



# Brightness Constancy

16-385 Computer Vision (Kris Kitani)  
**Carnegie Mellon University**

# Optical Flow

## **Problem Definition**

Given two consecutive image frames,  
estimate the motion of each pixel

## **Assumptions**

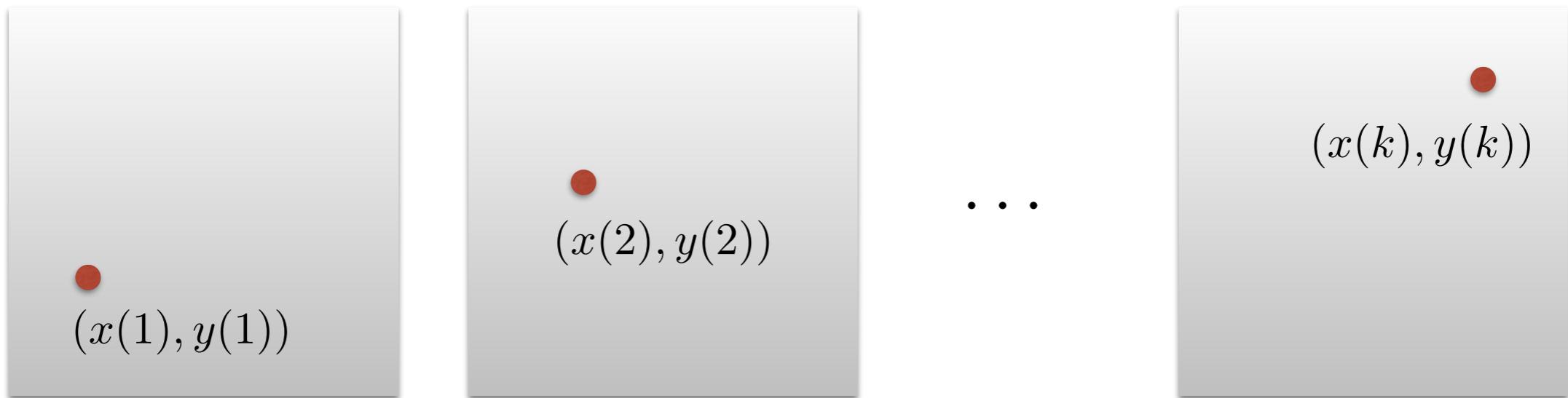
Brightness constancy

Small motion

Assumption 1

# Brightness constancy

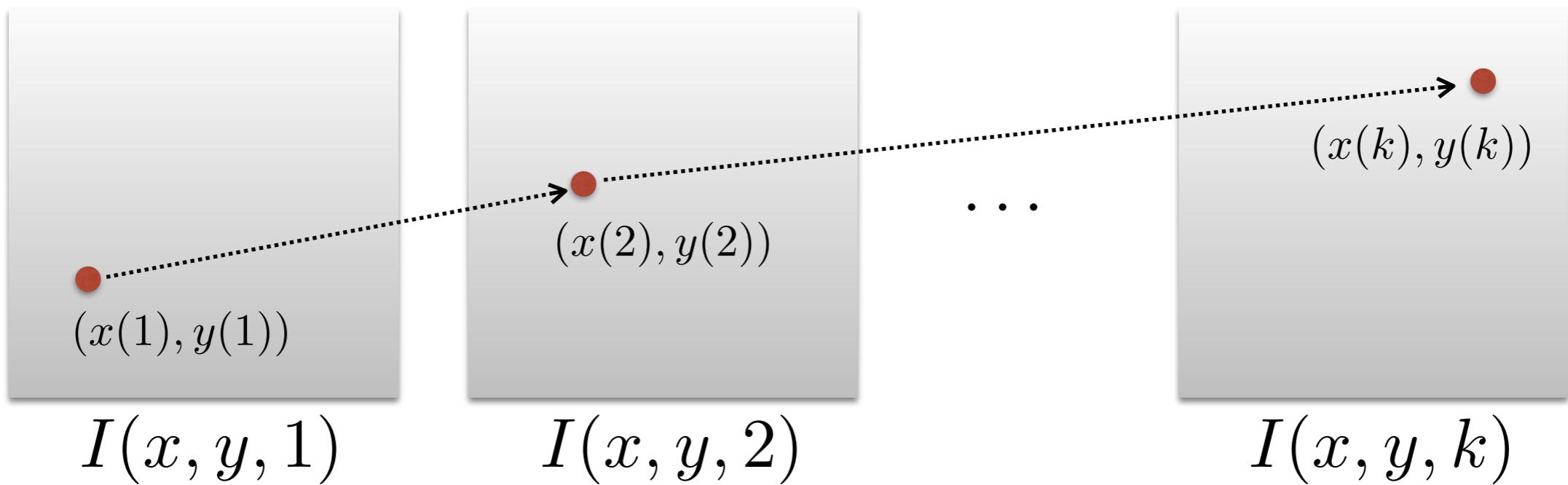
Scene point moving through image sequence



