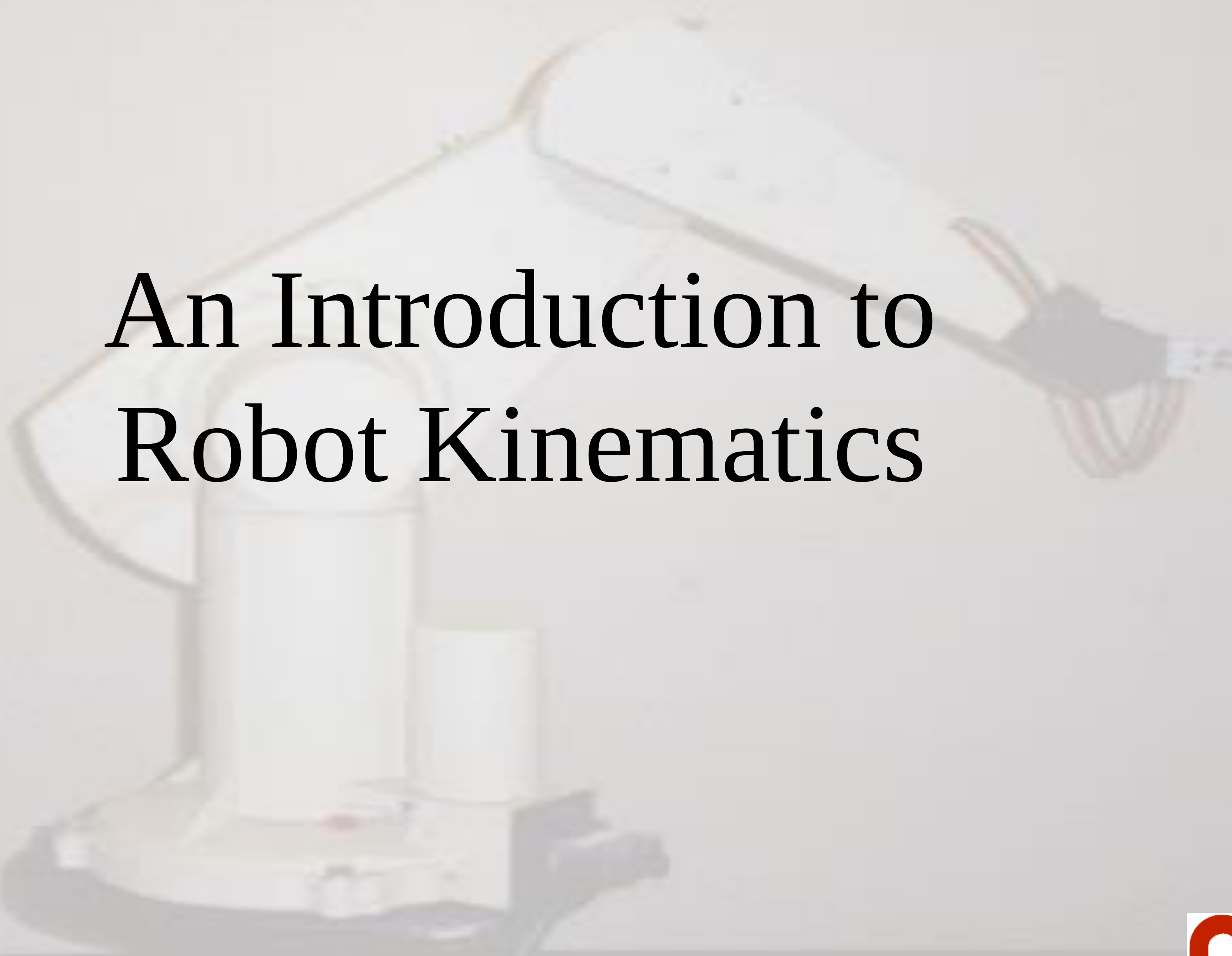
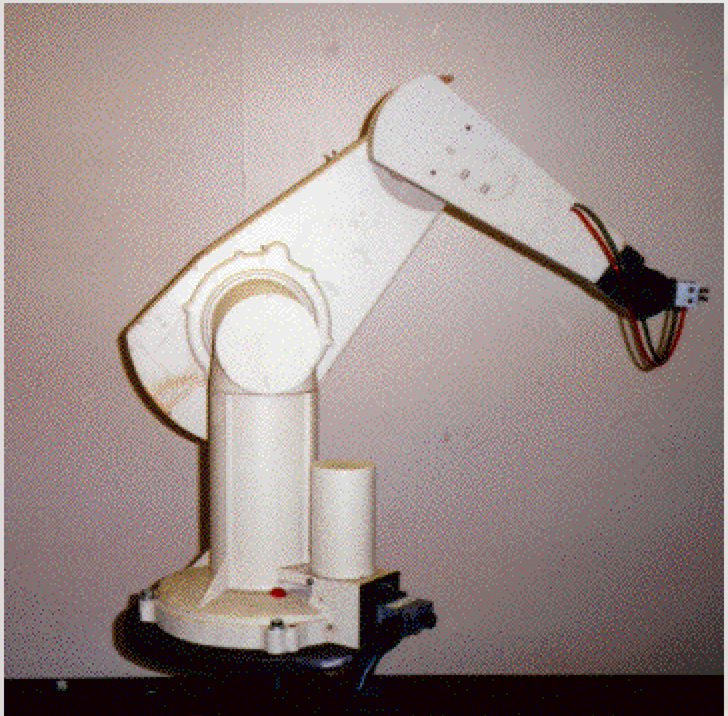
