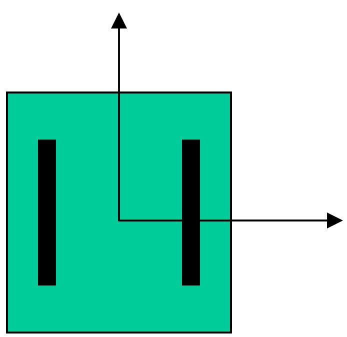
Non-holonomic Constraints and Lie brackets

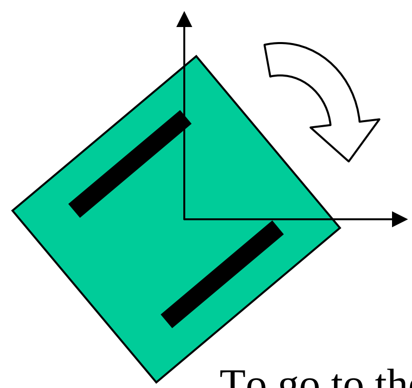
Definition: A non-holonomic constraint is non-integrable constraint

Example: A constraint on velocity does not induce a constraint on position

For a wheeled robot, it can instantaneously move in some directions (forwards and backwards), but not others (side to side).



The robot can instantly move forward and back, but can not move to the right or left without the wheels slipping.

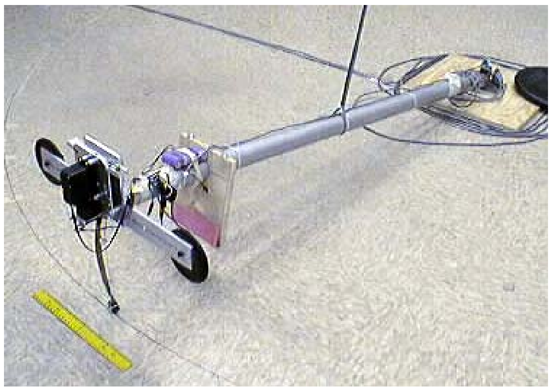


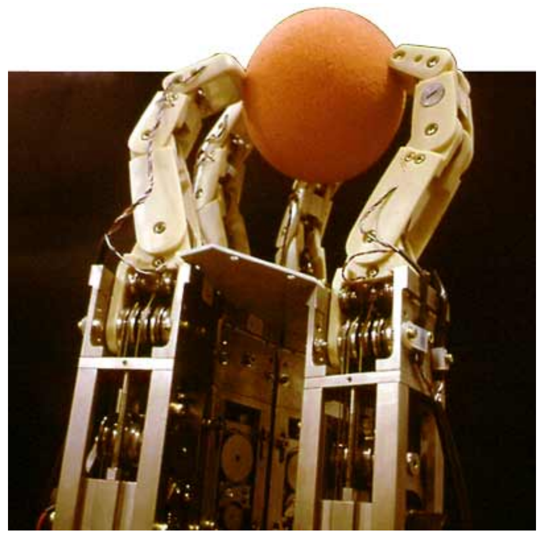
To go to the right, the robot must first turn, and then drive forward

Other examples of systems with non-holonomic constraints

Hopping robots

RI's bow leg hopper





Manipulation with a robotic hand

Multi-fingered hand from Nagoya University

Untethered space robots (conservation of angular momentuem is the constraint)

AERcam, NASA

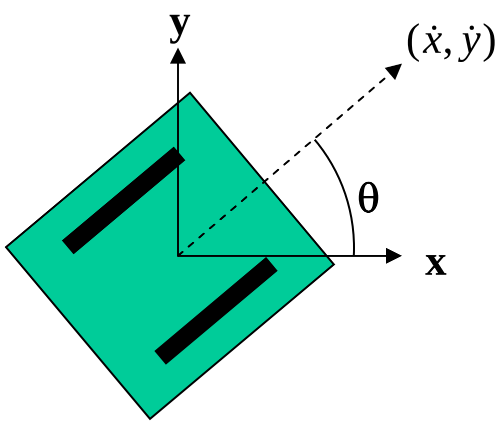
What about holonimc systems?

- A person walking is an example of a holonomic system- you can instantly step to the right or left, as well as going forwards or backwards. In other words, your velocity in the plane is not restricted.
- An Omni-wheel is a holonomic system- it can roll forwards and sideways.

How do we represent the constraint mathematically?

We write a constraint equation

For a differential drive, this is: $\dot{y}\cos\theta - \dot{x}\sin\theta = 0$



•What does this equation tell us? *The direction we can't move in*

So if θ=0, then the velocity in y = 0

if θ=90, then the velocity in x = 0

•We can also write the constraint in matrix form, with q the position vector and q dot the velocity, we can write a constraint vector $w_1(q)$

$$q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}, \quad \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \qquad w_1(q) = [-\sin\theta \ \ \cos\theta \ \ 0]$$

$$w1(q)\cdot\dot{q} = 0 = [-\sin\theta \ \ \cos\theta \ \ 0]\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \iff -\dot{x}\sin\theta + \dot{y}\cos\theta = 0$$

Lie Brackets

$$let\ g = \begin{bmatrix} a \\ b \\ c \end{bmatrix},\ q = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$$\frac{\partial g}{\partial q} = \begin{bmatrix} \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial a}{\partial \theta} \\ \frac{\partial b}{\partial x} & \frac{\partial b}{\partial y} & \frac{\partial b}{\partial \theta} \\ \frac{\partial c}{\partial x} & \frac{\partial c}{\partial y} & \frac{\partial c}{\partial \theta} \end{bmatrix}$$

$$\text{Lie Bracket}: [g_1,\ g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$$

