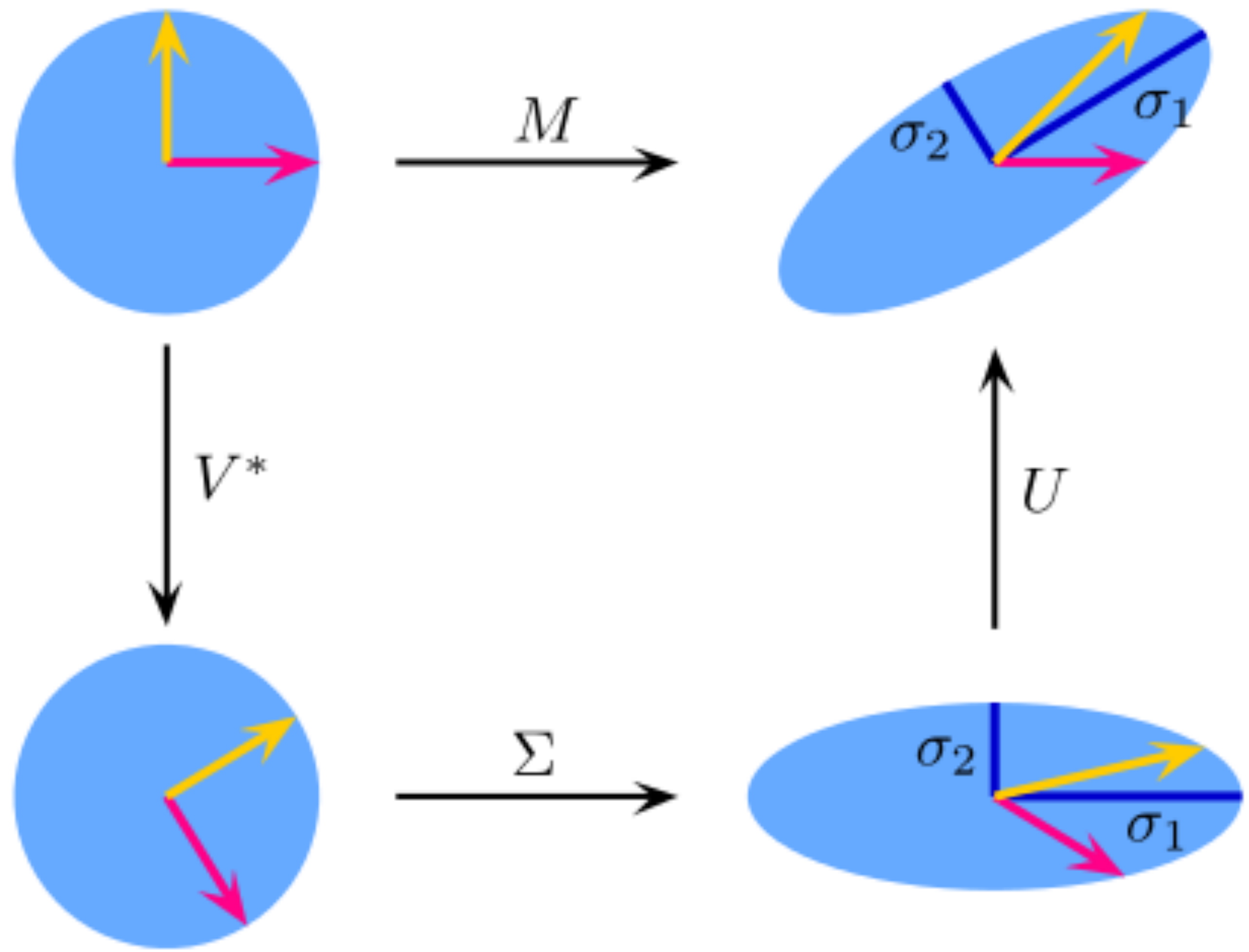

$$M = U \cdot \Sigma \cdot V^*$$

SVD for Total Least Squares

16-385 Computer Vision (Kris Kitani)

Carnegie Mellon University

General form of Total Least Squares

$$E_{\mathrm{TLS}} = \sum_i (\boldsymbol{a}_i \boldsymbol{x})^2$$

$$= \|\mathbf{A}\boldsymbol{x}\|^2$$ (matrix form)

$$\|\boldsymbol{x}\|^2 = 1$$ constraint

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

→

(Rayleigh quotient)

minimize $\dfrac{\|\mathbf{A}\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2}$

Solution is the eigenvector corresponding to smallest eigenvalue of

$$\mathbf{A}^\top \mathbf{A}$$

(equivalent)

Solution is the column of **V** corresponding to smallest singular value

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$$

Singular Value Decomposition

$$\underset{n \times m}{\mathbf{A}} = \overset{\text{ortho-normal}}{\underset{n \times n}{\mathbf{U}}} \overset{\text{diagonal}}{\underset{n \times m}{\mathbf{\Sigma}}} \overset{\text{ortho-normal}}{\underset{m \times m}{\mathbf{V}}}^\top$$

unit norm constraint

orthogonal: inner (dot) product between columns/rows is zero

norm (unit vector): magnitude of each column/row is equal to 1

last column of V is the solution to TLS … buy why?

Why does V give us the solution to the total least squares problem?

minimize $\|\mathbf{A}\boldsymbol{x}\|^2$ subject to $\|\boldsymbol{x}\|^2 = 1$

Minimize $\|\mathbf{A}\boldsymbol{x}\|^2$

subject to $\|\boldsymbol{x}\|^2 = 1$

can be written as…

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} ||\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \boldsymbol{x}||^2$$

suppressing constraint
from notation for simplicity

due to orthonormality

$$||\mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top \boldsymbol{x}||^2 = ||\mathbf{\Sigma}\mathbf{V}^\top \boldsymbol{x}||^2$$

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} ||\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top \boldsymbol{x}||^2$$

due to orthonormality

$$||\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top \boldsymbol{x}||^2 = ||\boldsymbol{\Sigma}\mathbf{V}^\top \boldsymbol{x}||^2$$

just rotates contents and doesn't scale it so magnitude is unchanged

can be written as …

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} ||\boldsymbol{\Sigma}\mathbf{V}^\top \boldsymbol{x}||^2$$

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} ||\boldsymbol{\Sigma}\mathbf{V}^\top \boldsymbol{x}||^2$$

from orthonormality

substitute $\boldsymbol{y} = \mathbf{V}^\top \boldsymbol{x}$ $||\mathbf{V}^\top \boldsymbol{x}||^2 = ||\boldsymbol{x}||^2$

$$\hat{\boldsymbol{y}} = \arg\min_{\boldsymbol{y}} ||\boldsymbol{\Sigma}\boldsymbol{y}||^2$$

if diagonals are sorted in decreasing order:

$$\boldsymbol{y} = [0, 0, \cdots, 1]^\top$$

$$\hat{\boldsymbol{y}} = \arg\min_{\boldsymbol{y}} ||\boldsymbol{\Sigma}\boldsymbol{y}||^2$$

if diagonals are sorted in decreasing order:

$$\boldsymbol{y} = [0, 0, \cdots, 1]^\top$$

From substitution we know that:

$$\boldsymbol{y} = \mathbf{V}^\top \boldsymbol{x} \qquad \boldsymbol{x} = \mathbf{V}\boldsymbol{y}$$

Therefore:

$\boldsymbol{y}$ is the last column of $\mathbf{V}$