Brightness Constancy

16-385 Computer Vision (Kris Kitani)

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Optical Flow

Problem Definition

Given two consecutive image frames, estimate the motion of each pixel

Assumptions

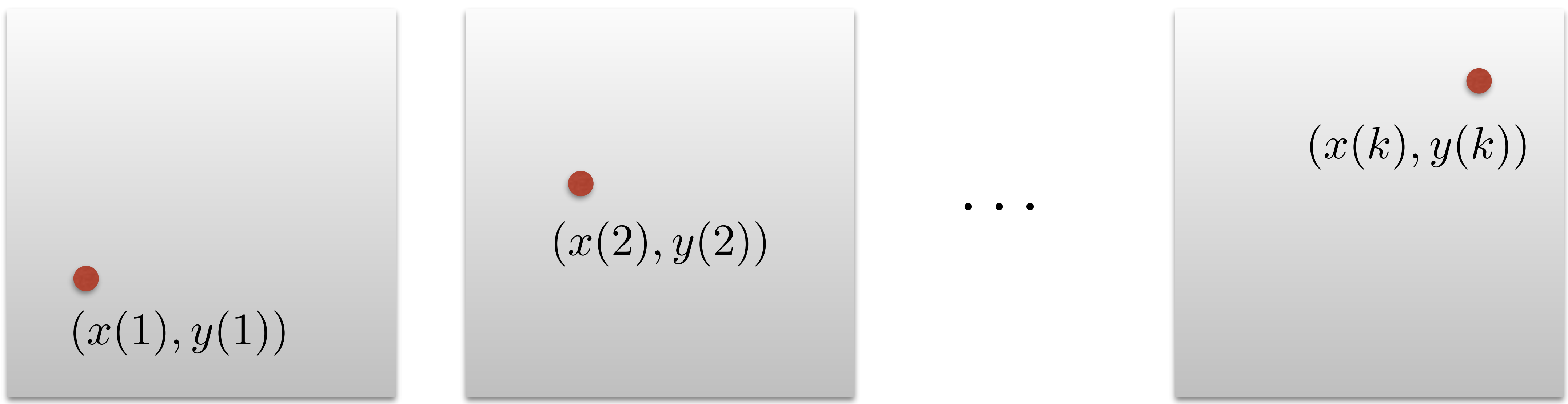
Brightness constancy

Small motion

Assumption 1

Brightness constancy

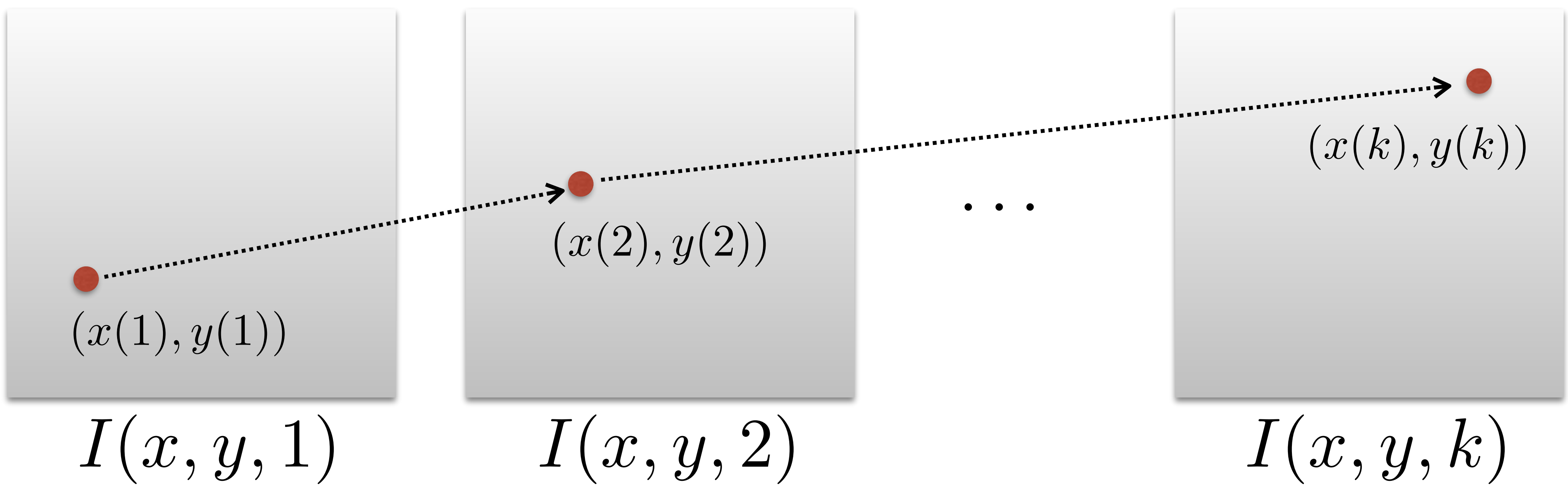
Scene point moving through image sequence



