection for Axial Non-Central Dioptric and Catadioptric Cameras

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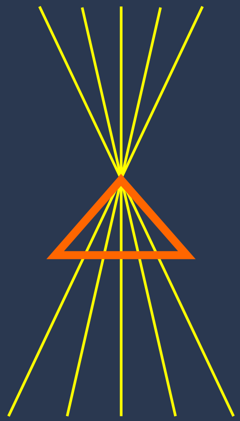
Perspective Cameras (Central)

Single Viewpoint
(Central)

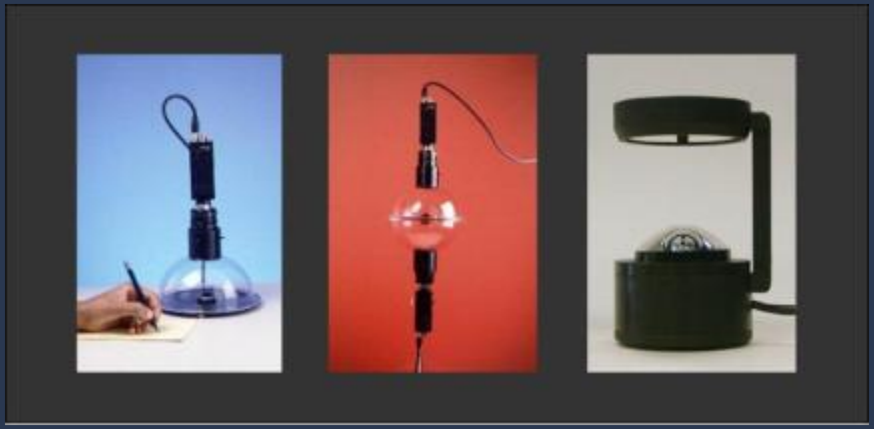


Perspective Camera

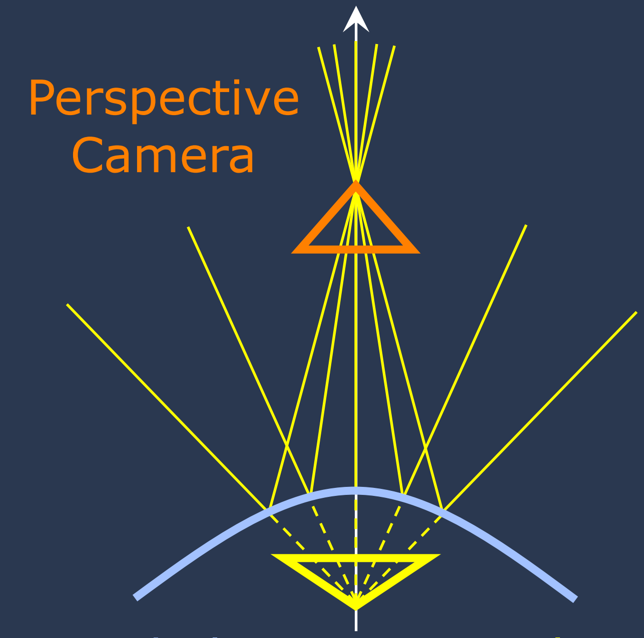
Single Viewpoint

Single-Viewpoint Catadioptric Cameras

- Mirror + Perspective Camera
- Wide Field of View

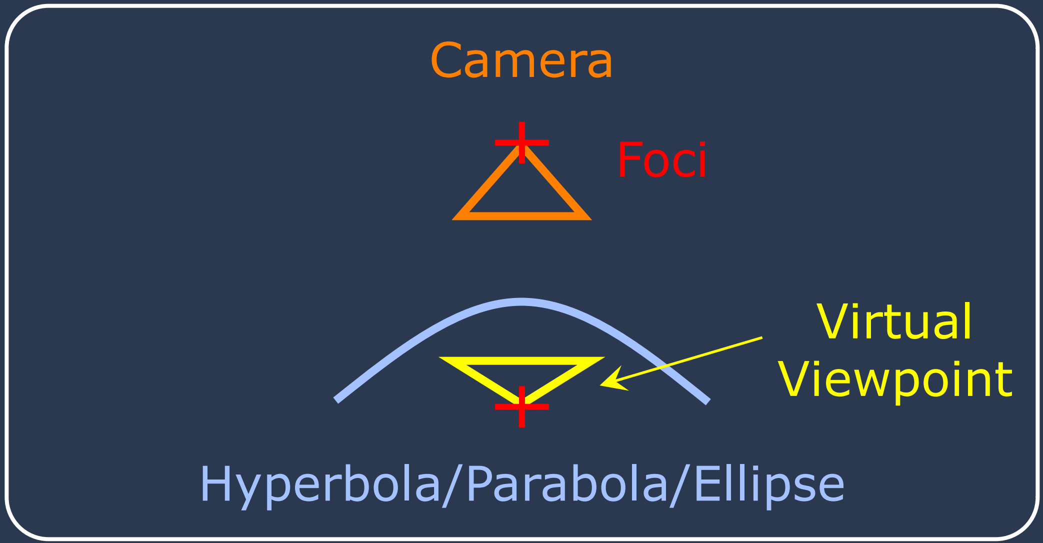


Single Viewpoint
(Central)



Single-Viewpoint Catadioptric Cameras

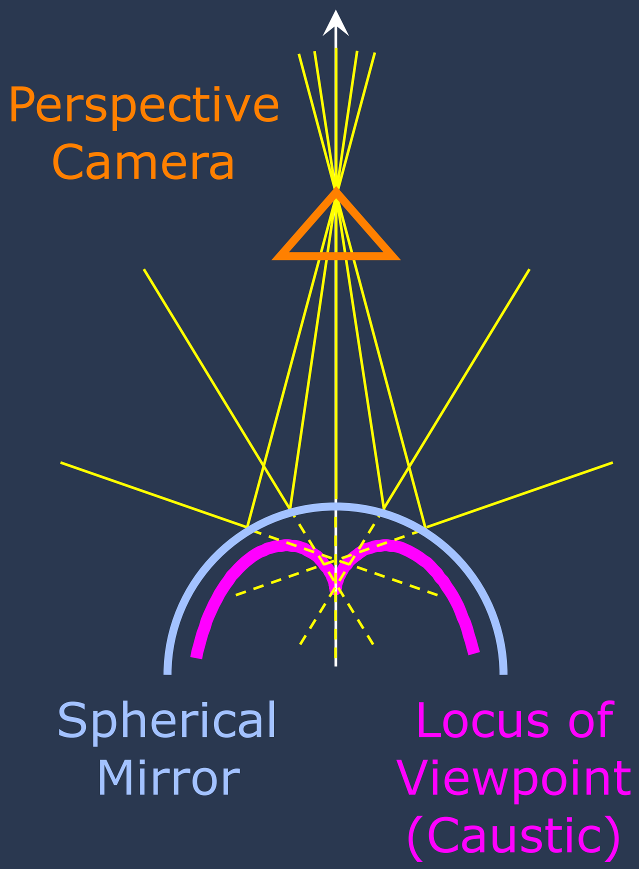
[Baker & Nayar 99]



- Only a few single-viewpoint configurations
- Other configurations lead to non-single viewpoint
  - Spherical mirror
  - Camera not on foci
  - Multiple mirrors

Non-Central Catadioptric Cameras

Camera looking into four spherical mirrors

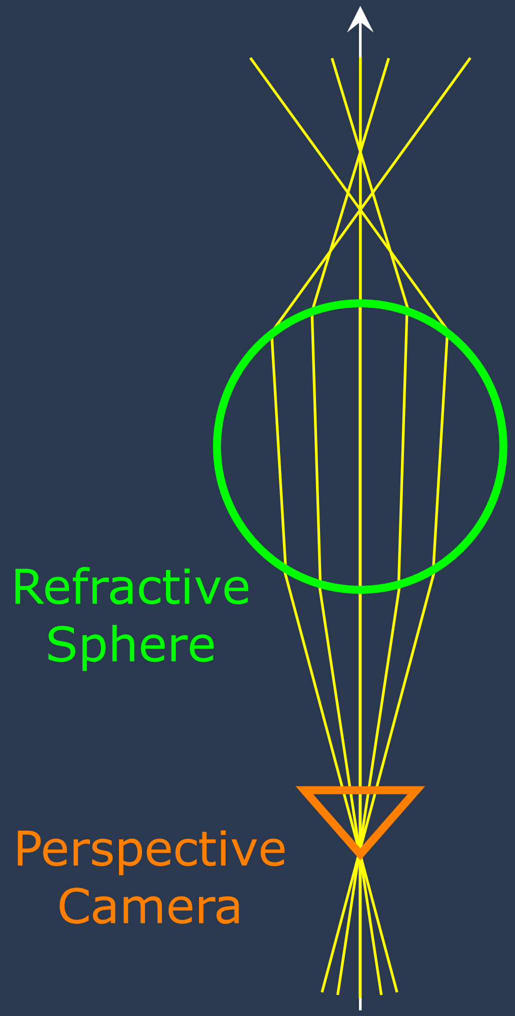


Can we analytically model the projection of 3D points to pixels?

