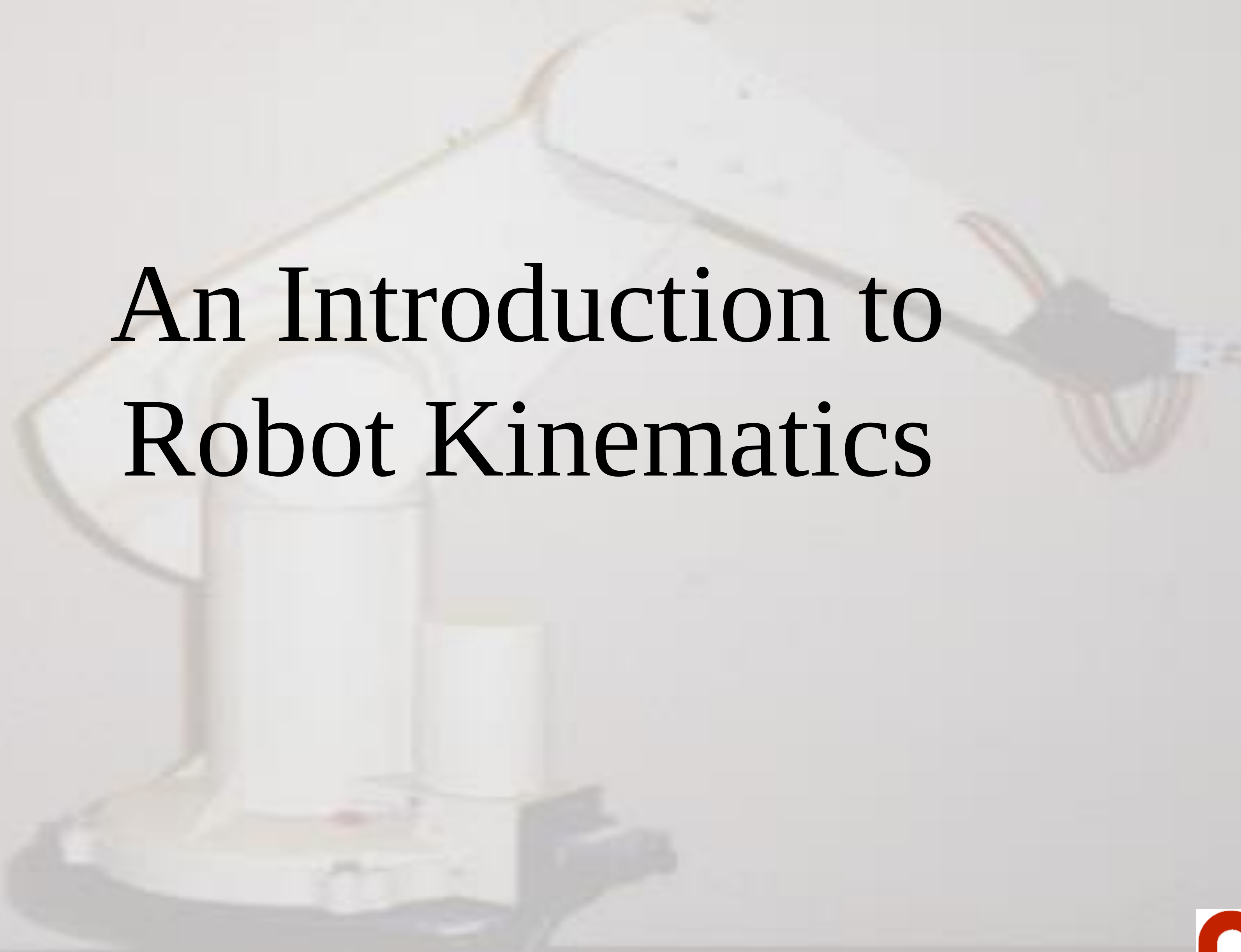


An Introduction to Robot Kinematics

We are interested in two kinematics topics

Forward Kinematics (angles to position)

What you are given: The length of each link
The angle of each joint

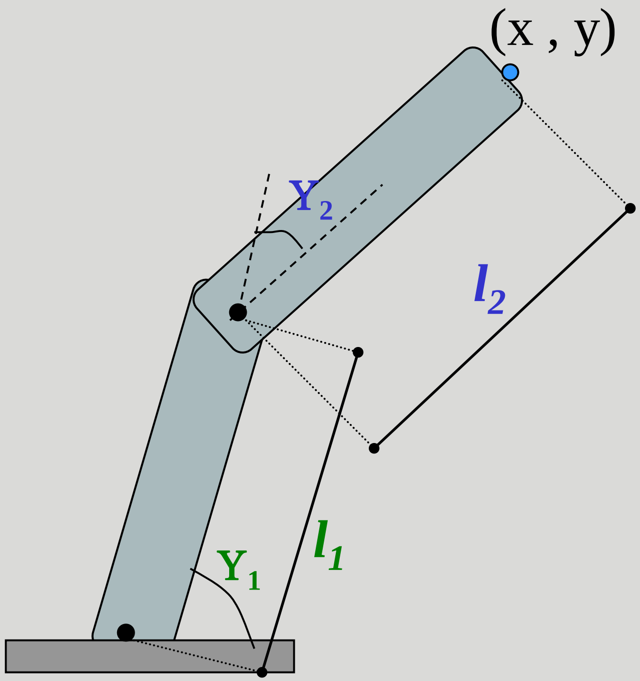
What you can find: The position of any point
(i.e. it's (x, y, z) coordinates

Inverse Kinematics (position to angles)

What you are given: The length of each link
The position of some point on the robot