Assumption 1

Brightness constancy

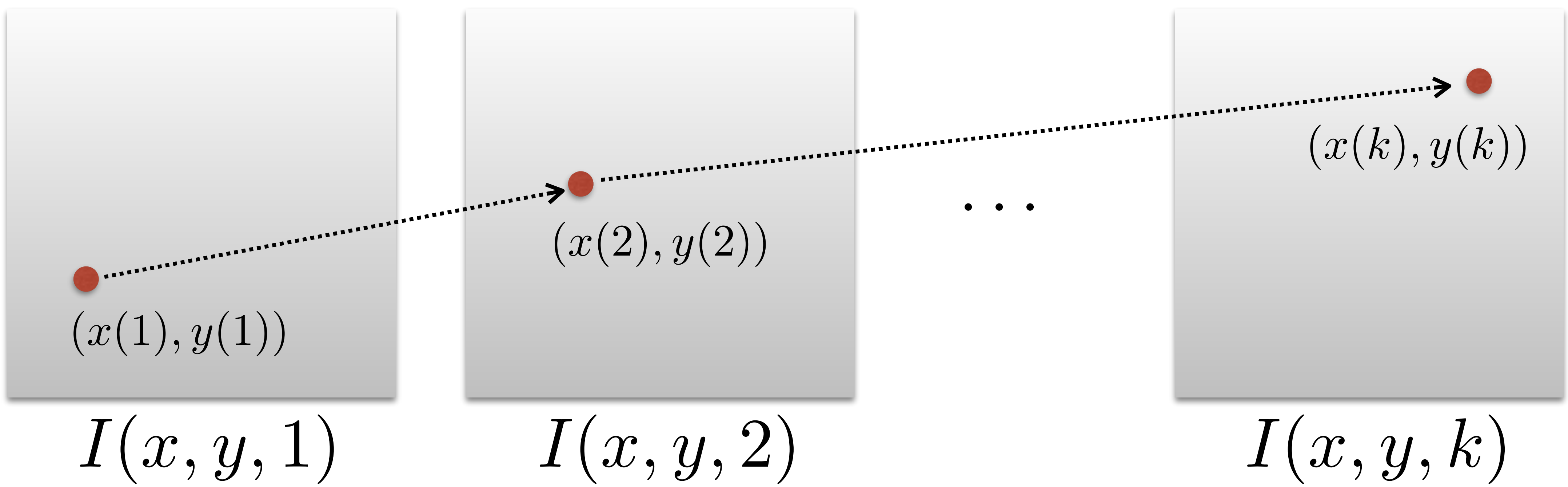
Scene point moving through image sequence