Assumption 1

# Brightness constancy

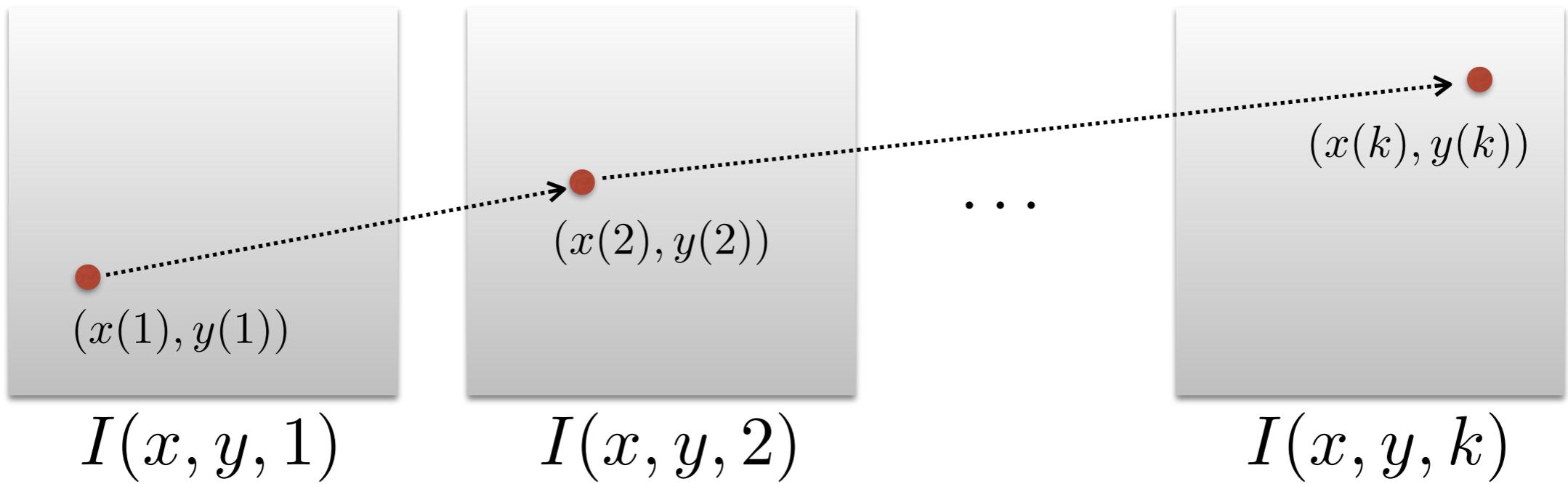
Scene point moving through image sequence