Yuichi

Non-Central Dioptric Camera

- Looking through a refractive glass sphere
- Google "Crystal Ball Photography"



Goal

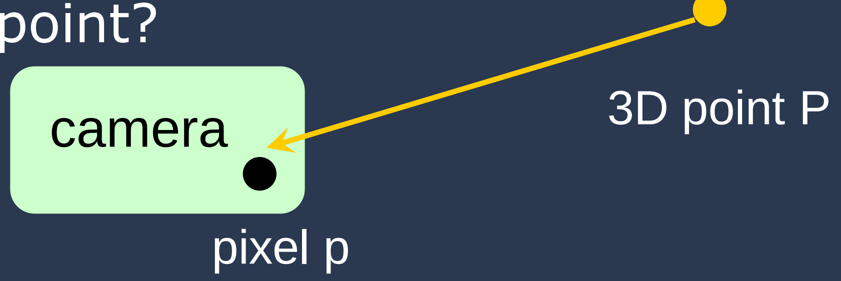
- Exact modeling of non-central cameras
  - Rotationally Symmetric Conic Mirrors & Refractive Sphere
  - Axial configuration: Camera placed on the axis
- Avoid approximations in modeling
  - Central Approximation
  - General linear cameras (GLC) approximation
    - Yu and McMillan, ECCV 2004
- Fast processing
  - Similar computational complexity as perspective camera

Why are non-central cameras difficult to model?

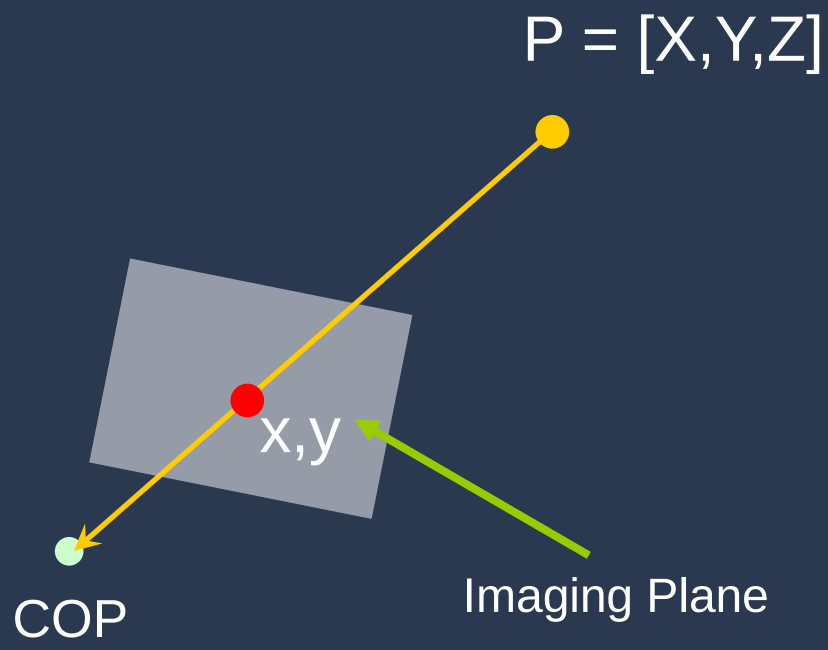
- Following two operations are essential for any camera

- Back Projection
  - What is the 3D ray corresponding to a pixel?
  - Generic Camera Calibration
    - Grossberg and Nayar, ICCV 2001
    - Sturm & Ramalingam, ECCV 2004
    - Ramalingam et al. CVPR 2005

- Forward Projection
  - What is the projection of a 3D point?
  - Inverse Ray Tracing
  - Compute the Light-Path



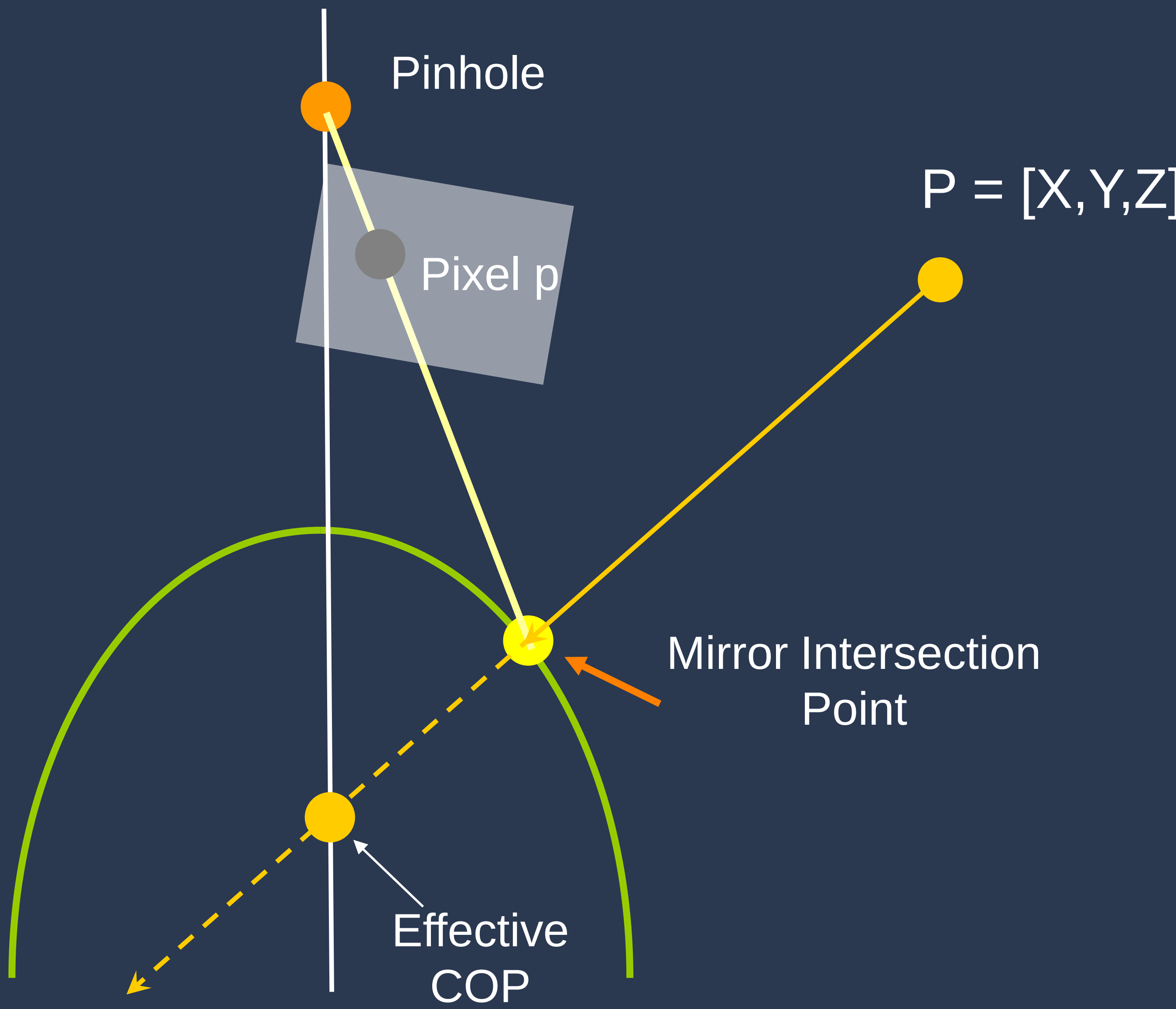
Forward Projection: Easy for Perspective Camera