What you can find: The angles of each joint needed to obtain that position

Inverse Kinematics of a Two Link Manipulator

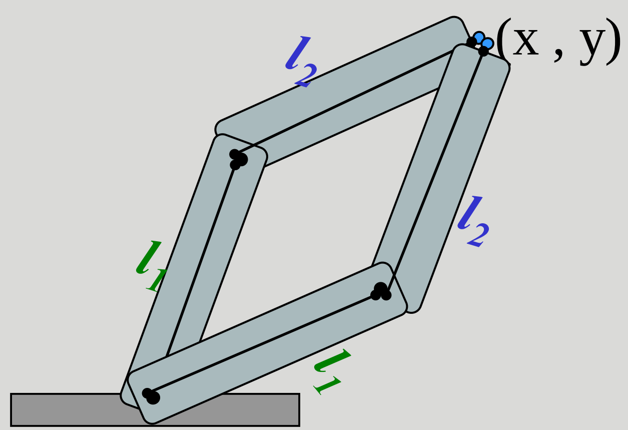


Given: l_1, l_2, x , y

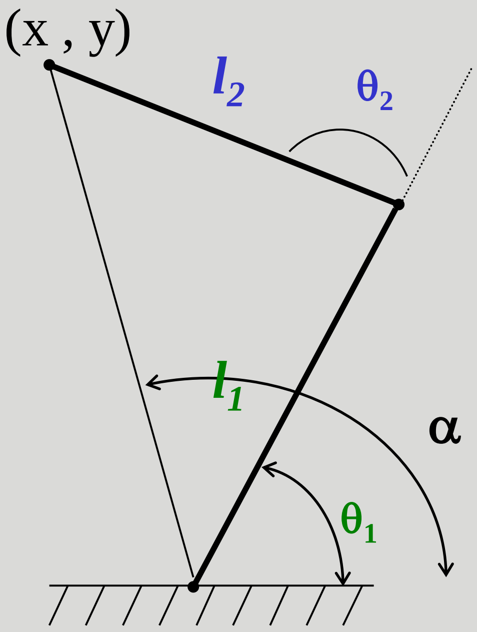
Find: Y_1, Y_2 really θ_1 & θ_2

Redundancy:

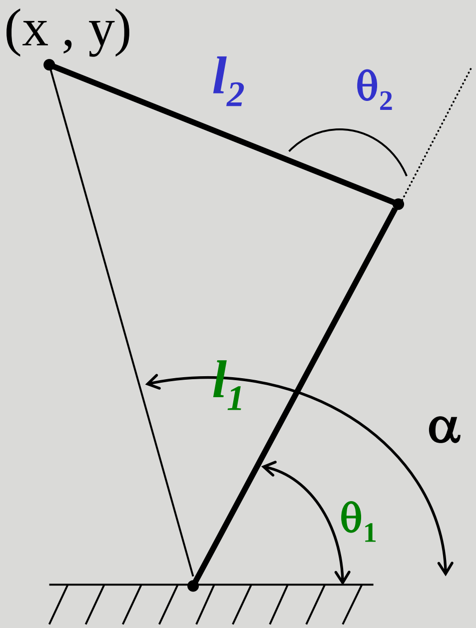
A unique solution to this problem does not exist. Notice, that using the "givens" two solutions are possible. Sometimes no solution is possible.



The Geometric Solution



The Geometric Solution



Using the Law of Sines:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin\bar{\theta}_1}{l_2} = \frac{\sin(180-\theta_2)}{\sqrt{x^2+y^2}} = \frac{\sin(\theta_2)}{\sqrt{x^2+y^2}}$$

$$\theta_1 = \alpha - \bar{\theta}_1$$

$$\alpha = \arctan 2\left(\frac{y}{x}\right)$$

Using the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$(x^2+y^2) = l_1^2 + l_2^2 - 2l_1l_2\cos(180-\theta_2)$$

$$\cos(180-\theta_2) = -\cos(\theta_2)$$

$$\cos(\theta_2) = \frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}$$

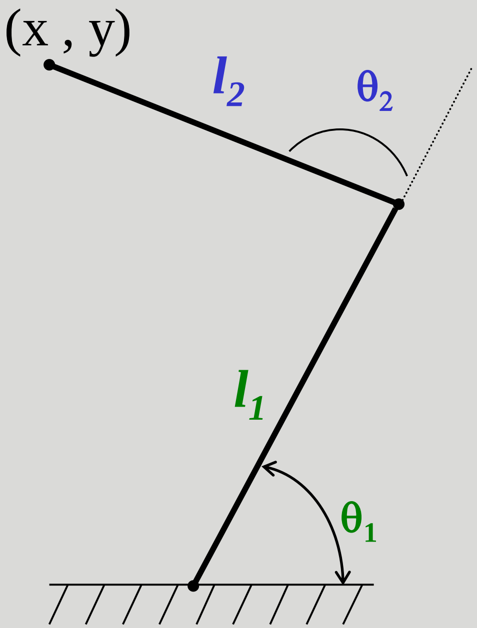
$$\theta_2 = \arccos\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}\right)$$

Redundant since θ_2 could be in the first or fourth quadrant.