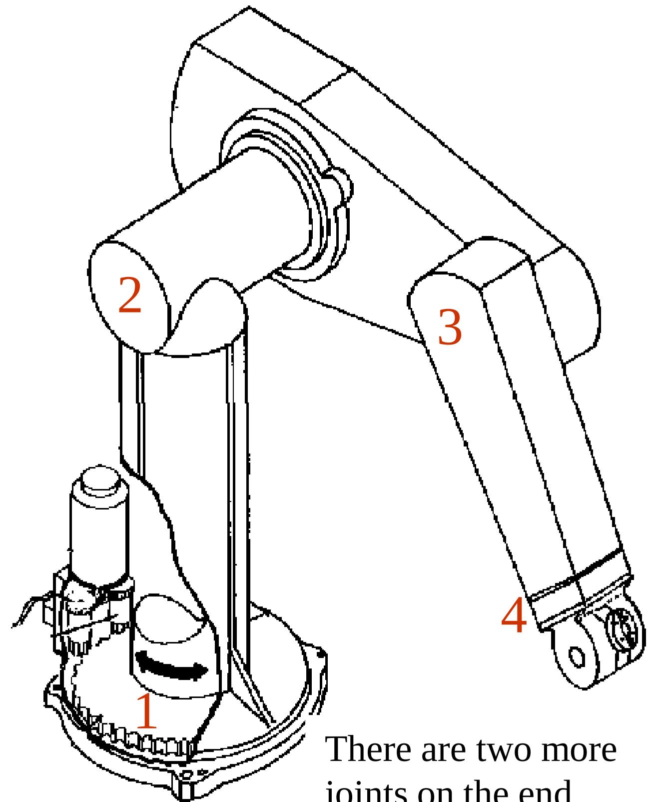