x = f*X/Z

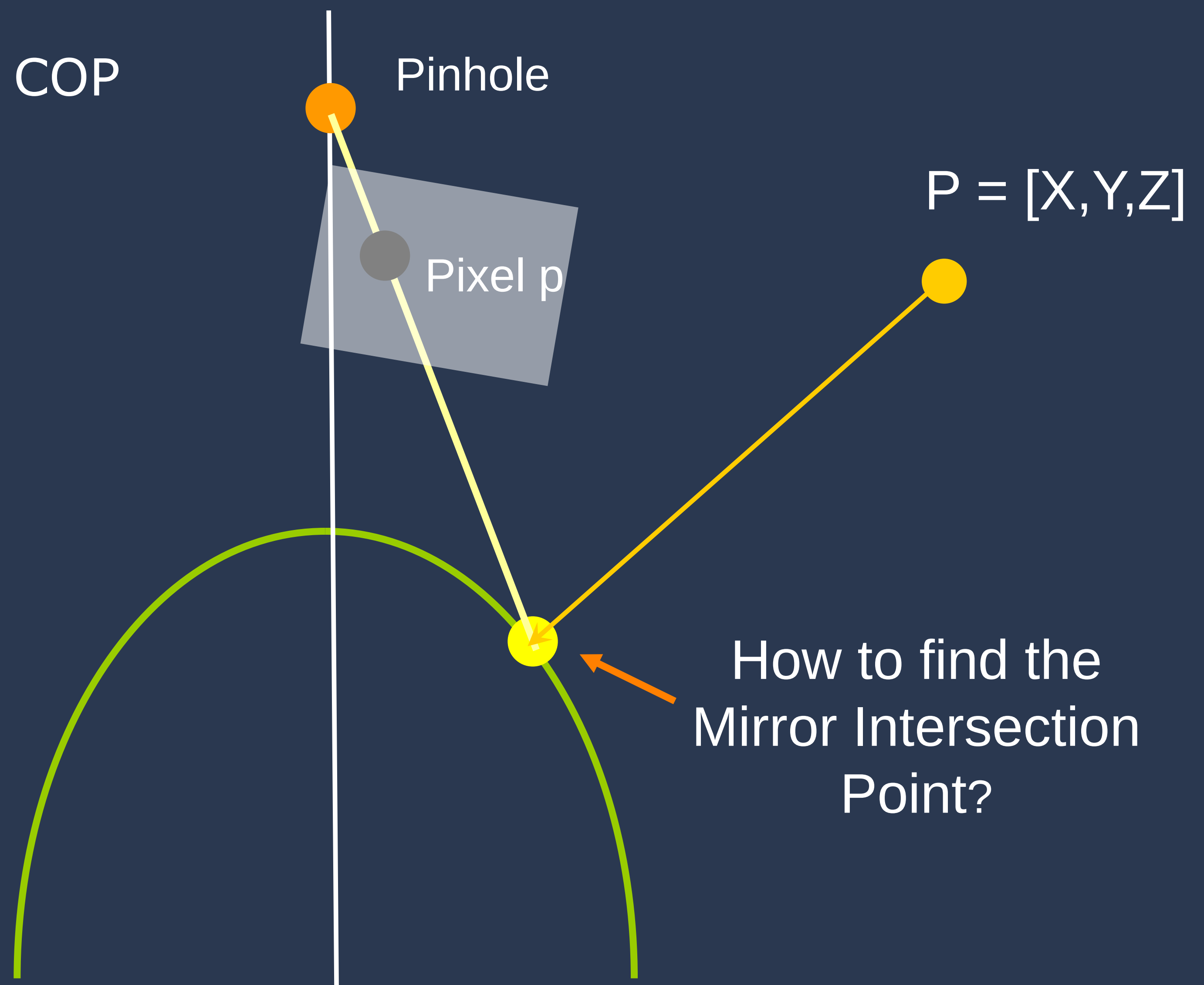
y = f*Y/Z

Easy for Central Catadioptric Camera



But, difficult for Non-Central Camera

- No effective COP



Forward Projection for Non-Central Camera

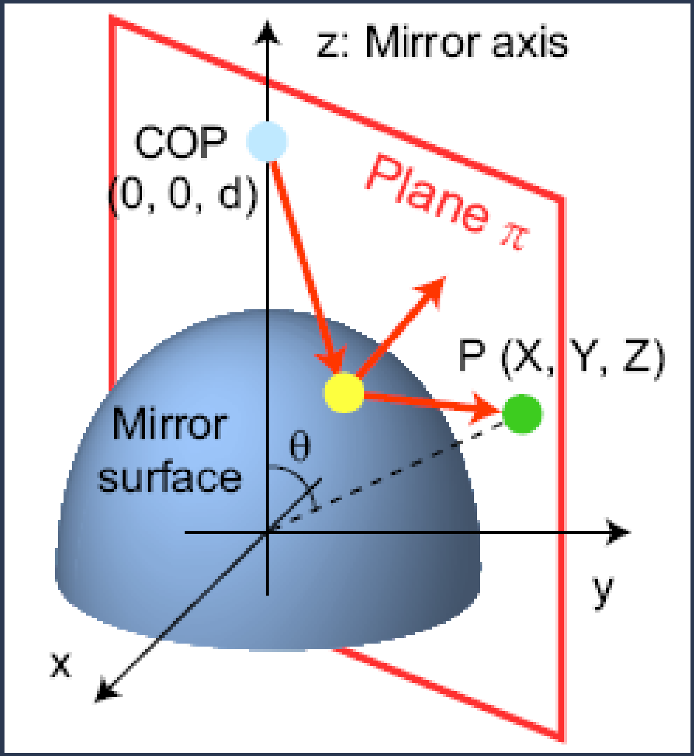
- Given a 3D point P
  - What is the corresponding image pixel p?
- Difficult, no closed form solution
- Can do optimization (Micusik and Pajdla, CVPR 2004)
  - Iterative Forward Projection
  - Slow

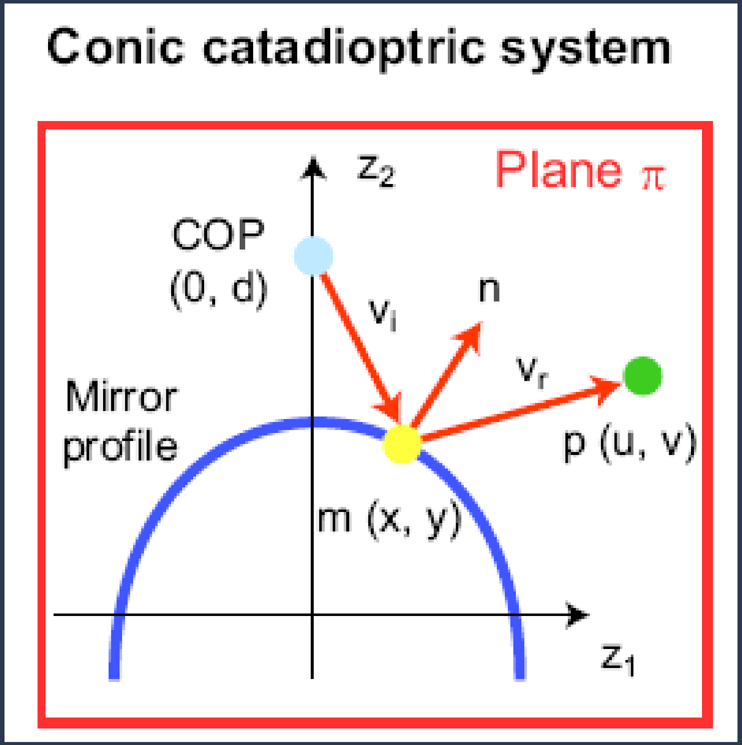
Can we obtain analytical solution for forward projection for non-central cameras?

Analytical Forward Projection for Non-Central Cameras

- Axial Configuration
  - Camera lies on the axis of rotationally symmetric mirror
- For conic mirrors
  - Solve 6^{th} degree equation in one unknown
  - Reduces to 4^{th} degree equation for spherical mirror
    - Closed Form Solution
- For refractive sphere
  - Solve 10^{th} degree equation in one unknown
- 100 times speed up
  - 3D reconstruction using bundle adjustment