Redundancy caused since θ_2 has two possible values

$$\theta_1 = \arctan 2(y, x) - \arcsin\left(\frac{l_2\sin(\theta_2)}{\sqrt{x^2+y^2}}\right)$$

The Algebraic Solution



$$c_1 = \cos\theta_1$$

$$c_{1+2} = \cos(\theta_2 + \theta_1)$$

$$(1)\ x = l_1\, c_1 + l_2\, c_{1+2}$$

$$(2)\ y = l_1\, s_1 + l_2\, \sin_{1+2}$$

$$(1)^2 + (2)^2 = x^2 + y^2 =$$

$$= \left(l_1^{\,2}\, c_1^{\,2} + l_2^{\,2}(c_{1+2})^2 + 2l_1 l_2\, c_1 (c_{1+2})\right) + \left(l_1^{\,2}\, s_1^{\,2} + l_2^{\,2}(\sin_{1+2})^2 + 2l_1 l_2\, s_1 (\sin_{1+2})\right)$$

$$= l_1^{\,2} + l_2^{\,2} + 2l_1 l_2 \left(c_1 (c_{1+2}) + s_1 (\sin_{1+2})\right)$$

$$= l_1^{\,2} + l_2^{\,2} + 2l_1 l_2\, c_2$$

← Only Unknown

$$\therefore \theta_2 = \arccos\left(\frac{x^2 + y^2 - l_1^{\,2} - l_2^{\,2}}{2l_1 l_2}\right)$$

Note:

$$\cos(a \pm b) = (\cos a)(\cos b) \mp (\sin a)(\sin b)$$

$$\sin(a \pm b) = (\cos a)(\sin b) \pm (\cos b)(\sin a)$$

$$
\begin{aligned}
x &= l_1 c_1 + l_2 c_{1+2} \\
&= l_1 c_1 + l_2 c_1 c_2 - l_2 s_1 s_2 \\
&= c_1 (l_1 + l_2 c_2) - s_1 (l_2 s_2)
\end{aligned}
$$

Note:

$$\cos(a \pm b) = (\cos a)(\cos b) \mp (\sin a)(\sin b)$$

$$\sin(a \pm b) = (\cos a)(\sin b) \pm (\cos b)(\sin a)$$

$$
\begin{aligned}
y &= l_1 s_1 + l_2 \sin_{1+2} \\
&= l_1 s_1 + l_2 s_1 c_2 + l_2 s_2 c_1 \\
&= c_1 (l_2 s_2) + s_1 (l_1 + l_2 c_2)
\end{aligned}
$$

We know what θ_2 is from the previous slide. We need to solve for θ_1 . Now we have two equations and two unknowns ($\sin \theta_1$ and $\cos \theta_1$)

$$
\begin{aligned}
x &= l_1 c_1 + l_2 c_{1+2} \\
&= l_1 c_1 + l_2 c_1 c_2 - l_2 s_1 s_2 \\
&= c_1 (l_1 + l_2 c_2) - s_1 (l_2 s_2)
\end{aligned}
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\end{aligned}
$$

We know what θ_2 is from the previous slide. We need to solve for θ_1 . Now we have two equations and two unknowns (sin θ_1 and cos θ_1)

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (l_1 + l_2 c_2) & (-l_2 s_2) \\ (l_2 s_2) & (l_1 + l_2 c_2) \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}
$$

$$\theta_1 = \arctan 2(s_1, c_1)$$

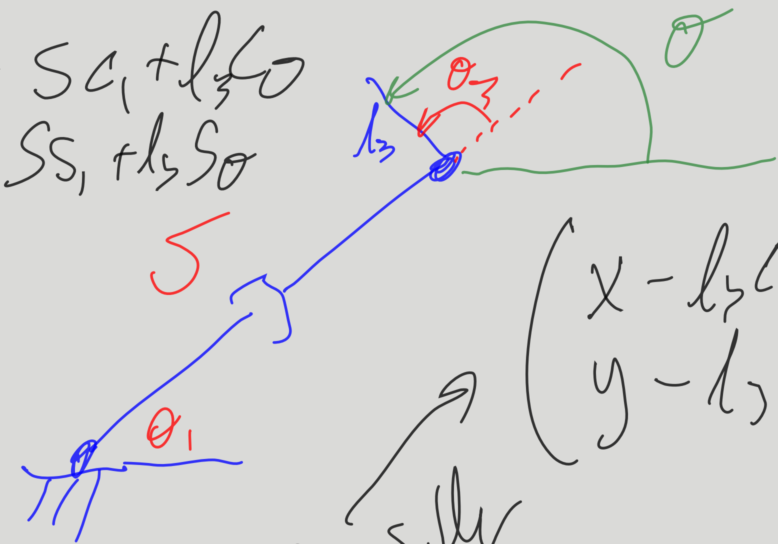
Three-link Manipulator IK

RPR Arm

$$x = S c_1 + l_3 c_{13}$$
$$y = S s_1 + l_3 s_{13}$$
$$\theta = \theta_1 + \theta_3$$

$$(1)^2 \Rightarrow x^2 = S^2 c_1^2 + 2 S l_3 c_\theta + l_3^2 c_\theta^2$$
$$y^2 = S^2 s_1^2 + 2 S l_3 s_\theta + l_3^2 s_\theta^2$$
$$x^2 + y^2 = S^2 + l_3^2 + 2 l_3 (s_\theta + c_\theta) S$$