Assumption 1

# Brightness constancy

Scene point moving through image sequence

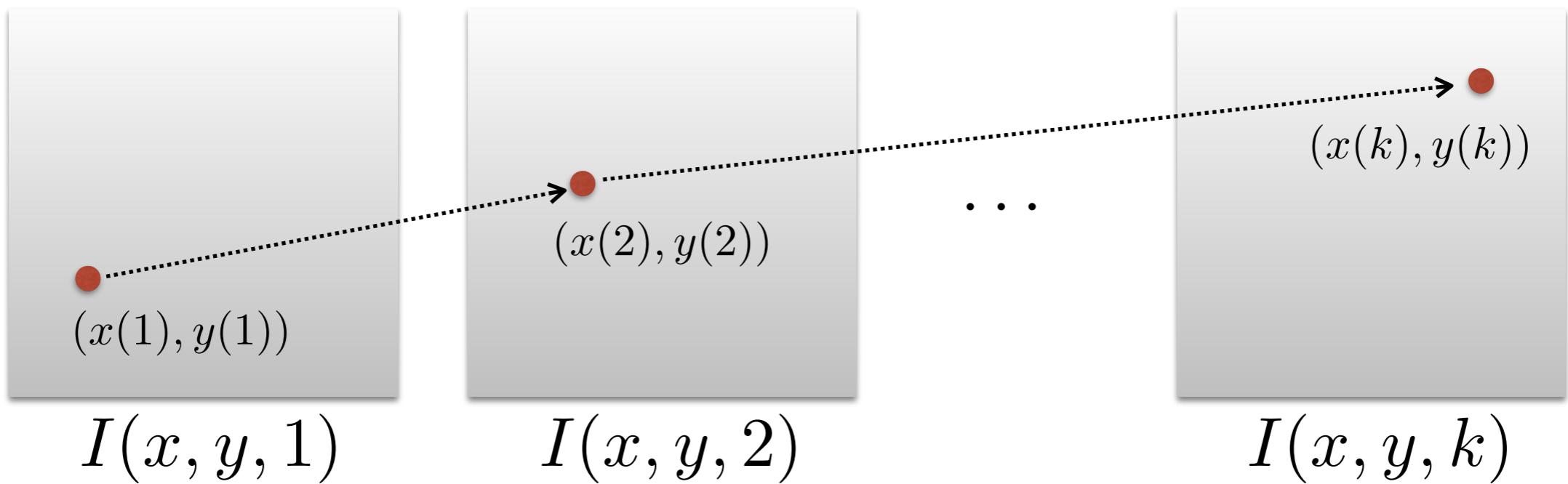


**Assumption: Brightness of the point will remain the same**

Assumption 1

# Brightness constancy

Scene point moving through image sequence



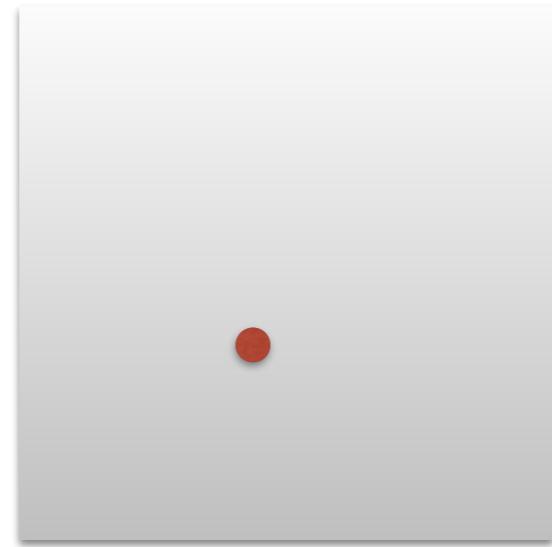
**Assumption: Brightness of the point will remain the same**