An Introduction to Robot Kinematics

An Example - The PUMA 560



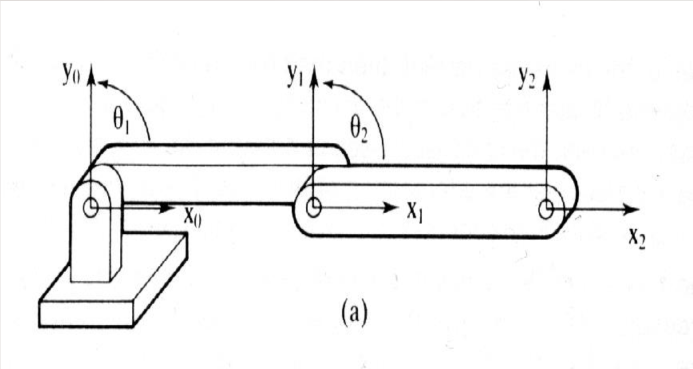


There are two more joints on the end effector (the gripper)

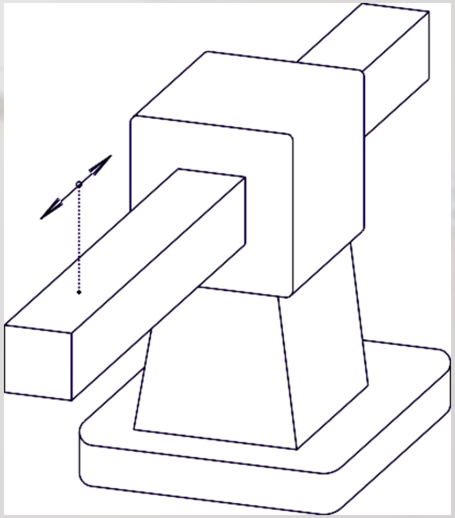
The PUMA 560 has SIX revolute joints

A revolute joint has ONE degree of freedom (1 DOF) that is defined by its angle

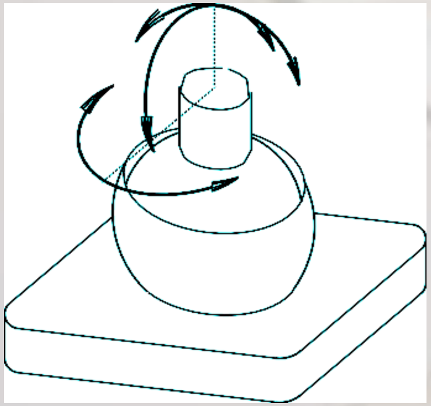
Other basic joints



Revolute Joint
1 DOF (Variable - Y)



Prismatic Joint
1 DOF (linear) (Variables - d)



Spherical Joint
3 DOF (Variables - Y_1, Y_2, Y_3)

We are interested in two kinematics topics

Forward Kinematics (angles to position)

What you are given:	The length of each link The angle of each joint
What you can find:	The position of any point (i.e. it’s (x, y, z) coordinates

Inverse Kinematics (position to angles)

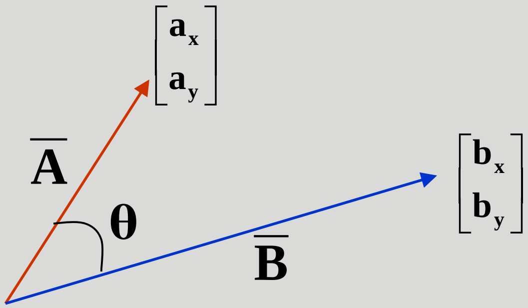
What you are given:	The length of each link The position of some point on the robot
What you can find:	The angles of each joint needed to obtain that position

Quick Math Review

Dot Product:

Geometric Representation:

$$\overline{\mathbf{A}} \bullet \overline{\mathbf{B}} = \|\mathbf{A}\|\|\mathbf{B}\|\cos\theta$$



Matrix Representation:

$$\overline{\mathbf{A}} \bullet \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{a_x} \\ \mathbf{a_y} \end{bmatrix} \bullet \begin{bmatrix} \mathbf{b_x} \\ \mathbf{b_y} \end{bmatrix} = \mathbf{a_x b_x} + \mathbf{a_y b_y}$$

Unit Vector

Vector in the direction of a chosen vector but whose magnitude is 1.

$$\overline{\mathbf{u}}_{\mathbf{B}} = \frac{\overline{\mathbf{B}}}{\|\mathbf{B}\|}$$

Quick Matrix Review

Matrix Multiplication:

An (m x n) matrix A and an (n x p) matrix B, can be multiplied since the number of columns of A is equal to the number of rows of B.

<u>Non-Commutative Multiplication</u>

AB is NOT equal to BA

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (ae+bg) & (af+bh) \\ (ce+dg) & (cf+dh) \end{bmatrix}$$