Assumption 1

Brightness constancy

Scene point moving through image sequence

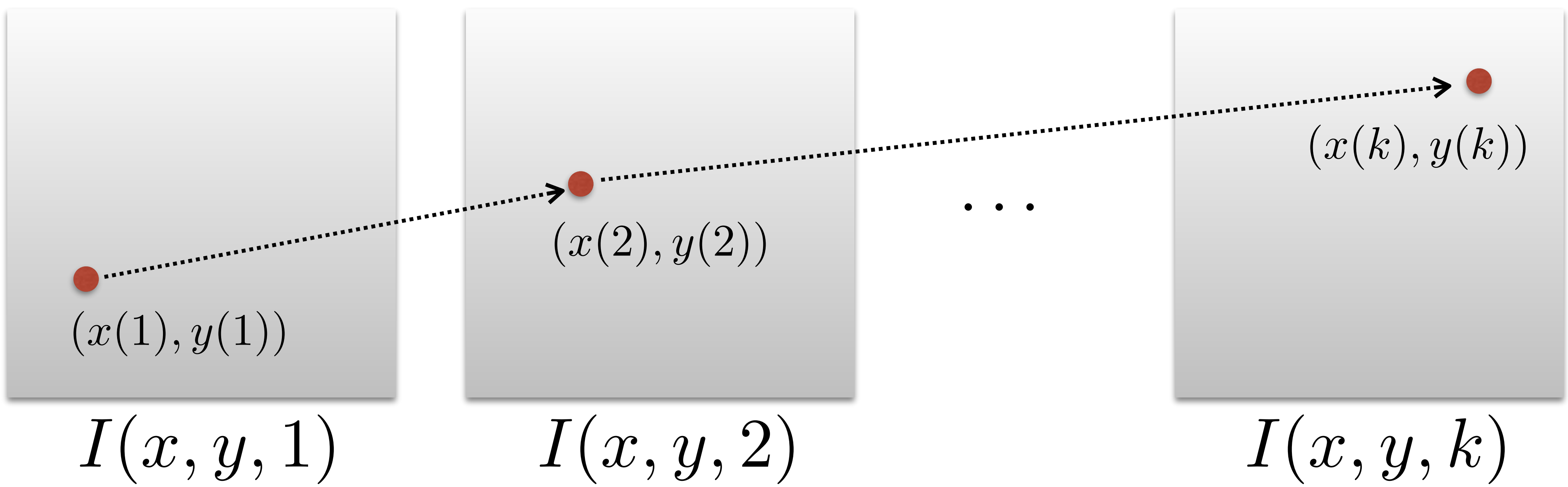


Assumption:Brightness of the point will remain the same

Assumption 1

Brightness constancy