$$I(x(t), y(t), t) = C$$

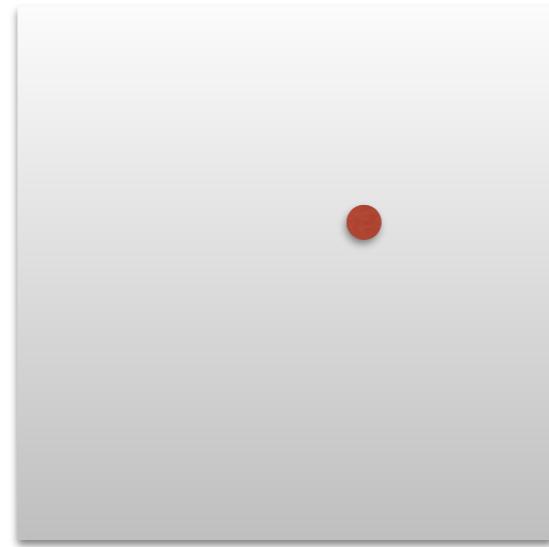
constant

# Assumption 2

# Small motion



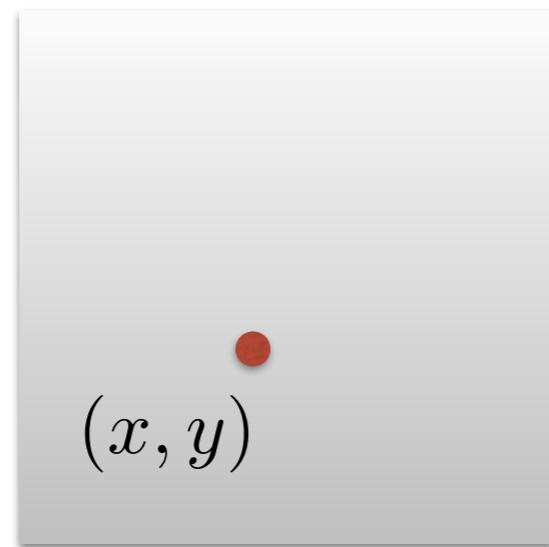
$I(x, y, t)$



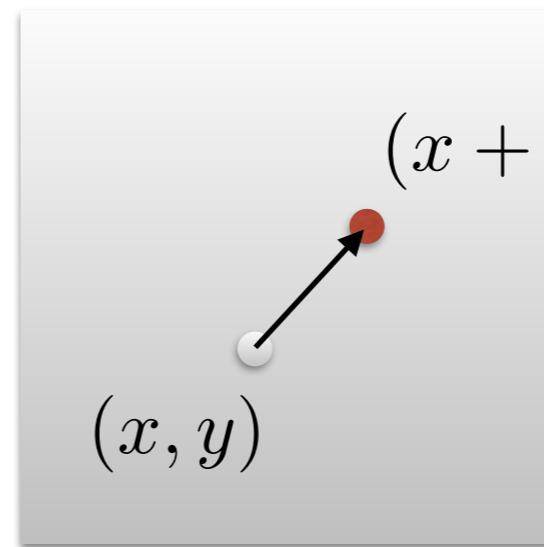
$I(x, y, t + \delta t)$

Assumption 2

# Small motion



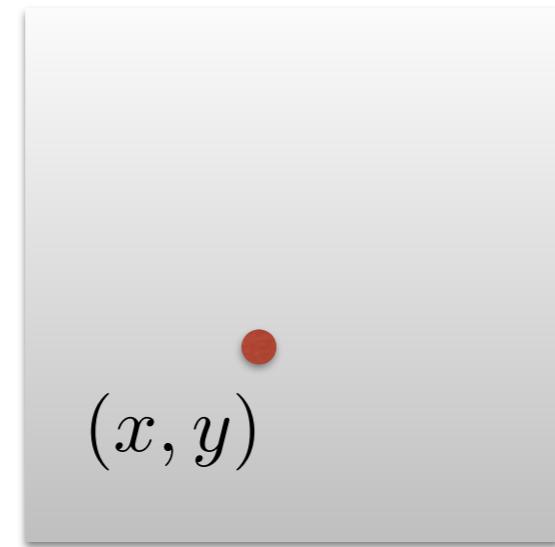
$I(x, y, t)$



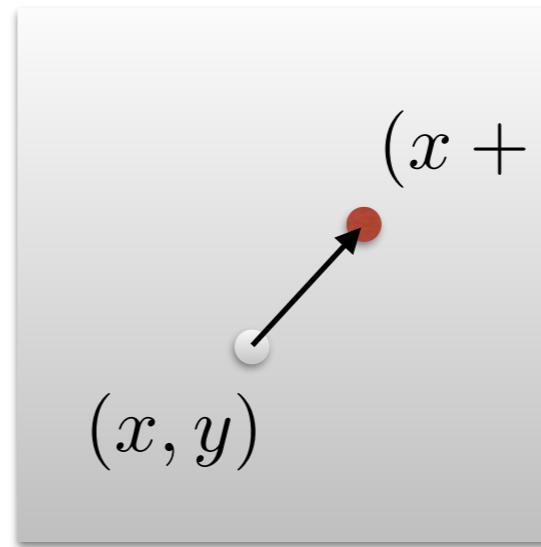
$I(x, y, t + \delta t)$

# Assumption 2

# Small motion



$$I(x, y, t)$$



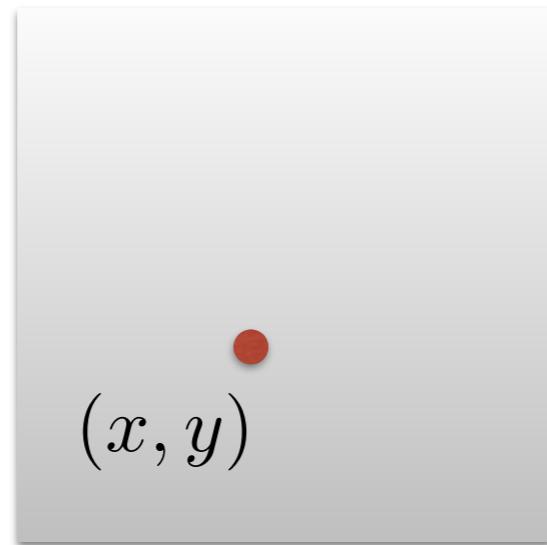
$$I(x, y, t + \delta t)$$

Optical flow (velocities):  $(u, v)$

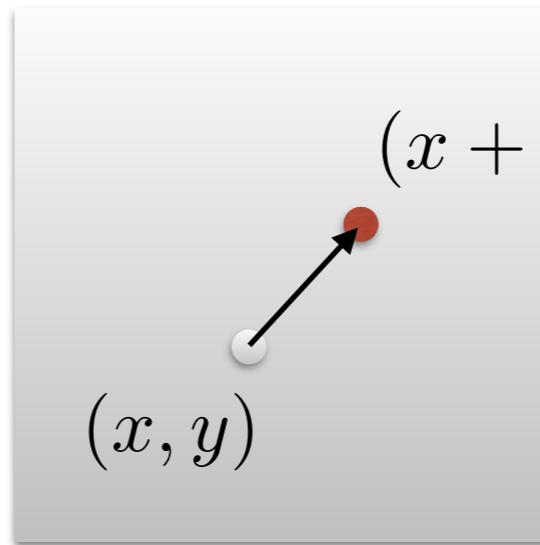
Displacement:  $(\delta x, \delta y) = (u\delta t, v\delta t)$

## Assumption 2

# Small motion



$$I(x, y, t)$$



$$I(x, y, t + \delta t)$$

Optical flow (velocities):  $(u, v)$

Displacement:  $(\delta x, \delta y) = (u\delta t, v\delta t)$

For a really small space-time step...