$$(1) \Rightarrow x = S c_1 + l_3 c_\theta$$
$$(2)\ y = S s_1 + l_3 s_\theta$$



$$S^2 + 2 l_3 (s_\theta + c_\theta) S + l_3^2 - x^2 - y^2 = 0$$

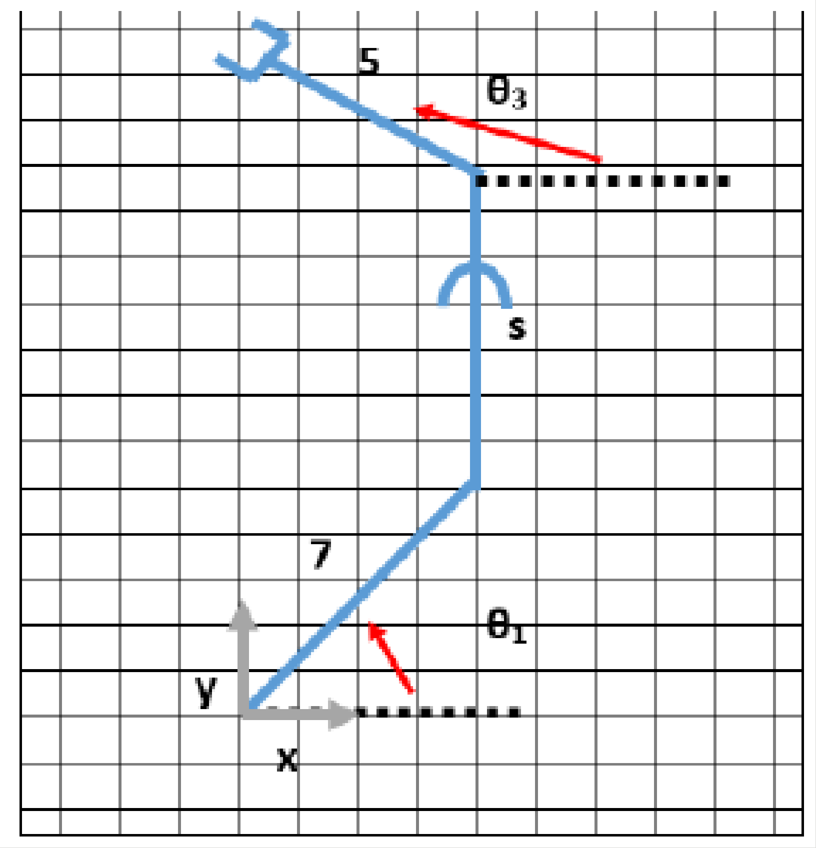
$$x - l_3 c_\theta = S c_1 \rightarrow x^2 - 2 x l_3 c_\theta + l_3^2 c_\theta^2 = S^2 c_1^2$$
$$y - l_3 s_\theta = S s_1 \rightarrow y^2 - 2 l_3 s_\theta + l_3^2 s_\theta^2 = S^2 s_1^2$$

consider

RPR Arm

RPR

Now we have a RPR arm with angles described from the horizontal. The prismatic link must always be vertical. Find θ_1, s, and θ_3 in terms of (x, y, and θ) of the end effector:



RPR

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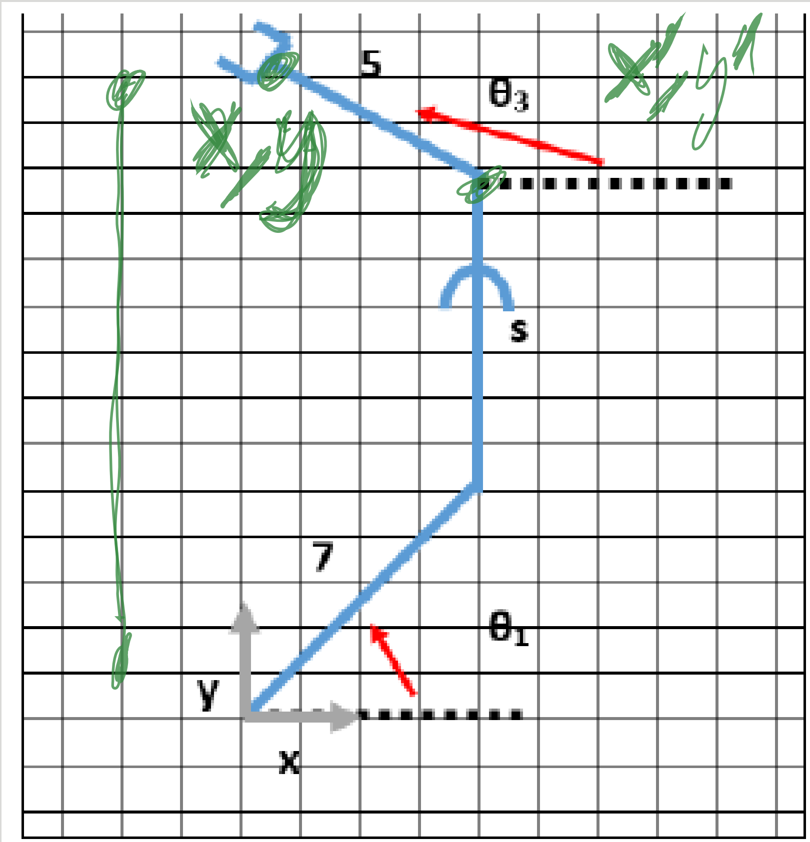


$\theta_3 = \theta$

$x' = x - 5c_3$

$y' = y - 5s_3$

$y = 7s_1 + s + 5s_3$

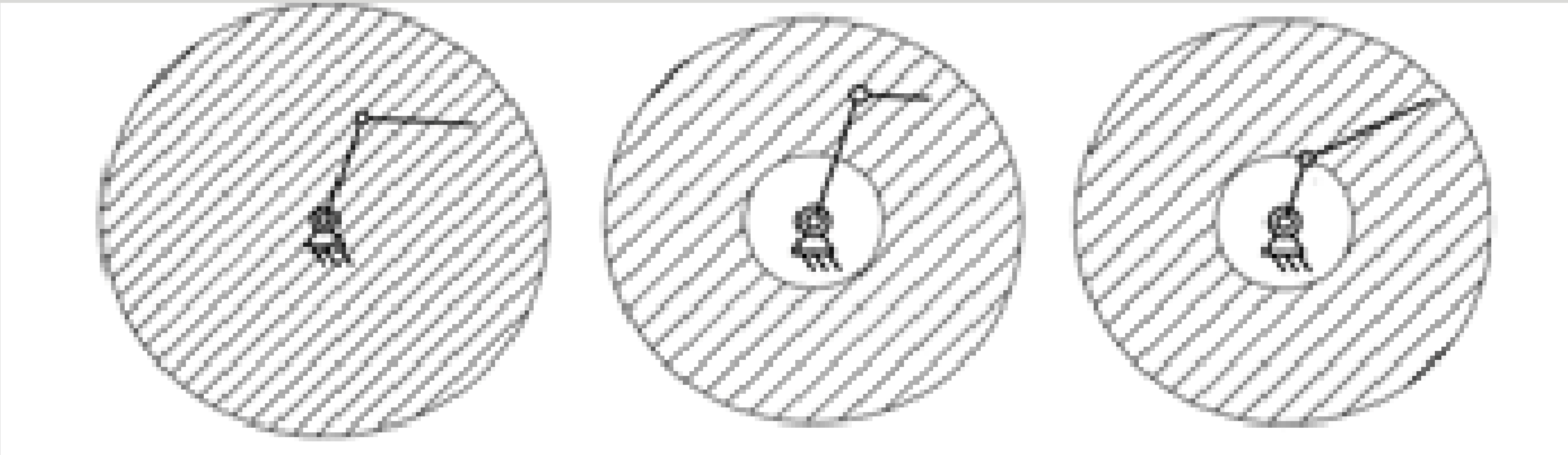
$\sin\theta_1 = \frac{x'}{7} = \frac{x - 5c_3}{7}$