Scene point moving through image sequence



Assumption:Brightness of the point will remain the same

$$I(x(t), y(t), t) = C$$

constant

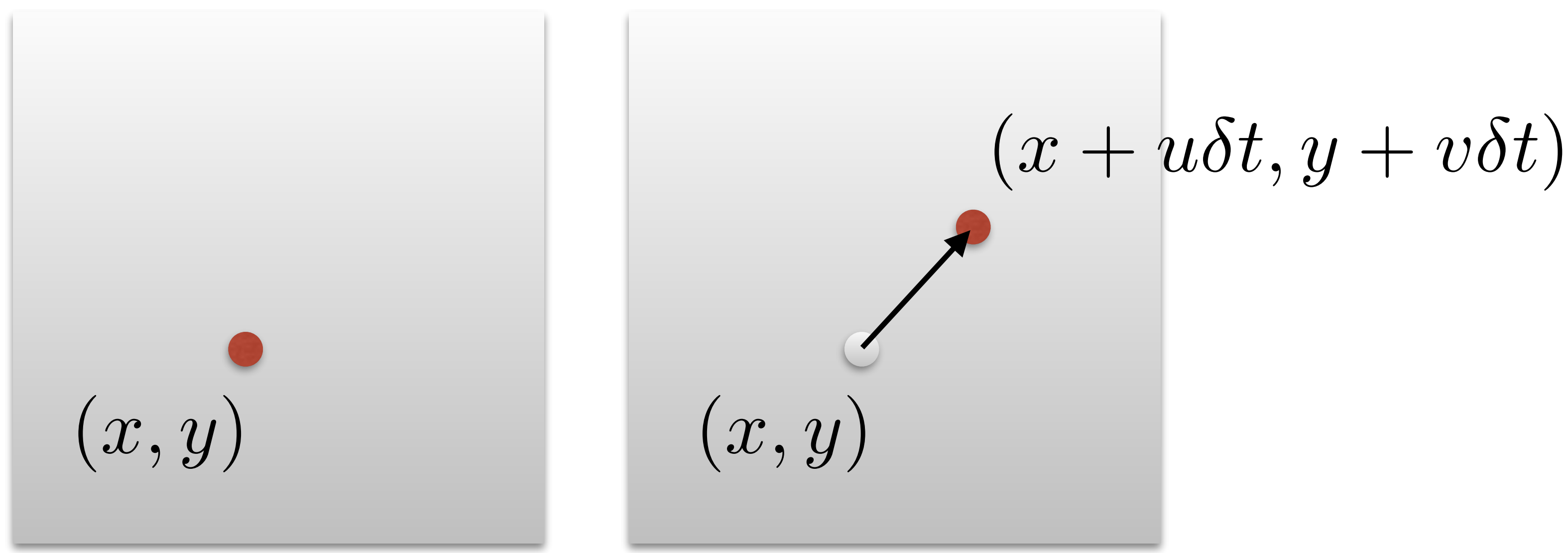
Assumption 2

Small motion

$I(x, y, t)$ $\quad$ $I(x, y, t + \delta t)$