A Lie Bracket takes two *n* dimensional vectors and returns a new *n*-vector

$$Example:$$

$$g_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix},\ g2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$so\ \frac{\partial g_1}{\partial q} = \begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix},\ \frac{\partial g_2}{\partial q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$LieBracket: [g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$$

$$[g_1, g_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Method for analyzing non-holonomic motion

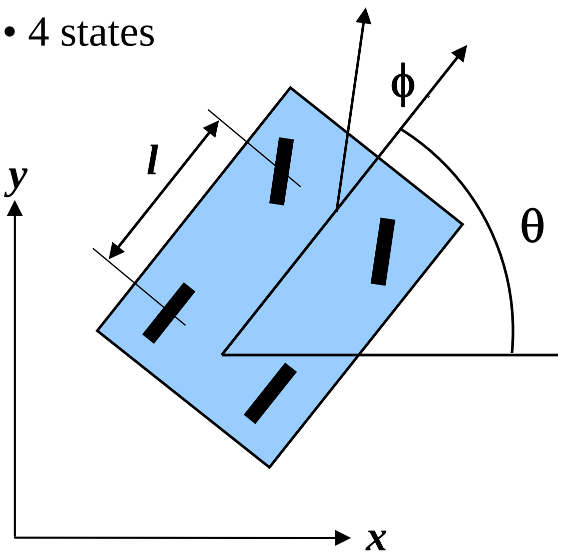
- Determine your constraints (w's)
- Convert the constraints into locally allowable motions, (w's –> g's)

  Must find allowable inputs g_1 and g_2 such that ($g_1 \perp w_1$) and ($g_2 \perp w_2$)
- Apply Lie Bracket to your g's to determine all possible motions

  If after you apply the Lie Bracket you find that you have *n* linearly independent columns, then you can control your robot in all *n* variables.

Ackerman steering example

- 2 constraints (front and rear wheels)
- 2 inputs (steering and gas pedal)
- 4 states



$$q = \begin{bmatrix} x \\ y \\ \theta \\ \phi \end{bmatrix}$$

$$w_1 = [-\sin\theta \;\; \cos\theta \;\; 0 \;\; 0]$$

$$w_2 = [-\sin(\theta+\phi) \;\; \cos(\theta+\phi) \;\; l\cos\phi \;\; 0]$$

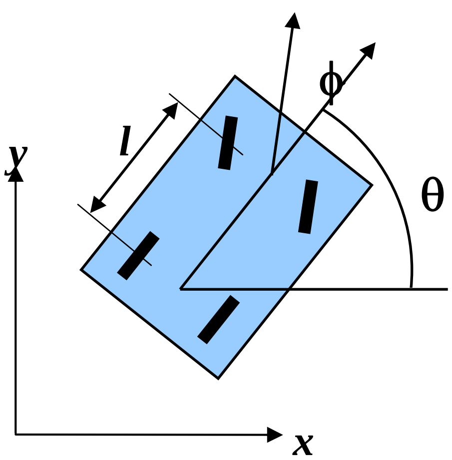
$$\dot{w}_1 \cdot \dot{q} = 0$$

Intuition tells us $g_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, which means the steering depends only on ϕ

now we want g_2 to tell us the direction we would go for a fixed ϕ

$$g_2 = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \frac{1}{l}\tan\phi \\ 0 \end{bmatrix}$$

Ackerman example cont.



We want four linearly independent g's. Since we already have two, we need to compute g_3 and g_4

$$g_3 = [g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2$$

We can immeadietly see that $\frac{\partial g_1}{\partial q} = 0$ since g_1 is constant.

And because the first three rows of g_1 are 0, we only need to find the last coulmn of $\frac{\partial g_2}{\partial q}$.

$$\frac{\partial g_2}{\partial q} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{l\cos^2\phi} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ and thus } \frac{\partial g_2}{\partial q} g_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{l\cos^2\phi} \\ 0 \end{bmatrix} = g_3$$

Ackerman example cont.

To find g_4 we do another Lie Bracket:

$$g_4 = [g_2, g_3] = \frac{\partial g_3}{\partial q} g_2 - \frac{\partial g_2}{\partial q} g_3$$

Again, there are a lot of zeroes that make things go quickly

$$\frac{\partial g_3}{\partial q} g_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} X \\ X \\ X \\ 0 \end{bmatrix} = 0 \ ; \ \frac{\partial g_2}{\partial q} g_3 = \begin{bmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ \frac{1}{l\cos^2\phi} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-\sin\theta}{l\cos^2\phi} \\ \frac{1}{l\cos\phi} \\ 0 \\ 0 \end{bmatrix}$$

so we have $g_4 = \begin{bmatrix} \frac{\sin\theta}{l\cos^2\phi} \\ \frac{-1}{l\cos\phi} \\ 0 \\ 0 \end{bmatrix}$, and $[g_1, g_2, g_3, g_4] = \begin{bmatrix} \cos\theta & 0 & 0 & \frac{\sin\theta}{l\cos^2\phi} \\ \sin\theta & 0 & 0 & \frac{-1}{l\cos\phi} \\ \frac{\tan\phi}{l} & 0 & \frac{-1}{l\cos^2\phi} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

Are we there yet?