$y = x - 5c_\theta + s + 5s_\theta$

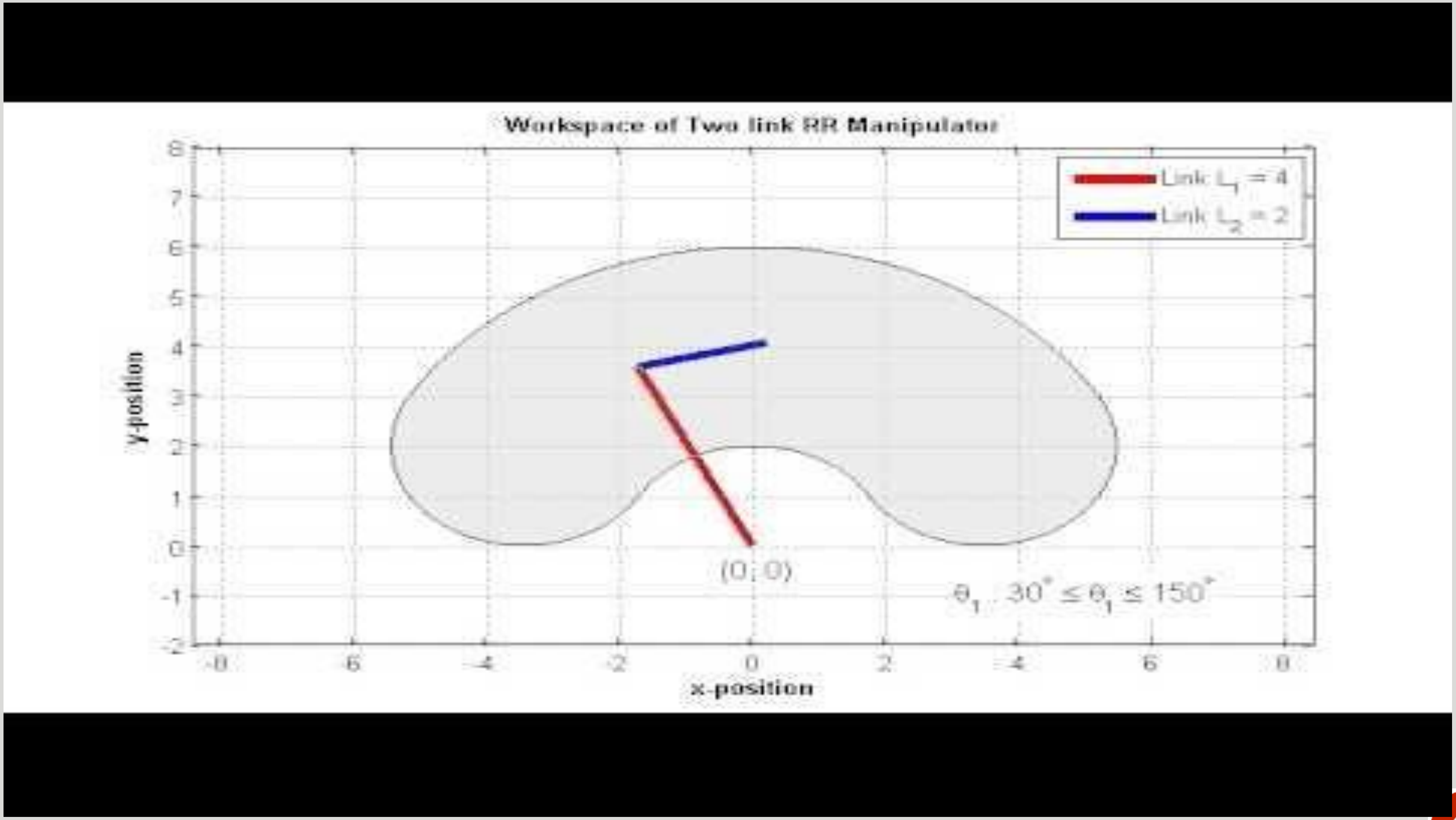
are we done?