Finding the Mirror Intersection Point





$$Az_1^2 + z_2^2 + Bz_1 = C$$

Mirror Equation

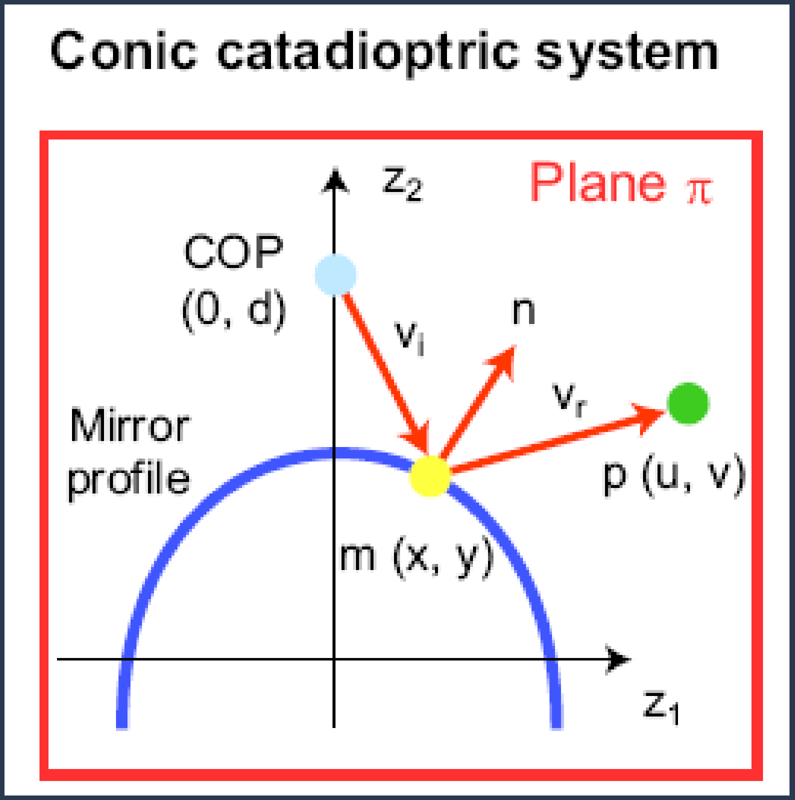
Constraints

- [x,y] lies on the mirror

$$Az_1^2 + z_2^2 + Bz_1 = C$$

Mirror Equation

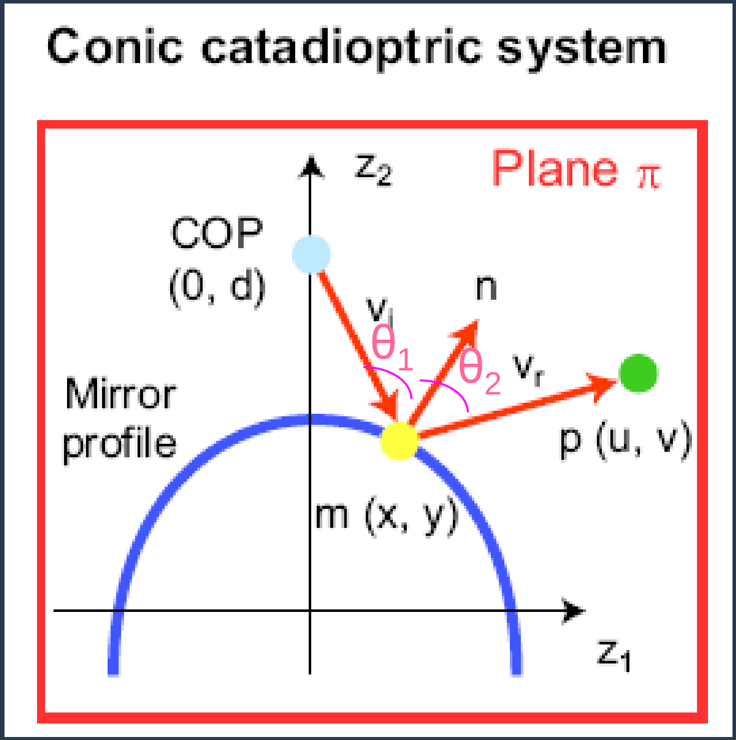
$$x = \pm\sqrt{C - By - Ay^2}$$



Constraints

- Planarity
  - Incoming ray (v_i), normal (n) and reflected ray (v_r) lie on same plane
- Angle constraint
  - $\theta_1 = \theta_2$
- Use vector form of law of reflection

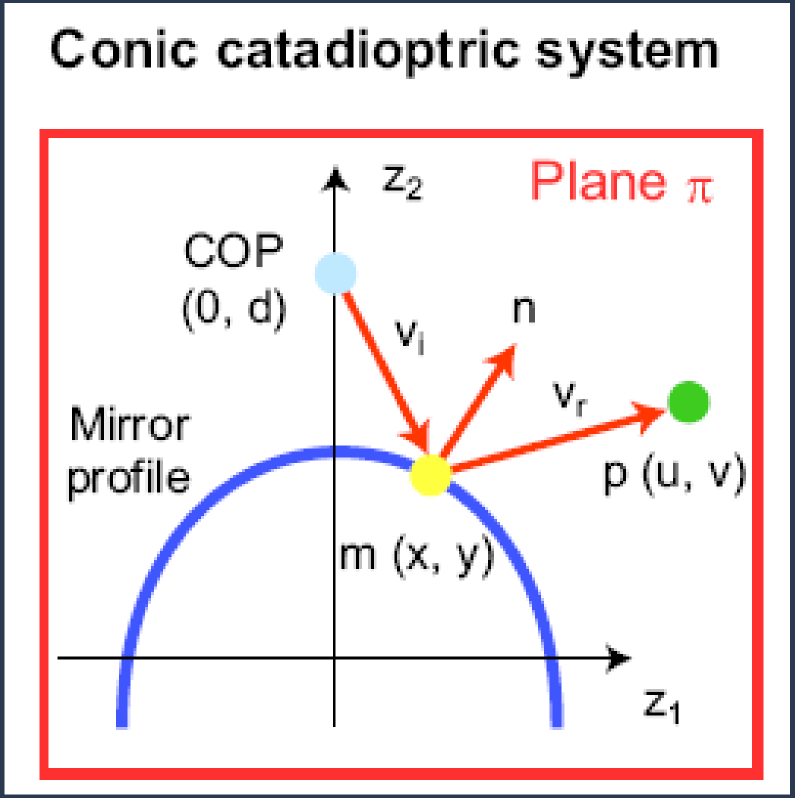
$$\mathbf{v}_r = \mathbf{v}_i - 2\mathbf{n}(\mathbf{n}^T\mathbf{v}_i)/(\mathbf{n}^T\mathbf{n}).$$



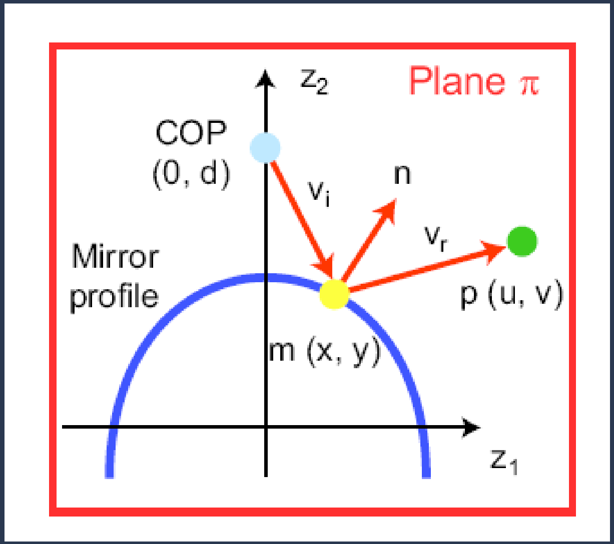
Constraint

- The reflected ray should pass through the given point

- Cross(v_r , p – m) = 0



Solution



$$Az_1^2 + z_2^2 + Bz_1 = C$$

Mirror Equation

$$x = \pm\sqrt{C - By - Ay^2}$$

$$\mathbf{n} = \begin{bmatrix} x \\ B/2 + Ay \end{bmatrix}, \quad \mathbf{v}_i = \begin{bmatrix} x \\ y - d \end{bmatrix}$$

$$\mathbf{v}_r = \mathbf{v}_i - 2\mathbf{n}(\mathbf{n}^T\mathbf{v}_i)/(\mathbf{n}^T\mathbf{n}).$$

$$\mathbf{v}_r \times (\mathbf{p} - \mathbf{m}) = 0$$

Solution

$$u^2 K_1^2(y) + K_2^2(y)(Ay^2 + By - C) = 0,$$

$$K_1(y) = K_{11}y^3 + K_{12}y^2 + K_{13}y + K_{14}$$

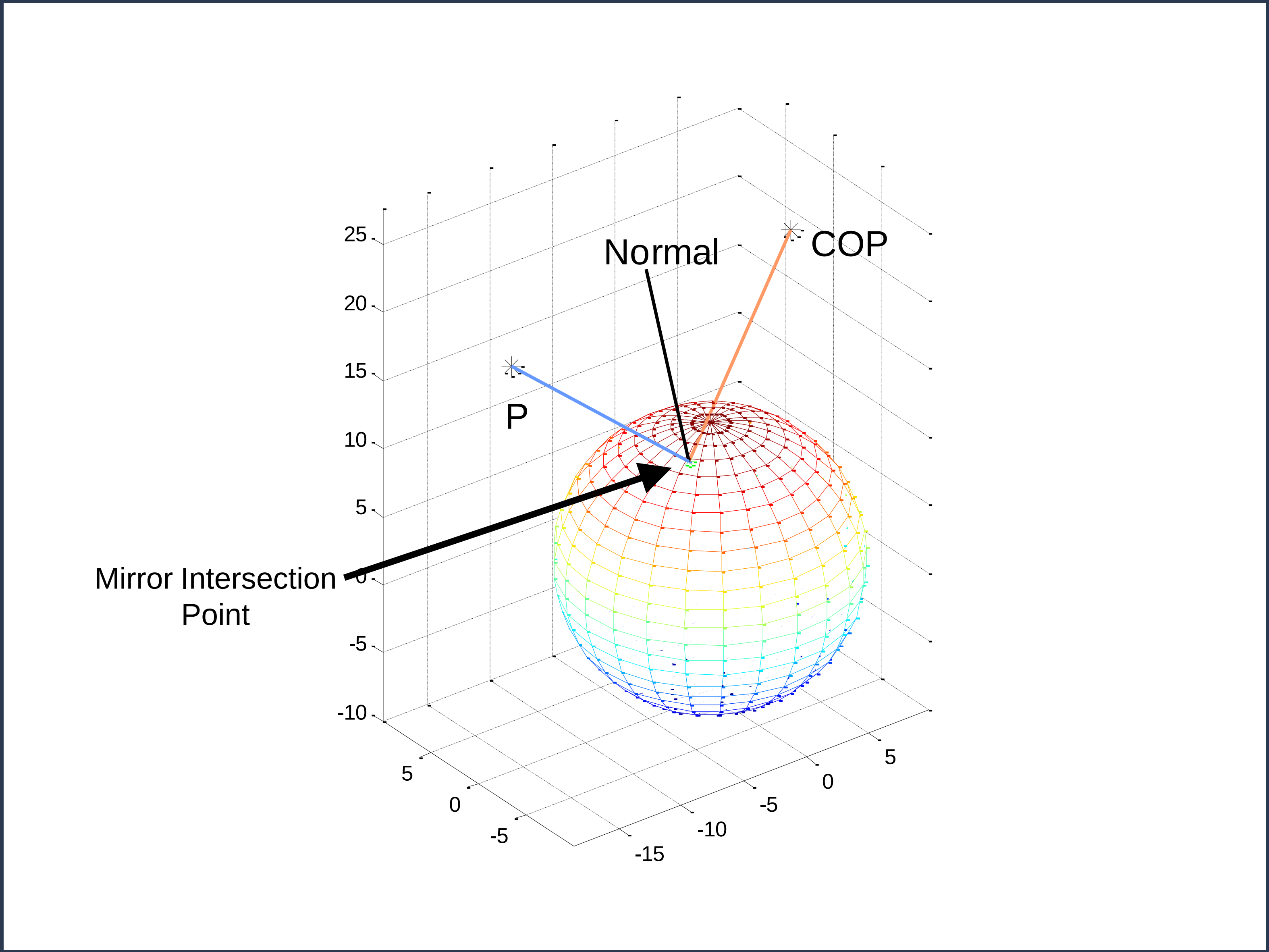
$$K_2(y) = K_{21}y^2 + K_{22}y + K_{23}$$