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

... the brightness between two consecutive image frames is the same

These assumptions yield the ...

## Brightness Constancy Equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

total derivative

partial derivative

*Equality is not obvious. Where does this come from?*

These assumptions yield the ...

## Brightness Constancy Equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

total derivative

partial derivative

*Where does this come from?*

proof!

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

For small space-time step, brightness of a point is the same

## **Insight:**

If the time step is really small,  
we can *linearize* the intensity function

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion  
(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

### Multivariable Taylor Series Expansion

(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = I(x, y, t) \quad \text{assuming small motion}$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

### Multivariable Taylor Series Expansion (First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

partial derivative

$$I(x, y, t) + \frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = I(x, y, t) \quad \text{assuming small motion}$$

cancel terms

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

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$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

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$$\frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = 0 \quad \text{cancel terms}$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

### Multivariable Taylor Series Expansion (First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

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$$\frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = 0 \quad \begin{aligned} &\text{divide by } \delta t \\ &\text{take limit } \delta t \rightarrow 0 \end{aligned}$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

### Multivariable Taylor Series Expansion (First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = I(x, y, t) \quad \text{assuming small motion}$$

$$\frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = 0 \quad \begin{matrix} \text{divide by } \delta t \\ \text{take limit } \delta t \rightarrow 0 \end{matrix}$$

$$\frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

$$I(x + u\delta t, y + v\delta t, t + \delta t) = I(x, y, t)$$

Multivariable Taylor Series Expansion  
(First order approximation, two variables)

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$I(x, y, t) + \frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = I(x, y, t) \quad \text{assuming small motion}$$

$$\frac{\partial I}{\partial x} \delta x + \frac{\partial I}{\partial y} \delta y + \frac{\partial I}{\partial t} \delta t = 0 \quad \begin{array}{l} \text{divide by } \delta t \\ \text{take limit } \delta t \rightarrow 0 \end{array}$$

$$\frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

**Brightness Constancy  
Equation**

$$\frac{\partial I}{\partial x} \frac{dx}{dt} + \frac{\partial I}{\partial y} \frac{dy}{dt} + \frac{\partial I}{\partial t} = 0$$

# Brightness Constancy Equation

$$I_x u + I_y v + I_t = 0$$

(x-flow) (y-flow)

## shorthand notation

$$\nabla I^\top v + I_t = 0$$

(1 x 2) (2 x 1)

## vector form

(putting the math aside for a second...)

What do the term of the  
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

(putting the math aside for a second...)

What do the term of the  
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$



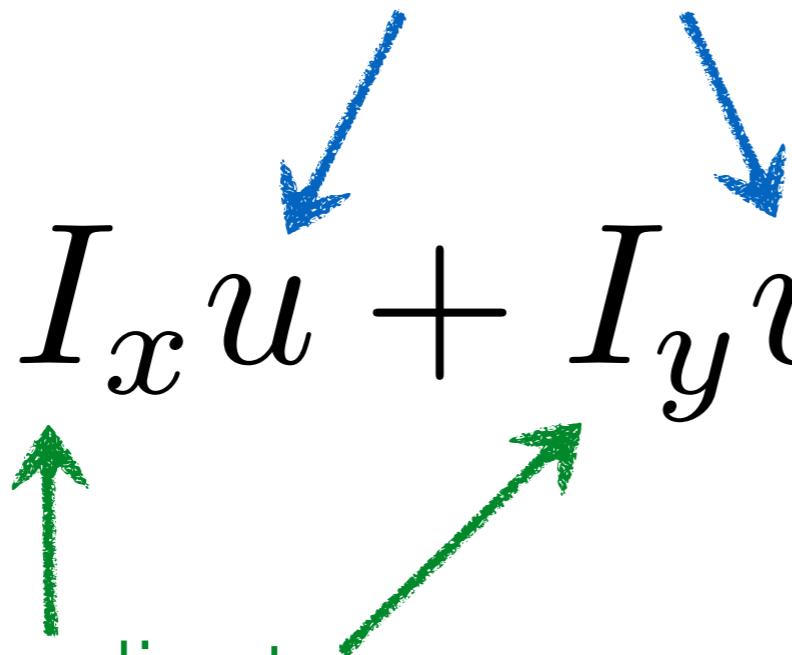
(putting the math aside for a second...)