$\sin\theta_1$, s, θ_3

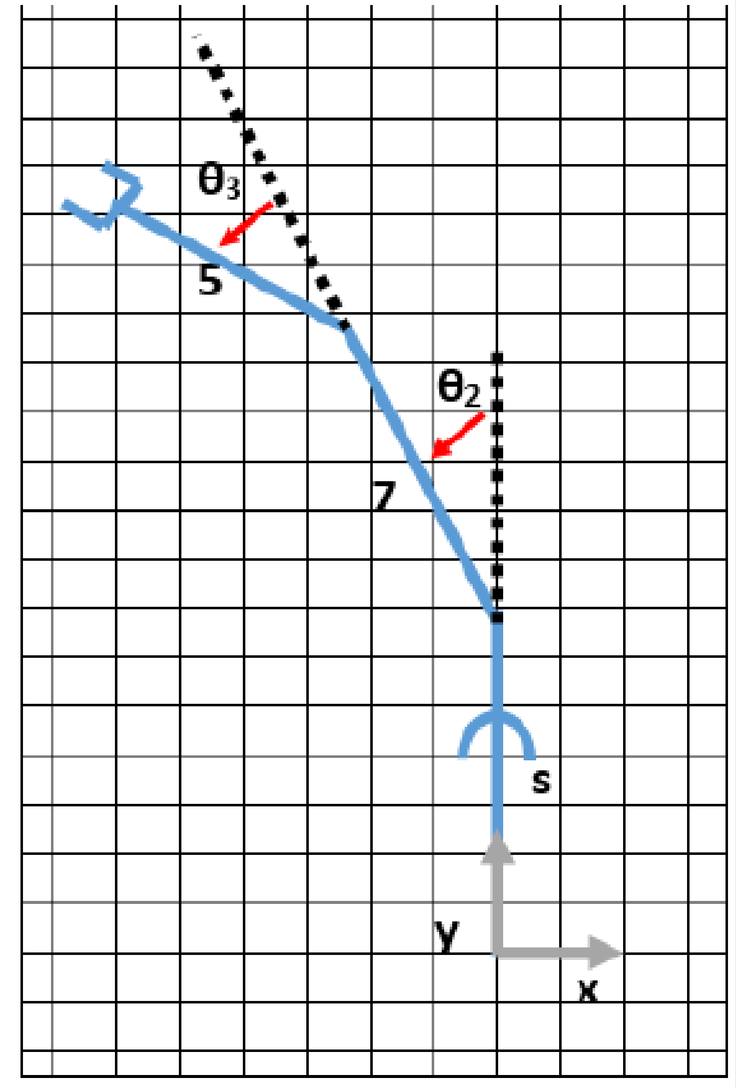
Workspace of Two-Link



Workspace of Two-Link



PRR Arm

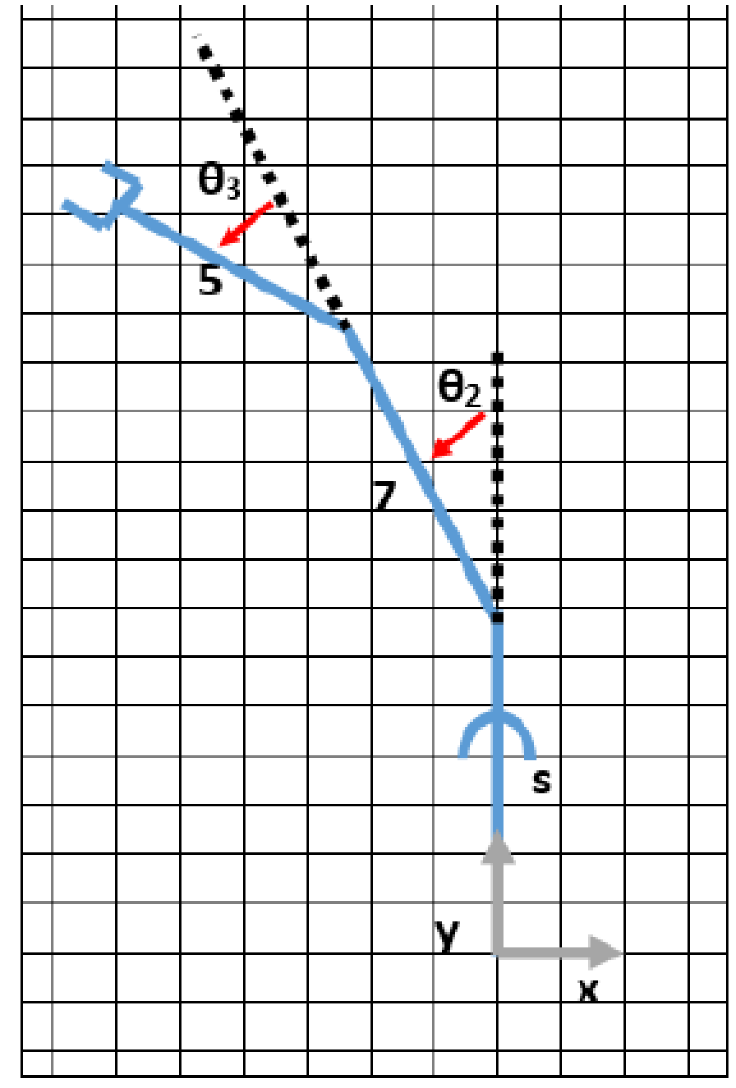


We have a PRR arm as shown below. The first joint is a prismatic joint starting the origin and continuing up along the y axis. It is of length s_1 and it is limited from 5 to 10 cm. The second and third joints are revolute joints with angles defined relative to the previous link where 0 is in line with the previous link and angles increasing counter-clockwise. The second link is of length 7. The third link is of length 5. The revolute joints (θ_2 and θ_3) are constrained between 0 and 180 degrees. The angle of the end effector is with respect to the positive x axis.

How many solutions are there to place the end effector at (0,22)?

How many solutions are there to place the end effector at (7,1)?