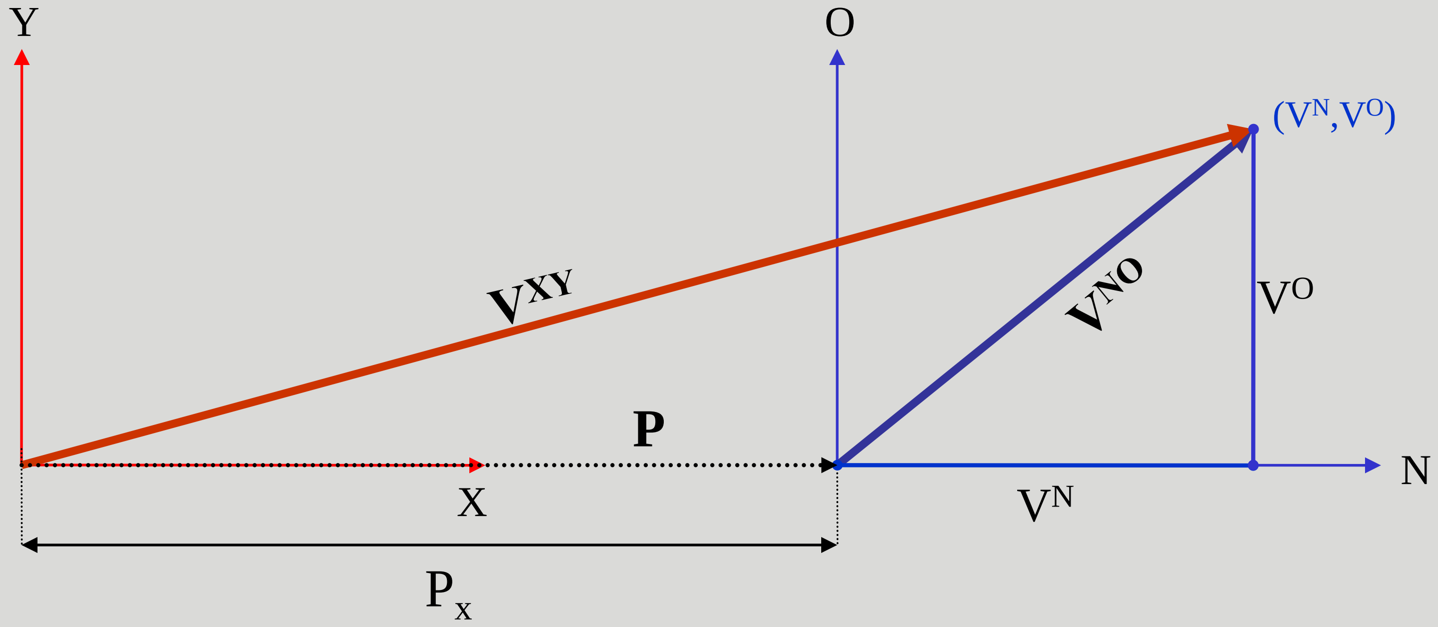
Matrix Addition:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix}$$

Basic Transformations

Moving Between Coordinate Frames

Translation Along the X-Axis

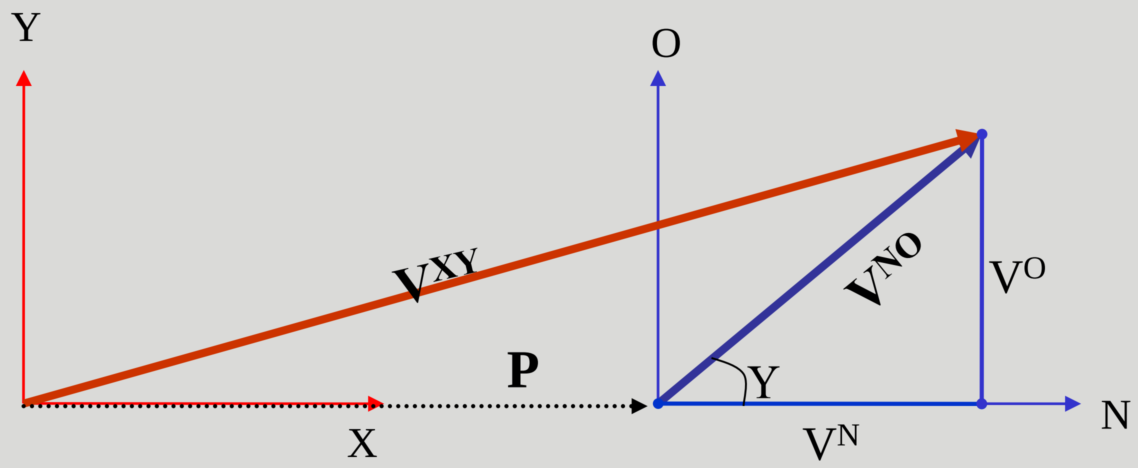


P_x = distance between the XY and NO coordinate planes

Notation:

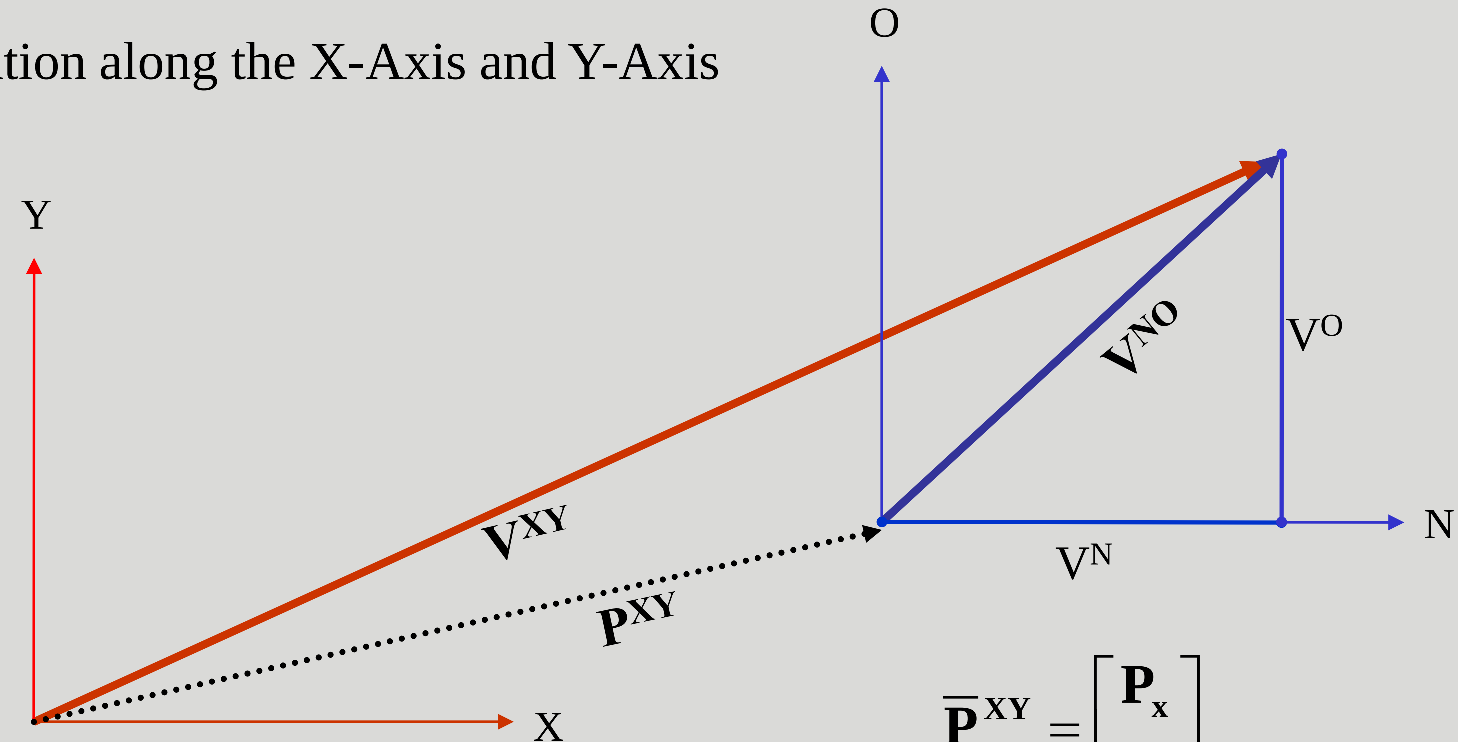
$$\overline{\mathbf{V}}^{XY} = \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \end{bmatrix} \qquad \overline{\mathbf{V}}^{NO} = \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \end{bmatrix} \qquad \overline{\mathbf{P}} = \begin{bmatrix} \mathbf{P_x} \\ \mathbf{0} \end{bmatrix}$$

Writing $\overline{\mathbf{V}}^{XY}$ in terms of $\overline{\mathbf{V}}^{NO}$



$$\overline{\mathbf{V}}^{XY} = \begin{bmatrix} \mathbf{P}_X + \mathbf{V}^N \\ \mathbf{V}^O \end{bmatrix} = \overline{\mathbf{P}} + \overline{\mathbf{V}}^{NO}$$

Translation along the X-Axis and Y-Axis



$$\overline{\mathbf{P}}^{XY} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_Y \end{bmatrix}$$

$$\overline{\mathbf{V}}^{XY} = \overline{\mathbf{P}} + \overline{\mathbf{V}}^{NO} = \begin{bmatrix} \mathbf{P}_X + \mathbf{V}^N \\ \mathbf{P}_Y + \mathbf{V}^O \end{bmatrix}$$