What do the term of the  
brightness constancy equation represent?

flow velocities

$$I_x u + I_y v + I_t = 0$$

Image gradients  
(at a point p)



(putting the math aside for a second...)

What do the term of the  
brightness constancy equation represent?

$$I_x u + I_y v + I_t = 0$$

flow velocities

Image gradients  
(at a point p)

temporal gradient

The diagram illustrates the components of the brightness constancy equation. The equation itself is  $I_x u + I_y v + I_t = 0$ . Above the first term,  $I_x u$ , the text "flow velocities" is written in blue, and two blue arrows point downwards. Below the first term, the text "Image gradients (at a point p)" is written in green, and a green arrow points upwards. Below the second term,  $I_y v$ , a purple arrow points upwards. The text "temporal gradient" is written in purple below the third term,  $I_t$ .

*How do you compute these terms?*

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

Forward difference

Sobel filter

Scharr filter

...

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference

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**spatial derivative**

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...

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

frame differencing

# Frame differencing

$t$

1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10

-

$t + 1$

1	1	1	1	1
1	1	1	1	1
1	10	10	10	10
1	10	10	10	10
1	10	10	10	10
1	1	10	10	10
1	1	10	10	10
1	1	10	10	10

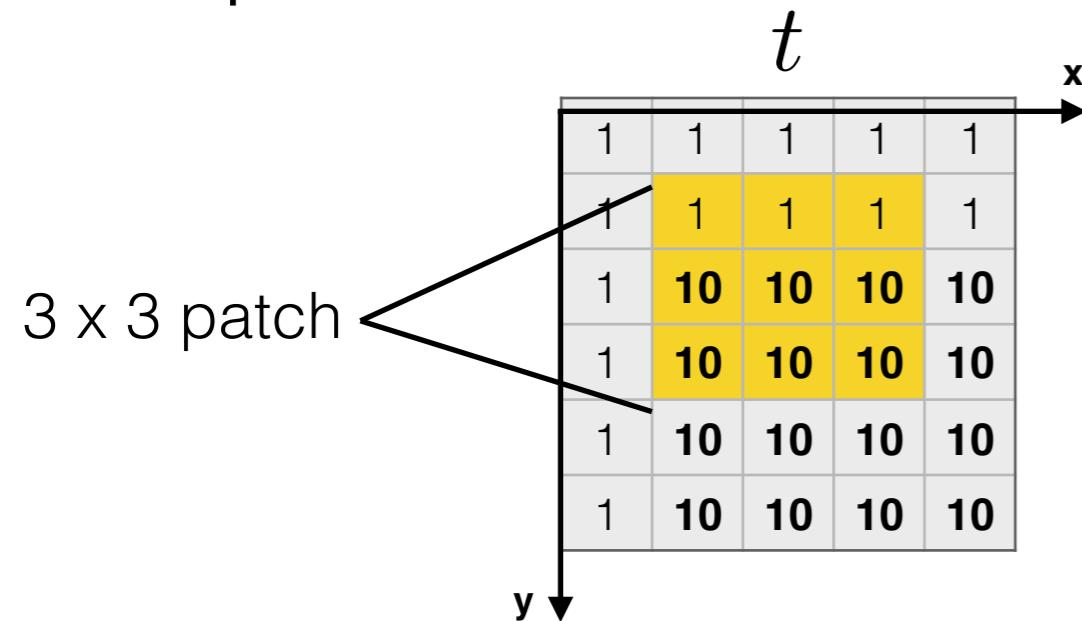
=

0	0	0	0	0
0	0	0	0	0
0	9	9	9	9
0	9	0	0	0
0	9	0	0	0
0	9	0	0	0

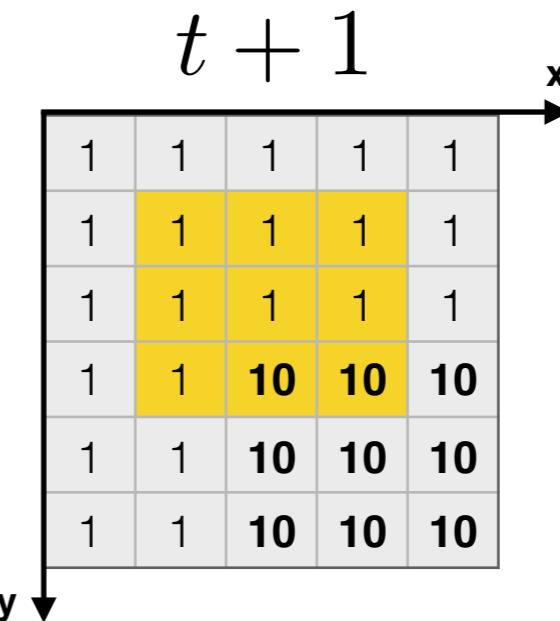
(example of a forward difference)