PRR Arm



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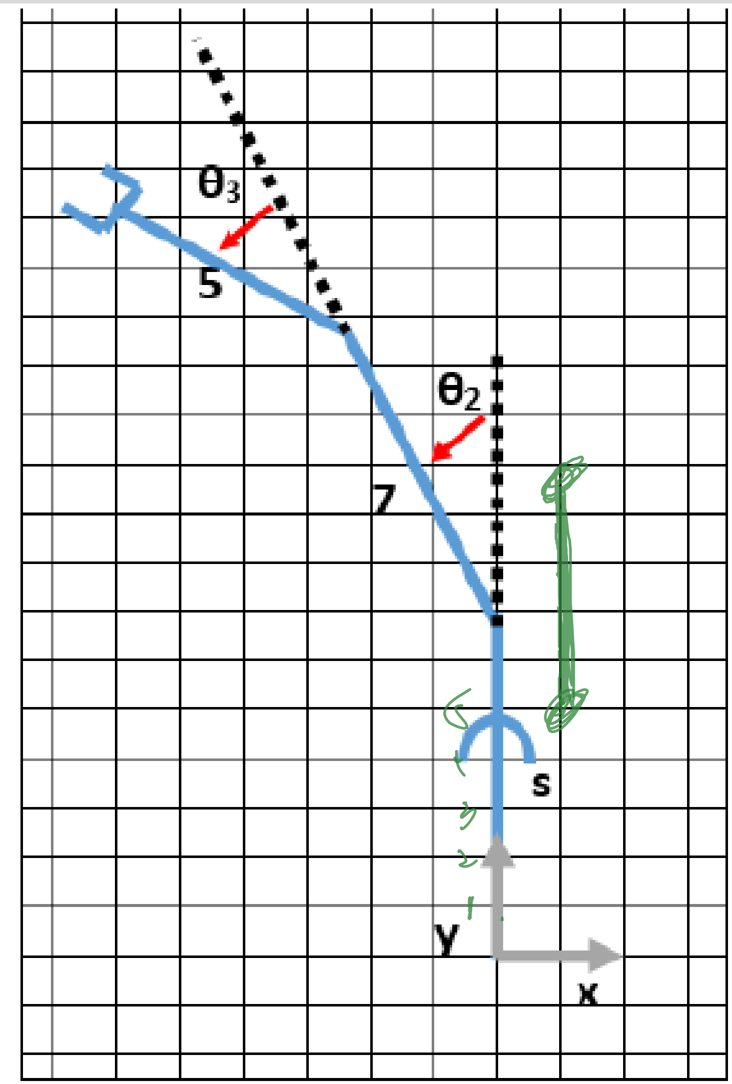
PRR Arm



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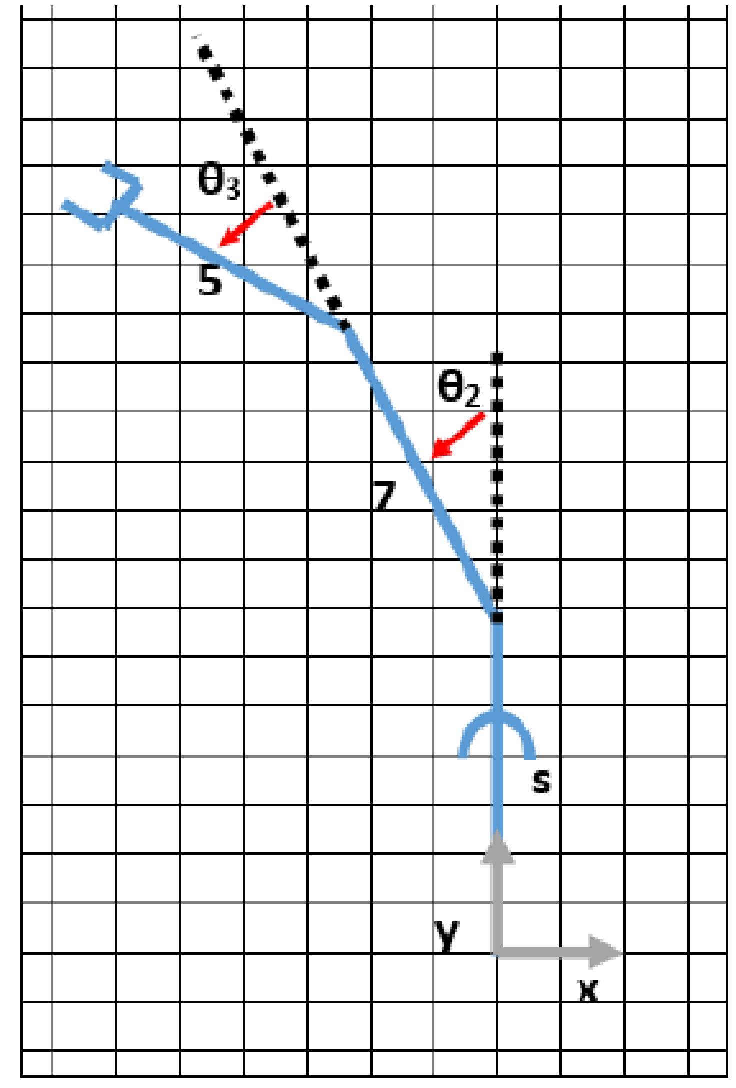
$x = -7s_2 - 5s_{23}$