- 6[th] degree equation in y
  - 6 solutions
  - Get the correct solution by checking law of reflection
- Obtain x using $x = \pm\sqrt{C - By - Ay^2}$
- 20 lines of Matlab Code

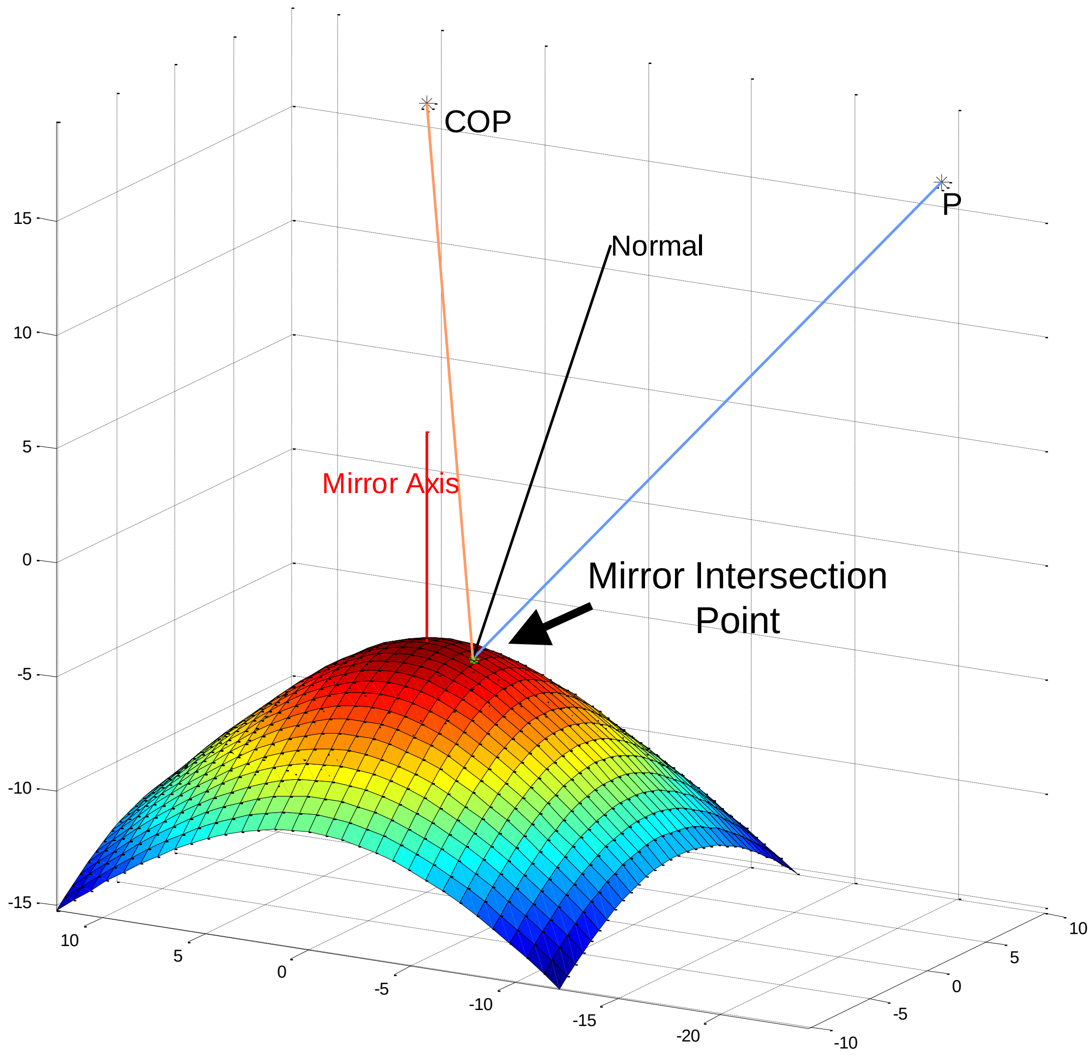
Mirror Shape $$Az_1^2 + z_2^2 + Bz_1 = C$$

Mirror Shape	Pinhole Placement	Parameters	Central	System Degree
General	On axis	A,B,C	No	6
Sphere	Any	$A = 1, B = 0, C > 0$	No	4
Elliptic	On axis, At Foci	$B = 0$	Yes	2
Elliptic	On axis, Not at Foci	$B = 0$	No	6
Hyperbolic	On axis, At Foci	$A < 0, C < 0$	Yes	2
Hyperbolic	On axis, Not at Foci	$A < 0, C < 0$	No	6
Parabolic	On axis, $d = \infty$	$A = 0, C = 0$	Yes	2
Parabolic	On axis, Finite d	$A = 0, C = 0$	No	5

Degree of Forward Projection Equation for Various Configurations



Visualization for Spherical Mirror



Visualization for Hyperbolic Mirror

Forward Projection for Spherical Mirror

- Also known as Alhazen problem or Circular Billiard Problem
  - One of the classical problem in geometry
  - Can be traced back to Ptolemy *Optics* (AD 150)
  - Described in *Book of Optics* (~1000 A.D) by Alhazen
- References
  - Marcus Baker, “Alhazen problem”, American Journal of Mathematics, Vol. 4, No. 1, 1881, pp. 327-331
  - *Heinrich Dorrie, 100 Great Problems of Elementary Mathematics*
- Four solutions
  - Intersection of circle and hyperbola
- Forward projection for general mirrors
  - Extension of Alhazen problem

Related Work

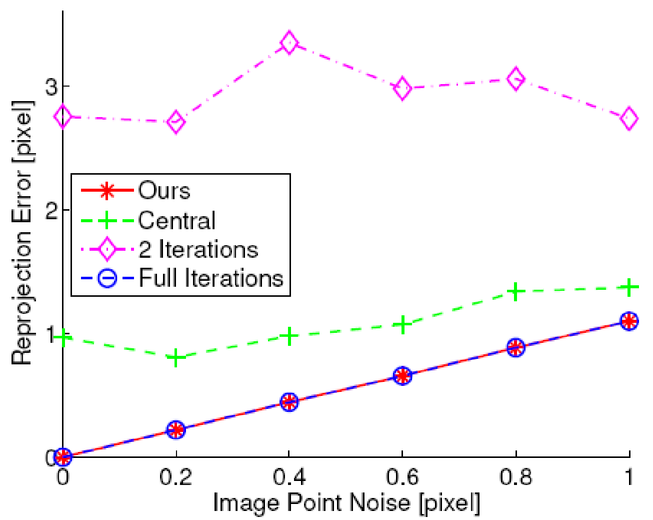
- Optimization for Forward Projection
  - Micusik and Pajdla, CVPR 2004
  - Iterative Forward Projection (IFP)
- Local perturbation method
  - Chen and Arvo, SIGGRAPH 2000
  - Approximation for fast rendering
- GLC Approximation
  - Ding et al. ICCV 2009
- Bertrand Vandeportaele Thesis, 2006 (in French)
  - Analysis in 3D
  - Higher degree (8^{th} degree) equation for ellipsoid