$$I_t = \frac{\partial I}{\partial t}$$

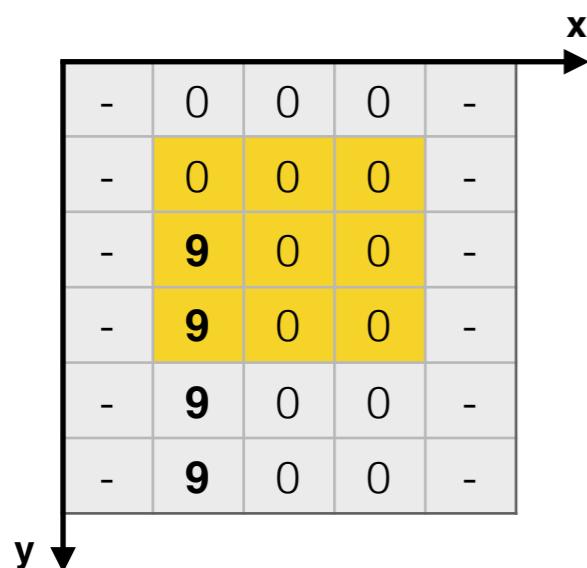
Example:



3 x 3 patch

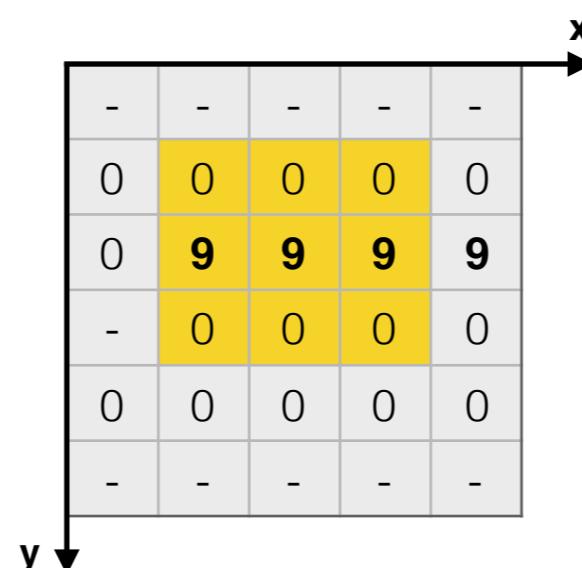


$$I_x = \frac{\partial I}{\partial x}$$



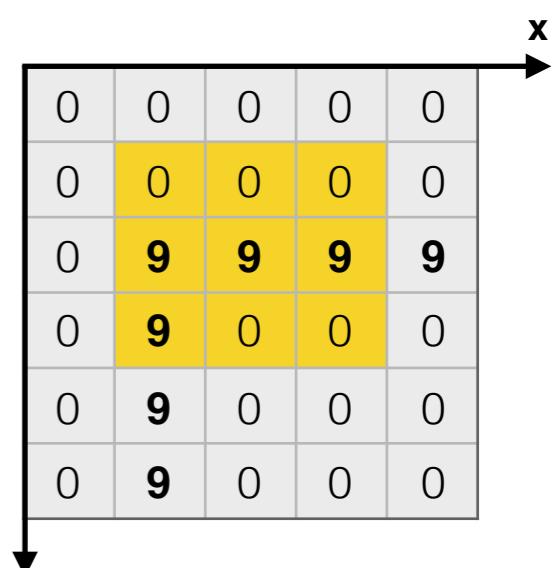
-1 0 1

$$I_y = \frac{\partial I}{\partial y}$$



-1  
0  
1

$$I_t = \frac{\partial I}{\partial t}$$



$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference

Sobel filter

Scharr filter

...

How do you compute this?

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

Forward difference

Sobel filter

Scharr filter

...

**We need to solve for this!**

(this is the unknown in the  
optical flow problem)

frame differencing

$$I_x u + I_y v + I_t = 0$$

How do you compute ...

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

Forward difference

Sobel filter

Scharr filter

...

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$(u, v)$

Solution lies on a line

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

frame differencing

Cannot be found uniquely  
with a single constraint

unknown

$$I_x u + I_y v + I_t = 0$$

known

The diagram illustrates a system of equations with two unknown variables,  $u$  and  $v$ . The equation is  $I_x u + I_y v + I_t = 0$ . The terms  $I_x u$  and  $I_y v$  are circled in green, indicating they are unknowns. The term  $I_t$  is not circled, indicating it is a known value. Green arrows point from the circled terms to the word "unknown" above the equation. Black arrows point from the word "known" below the equation to the term  $I_t$ .