Assumption 2

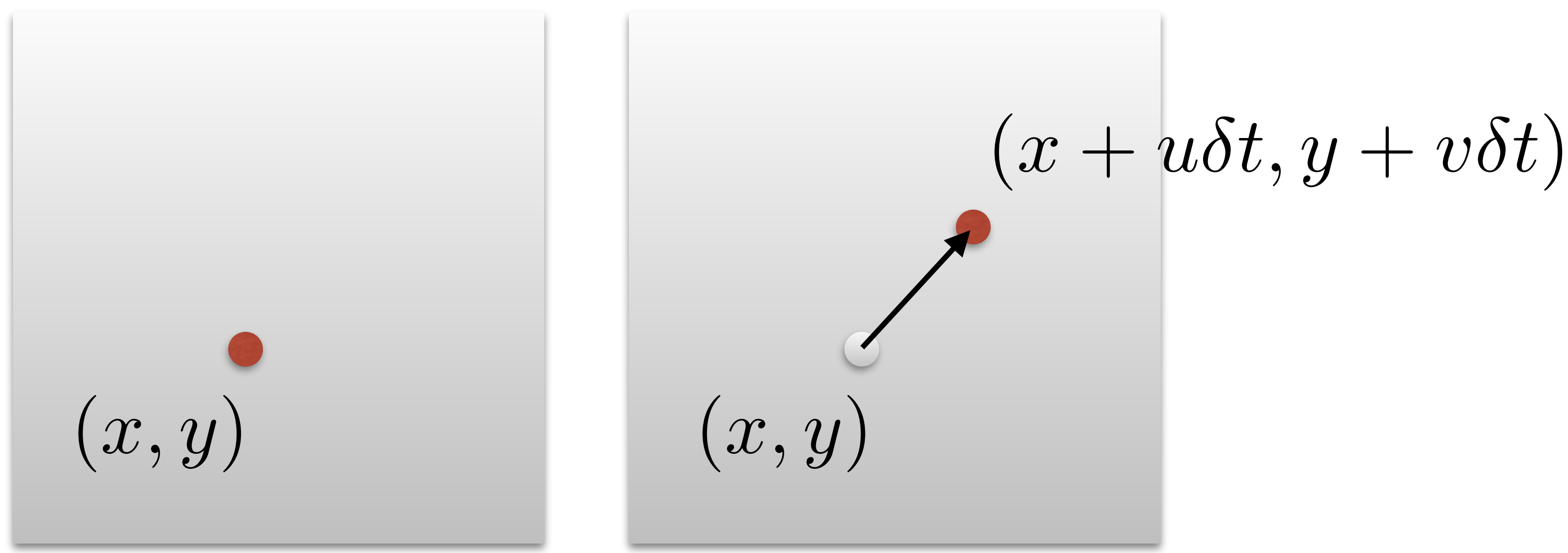
Small motion



$I(x, y, t)$ $\quad$ $I(x, y, t + \delta t)$

Assumption 2

Small motion



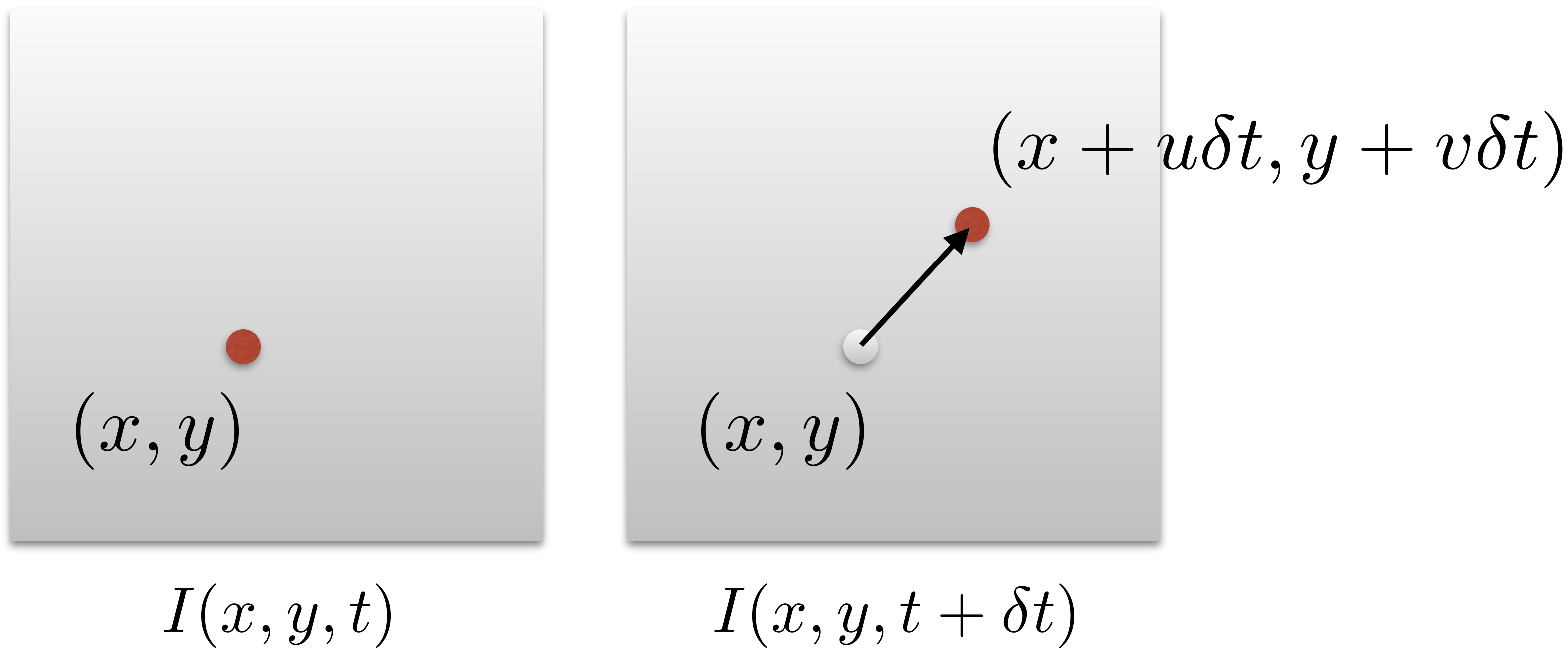
$I(x, y, t)$ $\quad$ $I(x, y, t + \delta t)$