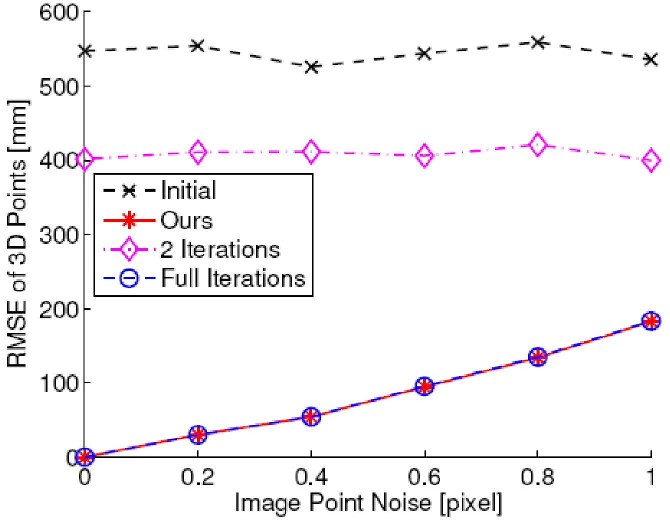
Results: Fast Projection of 3D Points

- Project randomly generated 100,000 points (spherical mirror)
- Matlab on standard PC
- 1120 seconds for IFP
- 13.8 seconds for AFP
- Speed up ~80
- Similar speed up for hyperbolic, elliptical and parabolic mirrors

Sparse 3D Reconstruction: Simulation

- Single perspective camera looking at 4 spherical mirrors
- Perturb sphere centers and add image noise
- Bundle Adjustment of sphere centers and 3D points

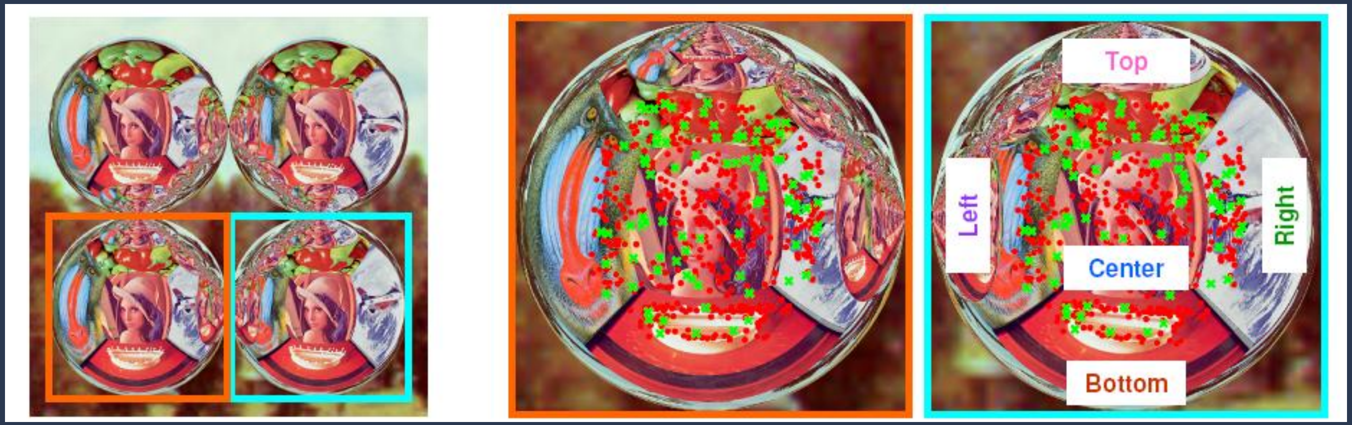


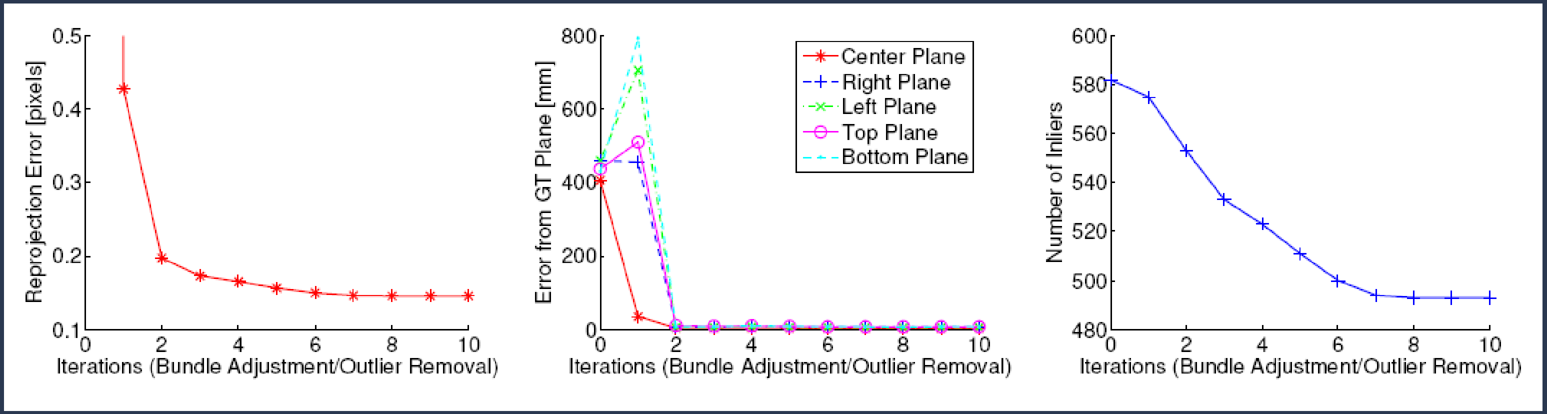


Run Time	Iterative FP	AFP (Without Jacobian)	AFP (With Jacobian)
$N = 100$	470	6.6	4.0
$N = 1000$	4200	68	48

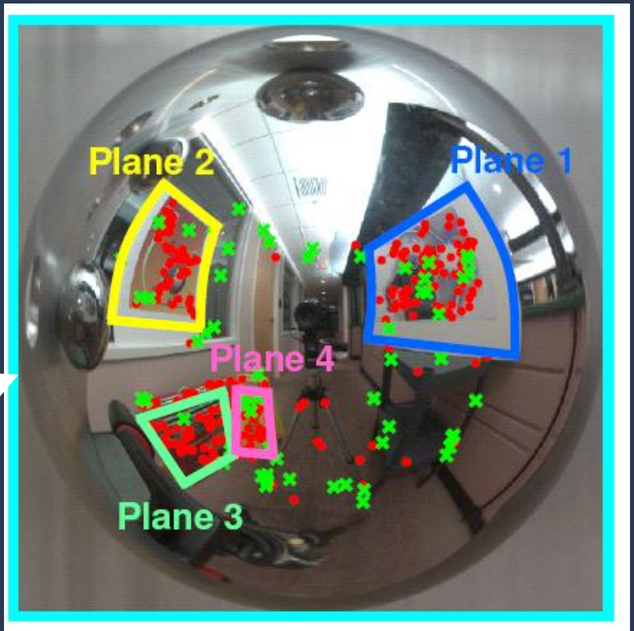
Sparse 3D Reconstruction: PovRay Simulation

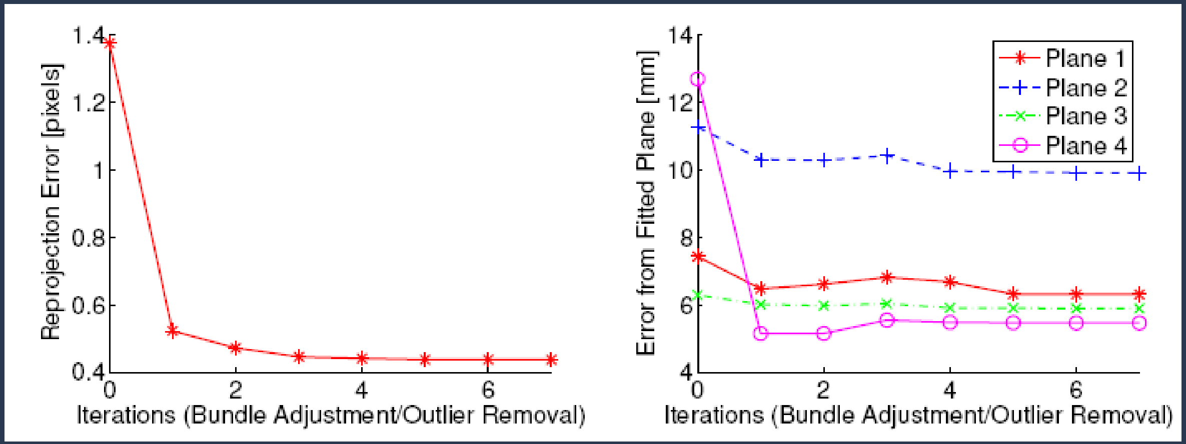
- Feature extraction using SIFT
- Iterate bundle adjustment and outlier removal