We need at least \_\_\_\_\_ equations to solve for 2 unknowns.

unknown

$$I_x u + I_y v + I_t = 0$$

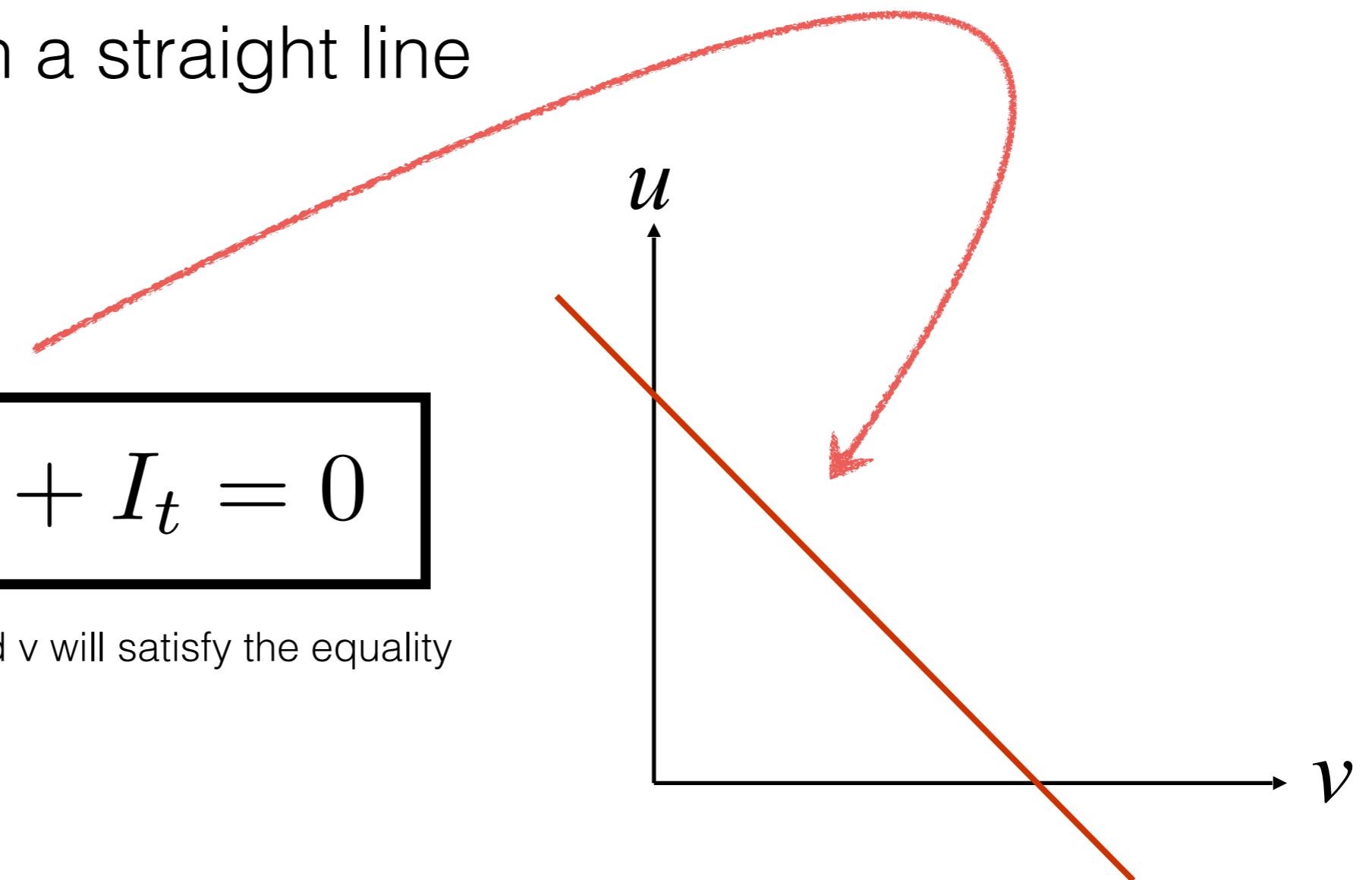
known

Where do we get more equations (constraints)?

Solution lies on a straight line

$$I_x u + I_y v + I_t = 0$$

many combinations of  $u$  and  $v$  will satisfy the equality



The solution cannot be determined uniquely with a single constraint (a single pixel)

$$I_x u + I_y v + I_t = 0$$

$$I_x = \frac{\partial I}{\partial x} \quad I_y = \frac{\partial I}{\partial y}$$

**spatial derivative**

$$u = \frac{dx}{dt} \quad v = \frac{dy}{dt}$$

**optical flow**

$$I_t = \frac{\partial I}{\partial t}$$

**temporal derivative**

*How can we use the brightness constancy equation to estimate the optical flow?*