$y = s + 7c_2 + 5c_{23}$

$\theta = \theta_2 + \theta_3 + 90$

PRR Arm



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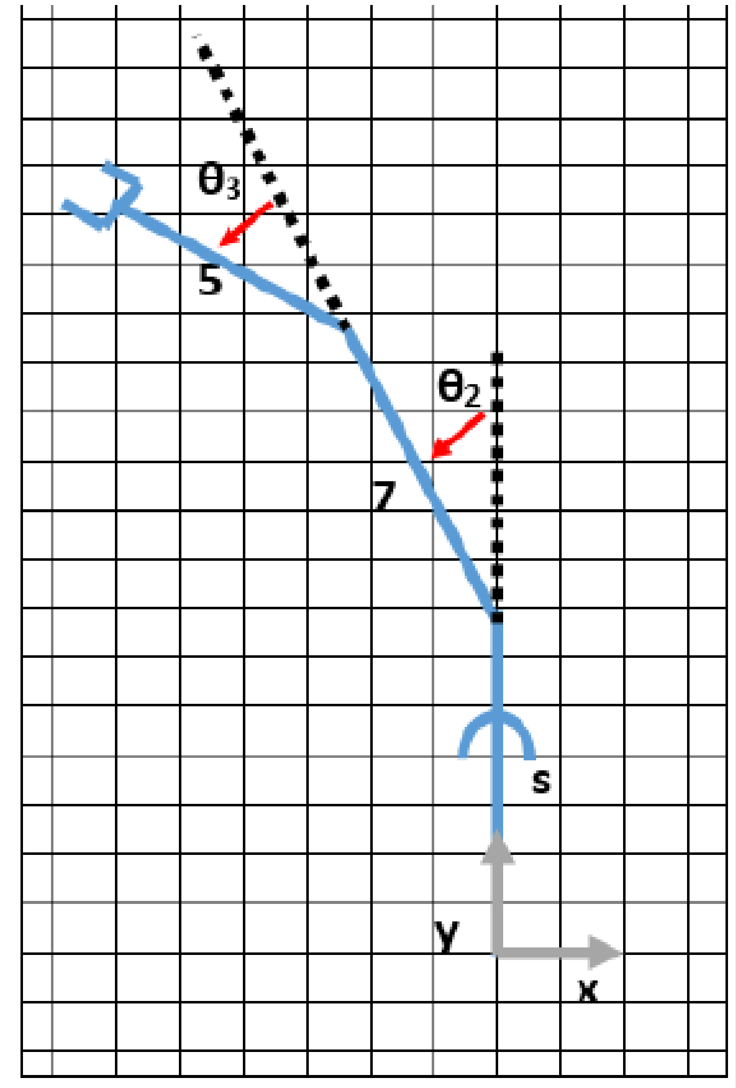
PRR Arm



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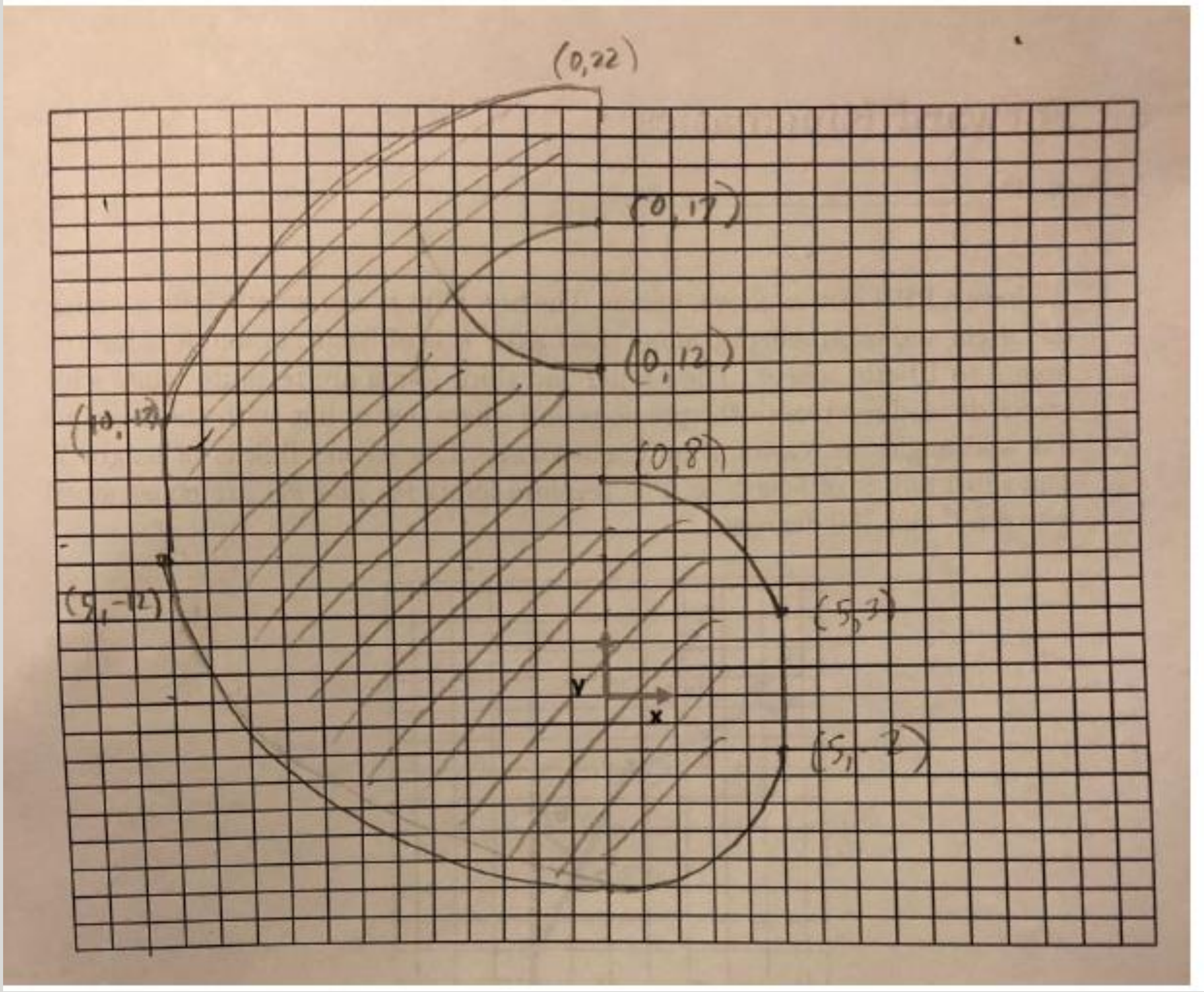
(0,22)
(0,17)
(0,12)
(10,17)
(0,8)
(5,-12)
(5,3)
Y
X
(5,-7)