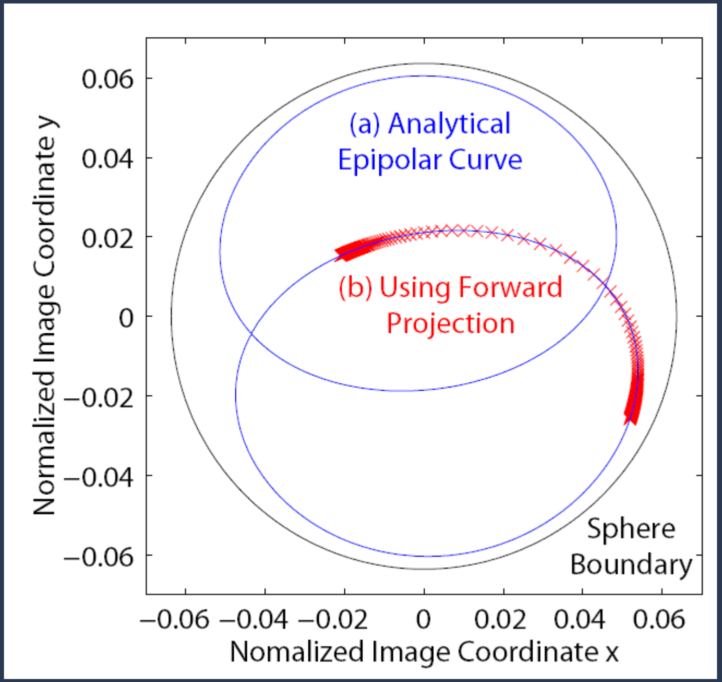
Sparse 3D Reconstruction: Real Data





Epipolar Curves (Spherical Mirror)

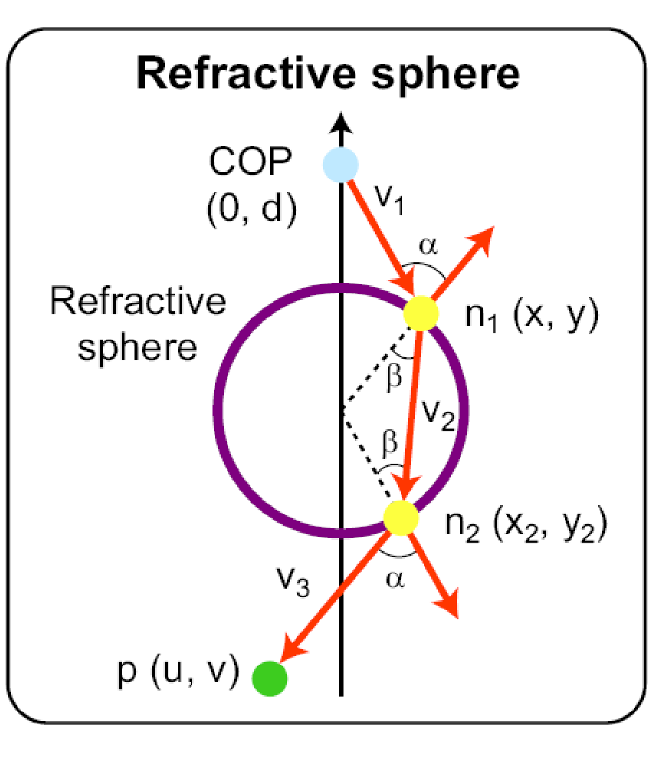
- Perspective Cameras
  - Epipolar geometry
  - Epipolar *lines*
- Central Catadioptric Cameras
  - Epipolar curves (2^{nd} order)
  - Svoboda & Pajdla, IJCV 2002



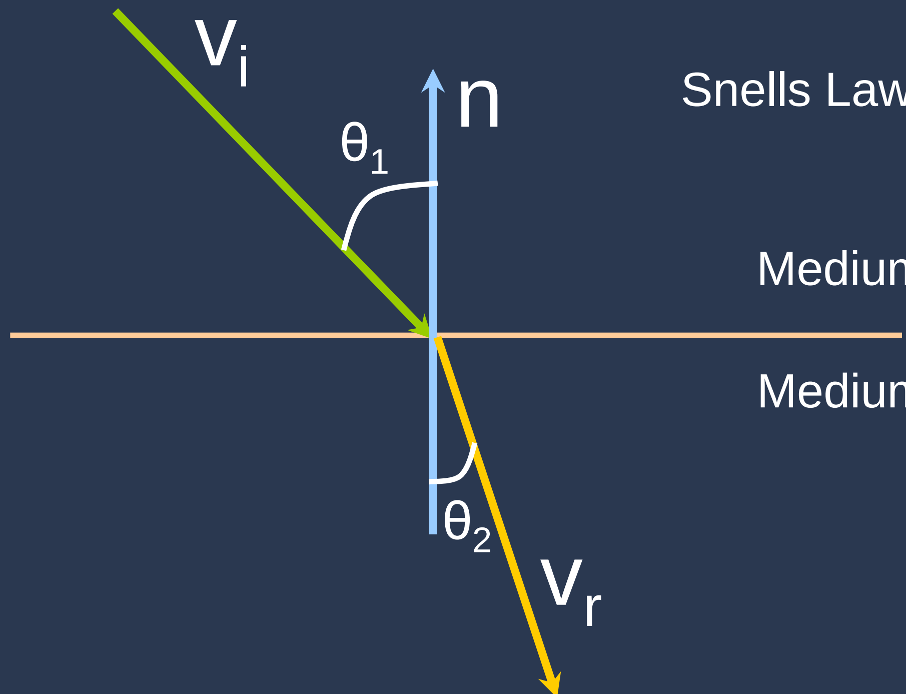
- Non-central Catadioptric Camera (Spherical Mirror)
  - Projection of a line == 4^{th} order curve (Quartic curves)
  - Derivation similar to Strum & Barreto, ECCV 2008
  - Back-projection matrix using lifted image coordinates

Forward Projection for Refractive Sphere

- Refractive index μ
- Key idea 1: Analysis can be done in a plane



Key idea 2



Snells Law: $\mu_1 \sin(\theta_1) = \mu_2 \sin(\theta_2)$

- Use vector form of the refraction equation $\mathbf{v}_r = a\mathbf{v}_i + b\mathbf{n}$

$$a = \frac{\mu_1}{\mu_2}, \quad b = \frac{-\mu_1 \mathbf{v}_i^T \mathbf{n} \pm \sqrt{\mu_1^2 (\mathbf{v}_i^T \mathbf{n})^2 - (\mu_1^2 - \mu_2^2)(\mathbf{v}_i^T \mathbf{v}_i)(\mathbf{n}^T \mathbf{n})}}{\mu_2 (\mathbf{n}^T \mathbf{n})}$$

Solution

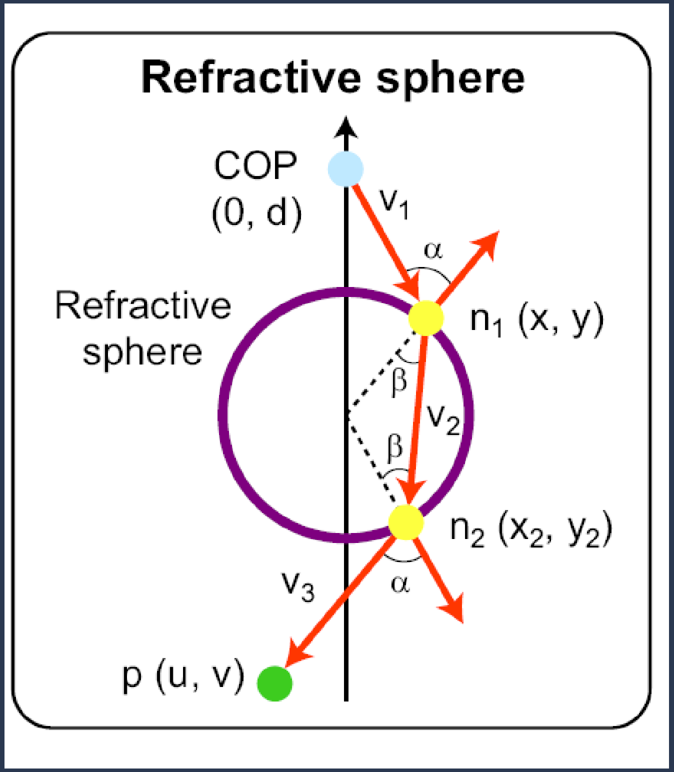
$$\mathbf{v}_1 = [x, y-d]^T$$

$$\mathbf{n}_1 = [x, y]^T$$

$$\mathbf{v}_2 = \frac{1}{\mu}\mathbf{v}_1 + \mathbf{n}_1 \frac{-\mathbf{v}_1^T\mathbf{n}_1 - \sqrt{(\mathbf{v}_1^T\mathbf{n}_1)^2 - r^2(1-\mu^2)(\mathbf{v}_1^T\mathbf{v}_1)}}{\mu r^2}$$