Optical flow (velocities): (u, v)

Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

Assumption 2

Small motion



Optical flow (velocities): (u, v) Displacement: $(\delta x, \delta y) = (u\delta t, v\delta t)$

For a <u>*really small space-time step*</u>…

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

… the brightness between two consecutive image frames is the same

These assumptions yield the …

Brightness Constancy Equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

total derivative

partial derivative

Equality is not obvious. Where does this come from?

These assumptions yield the …

Brightness Constancy Equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

total derivative

partial derivative

Where does this come from?

proof!

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

Insight:

If the time step is really small,

we can *linearize* the intensity function

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x+u\delta t, y+v\delta t, t+\delta t) = I(x,y,t)$$

Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) - f_y(a,b)(y-b)$$

$$I(x,y,t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x,y,t)$$

assuming small motion

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Multivariable Taylor Series Expansion
(First order approximation, two variables)

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partial derivative

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

fixed point

assuming small motion

cancel terms

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) - f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = I(x, y, t)$$

assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

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assuming small motion

$$\frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t = 0$$

divide by δt

take limit $\delta t \rightarrow 0$

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assuming small motion

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Brightness Constancy Equation

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \frac{\partial I}{\partial y}\frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

Brightness Constancy Equation

$$I_x u + I_y v + I_t = 0$$

(x-flow) (y-flow)

shorthand notation

$$\nabla I^\top \boldsymbol{v} + I_t = 0$$

(1 x 2) (2 x 1)

vector form

(putting the math aside for a second…)

What do the term of the brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

(putting the math aside for a second…)

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brightness constancy equation represent?

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Image gradients
(at a point p)

(putting the math aside for a second…)

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flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

(putting the math aside for a second…)

What do the term of the
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients
(at a point p)

temporal gradient

How do you compute these terms?

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

Forward difference
Sobel filter
Scharr filter
…

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

Forward difference
Sobel filter
Scharr filter
…

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

frame differencing

Frame differencing

t

1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10

-

$t+1$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	10	10	10
1	1	10	10	10
1	1	10	10	10

=

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	9	9	9	9
0	9	0	0	0
0	9	0	0	0
0	9	0	0	0

(example of a forward difference)

Example:

t

3 x 3 patch

1	1	1	1	1
1	1	1	1	1
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**
1	**10**	**10**	**10**	**10**

$t+1$

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	**10**	**10**	**10**
1	1	**10**	**10**	**10**
1	1	**10**	**10**	**10**

$$I_x = \frac{\partial I}{\partial x}$$

-	0	0	0	-
-	0	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-
-	**9**	0	0	-

-1 0 1

$$I_y = \frac{\partial I}{\partial y}$$

-	-	-	-	-
0	0	0	0	0
0	**9**	**9**	**9**	**9**
-	0	0	0	0
0	0	0	0	0
-	-	-	-	-

-1
0
1

$$I_t = \frac{\partial I}{\partial t}$$

0	0	0	0	0
0	0	0	0	0
0	**9**	**9**	**9**	**9**
0	**9**	0	0	0
0	**9**	0	0	0
0	**9**	0	0	0

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

How do you compute this?

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute …

spatial derivative	optical flow	temporal derivative
$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$	$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$	$I_t = \frac{\partial I}{\partial t}$
Forward difference Sobel filter Scharr filter …	**We need to solve for this!** (this is the unknown in the optical flow problem)	frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute …

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

Forward difference
Sobel filter
Scharr filter
…

(u, v)
Solution lies on a line

Cannot be found uniquely
with a single constraint

frame differencing

unknown

$$I_x u + I_y v + I_t = 0$$

known

We need at least ____ equations to solve for 2 unknowns.