$$\mathbf{v}_3 = \mu \mathbf{v}_2 + b_3 \mathbf{n}_2$$

$$\mathbf{v}_3 \times (\mathbf{p} - \mathbf{n}_2) = 0$$



Solution

$$0 = K_1(x,y) + K_2(x,y)\sqrt{A} + K_3(x,y)A^{3/2}$$

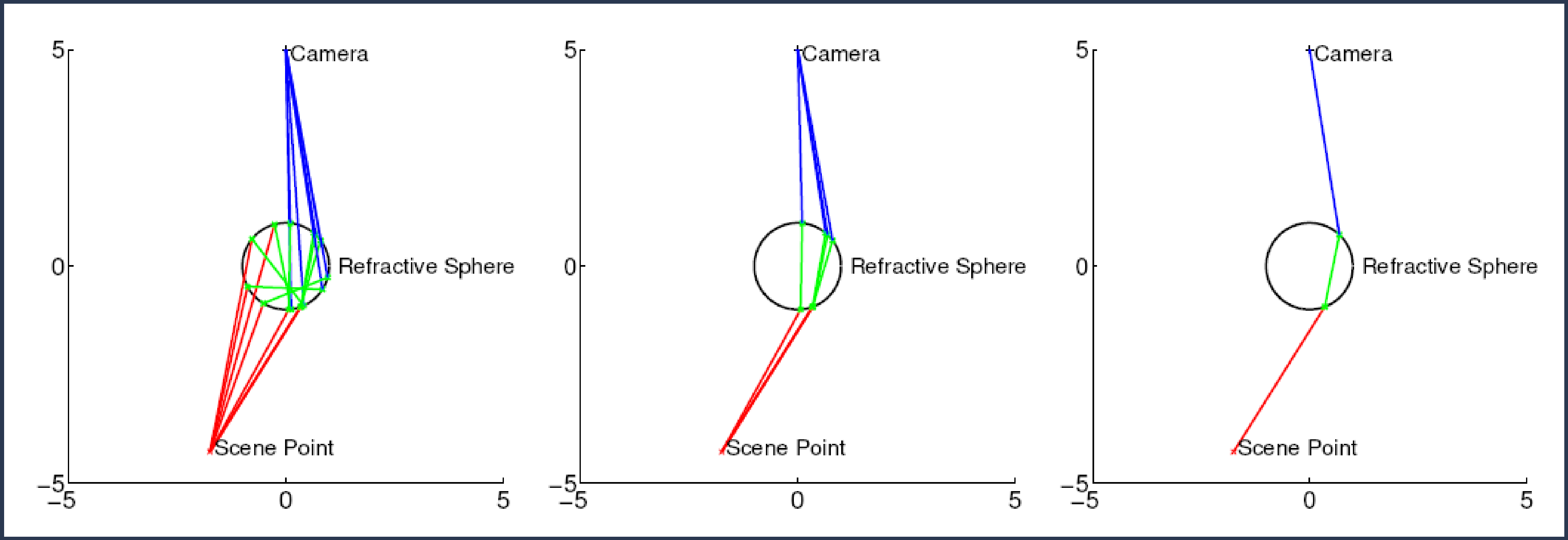
$$A = d^2\mu^2r^2 - d^2x^2 - 2d\mu^2r^2y + \mu^2r^4$$

$$x^2 = r^2 - y^2$$

10th degree equation in y

Visualizing Solutions

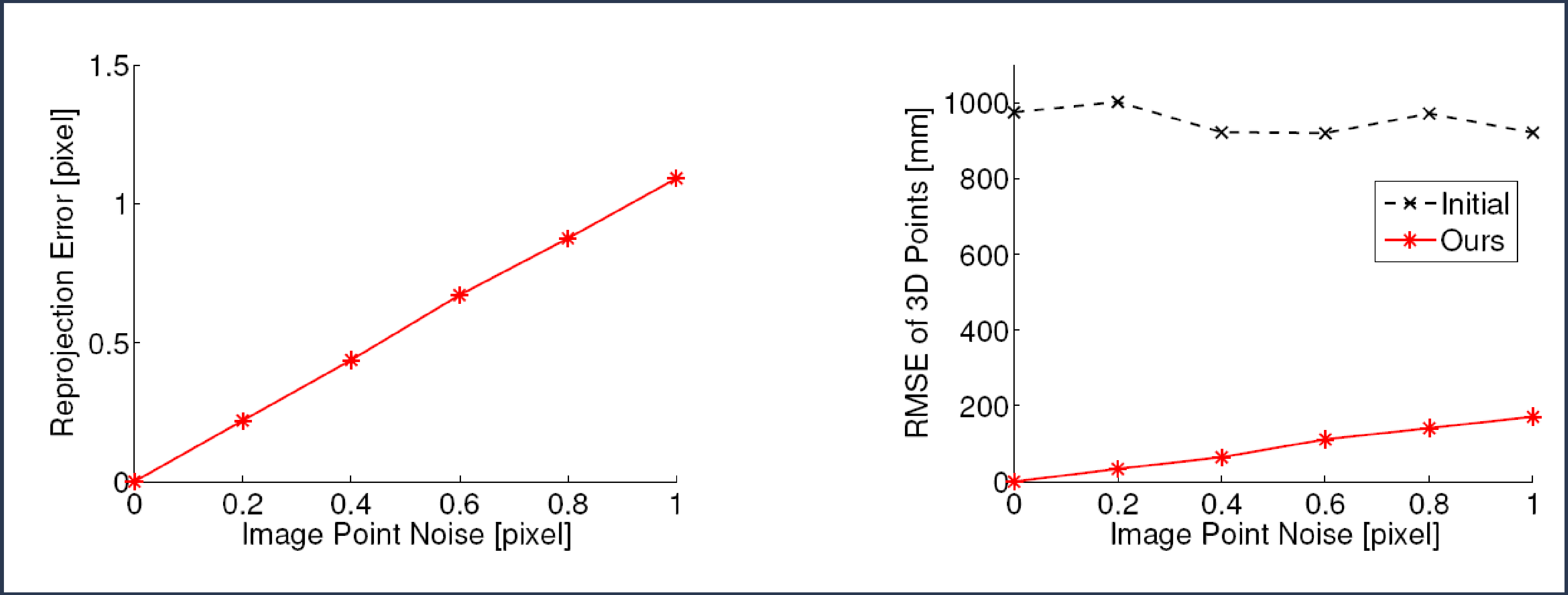
- Radius r=1
- Refractive Index μ = 1.5
- D = 5 (distance of camera from refractive sphere)



8 real solutions | Removing invalid refraction points | Checking Snell's Law

3D Reconstruction using Refractive Sphere

- Single camera looking through 4 refractive spheres
- Perturb sphere centers and add image noise
- Jointly optimize sphere centers and 3D points



Summary

- Analytical Forward Projection
  - Axial Non-Central Catadioptric cameras
    - Hyperbolic, Elliptical, Spherical and Parabolic Mirrors
  - Refractive Sphere
    - Poor man's fish-eye lens
- Avoid central and GLC approximation
  - Can use exact non-central model
- 100 times speed up over iterative approach
  - Sparse 3D reconstruction using bundle adjustment
- Epipolar curves
  - Quartic curves for Spherical Mirror

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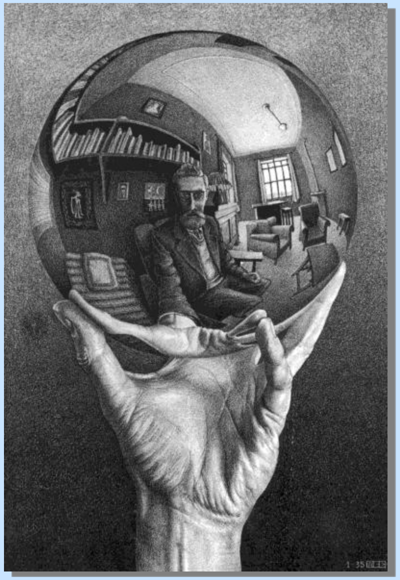
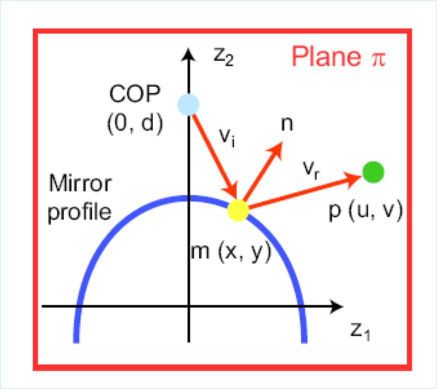
Analytical Forward Projection for Non-Central Cameras

Axial Catadioptric Camera

Solve 6th degree Equation





Refractive Sphere

Solve 10th degree Equation