unknown

$$I_x u + I_y v + I_t = 0$$

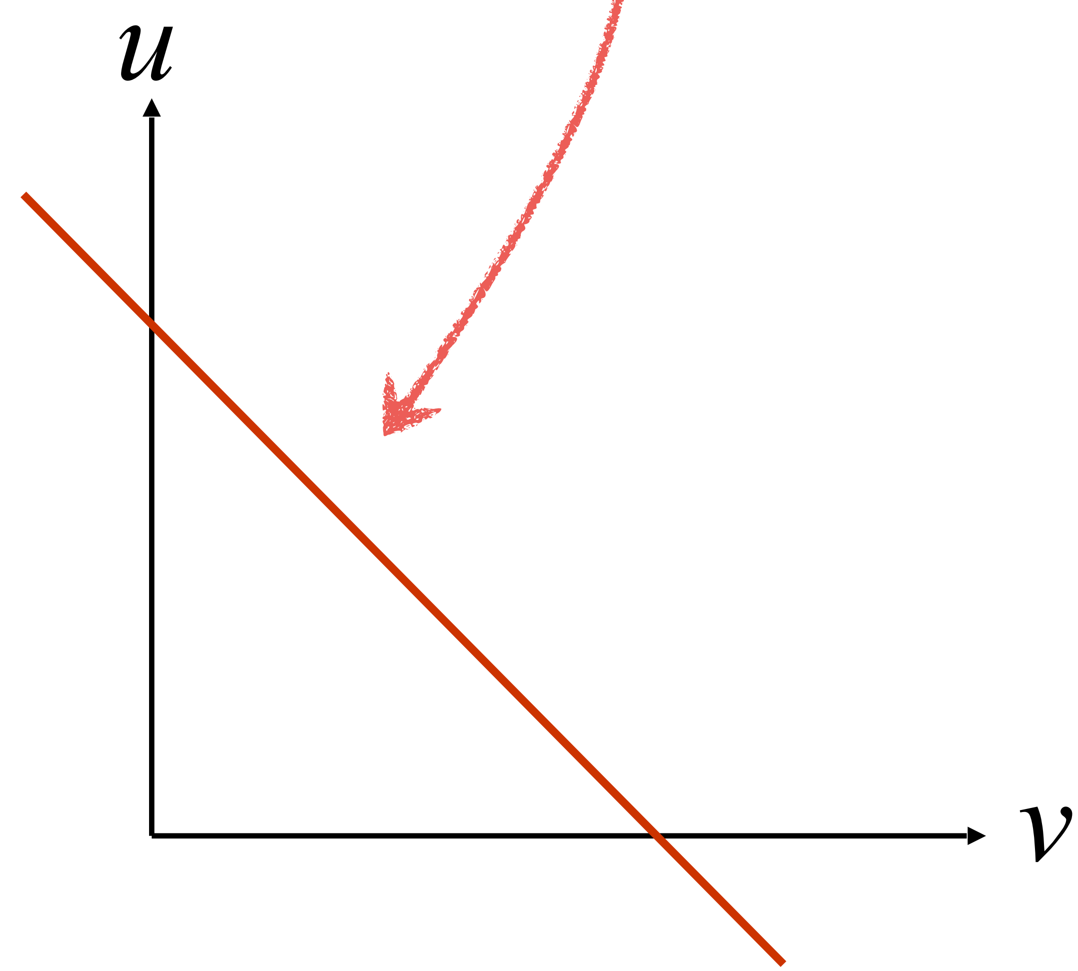
known

Where do we get more equations (constraints)?

Solution lies on a straight line

$$I_x u + I_y v + I_t = 0$$

many combinations of u and v will satisfy the equality



The solution cannot be determined uniquely with a single constraint (a single pixel)

$$I_x u + I_y v + I_t = 0$$

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

spatial derivative

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

optical flow

$$I_t = \frac{\partial I}{\partial t}$$

temporal derivative

How can we use the brightness constancy equation to estimate the optical flow?