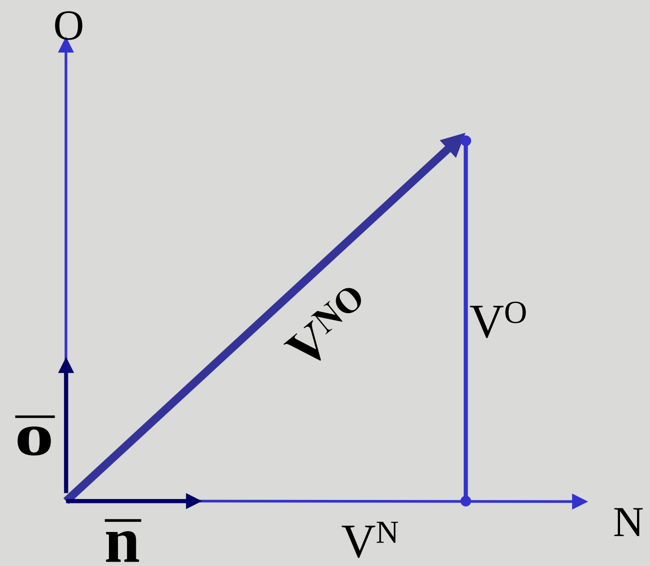
Using Basis Vectors

Basis vectors are unit vectors that point along a coordinate axis

$\overline{\mathbf{n}}$ Unit vector along the N-Axis

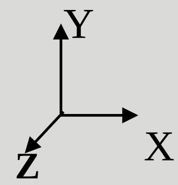
$\overline{\mathbf{o}}$ Unit vector along the N-Axis

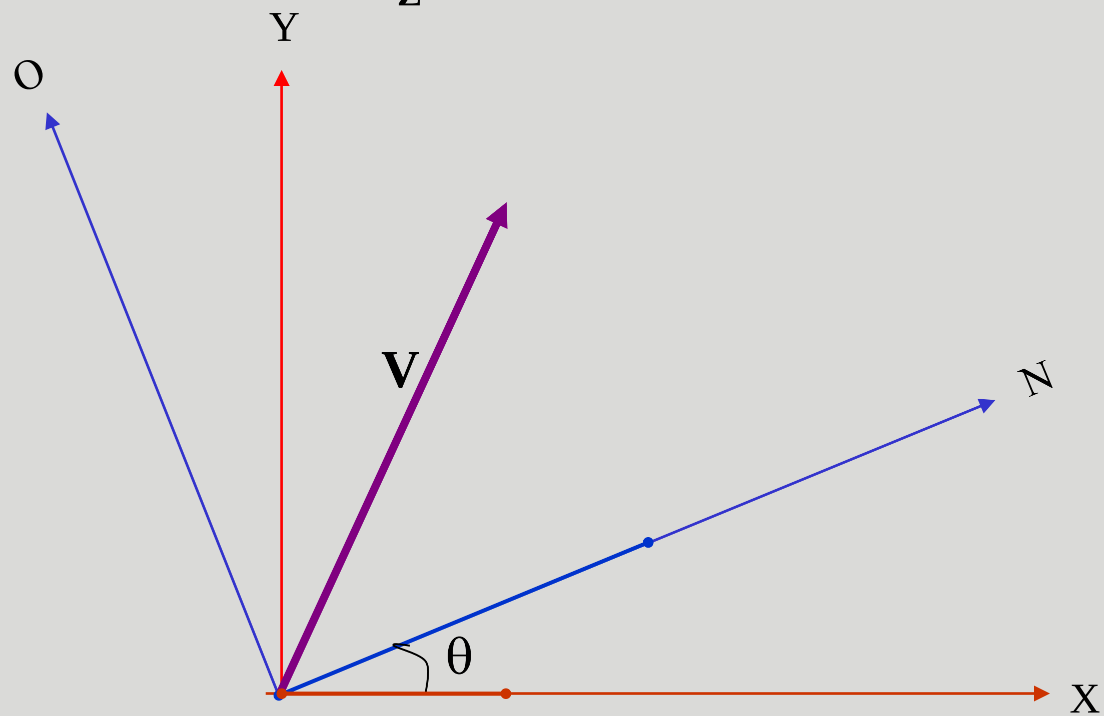
$\left\| \mathbf{V}^{NO} \right\|$ Magnitude of the V^{NO} vector



$$\overline{\mathbf{V}}^{NO} = \begin{bmatrix} \mathbf{V}^{N} \\ \mathbf{V}^{O} \end{bmatrix} = \begin{bmatrix} \left\| \mathbf{V}^{NO} \right\| \mathbf{cos\theta} \\ \left\| \mathbf{V}^{NO} \right\| \mathbf{sin\theta} \end{bmatrix} = \begin{bmatrix} \left\| \mathbf{V}^{NO} \right\| \mathbf{cos\theta} \\ \left\| \mathbf{V}^{NO} \right\| \mathbf{cos(90-\theta)} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{V}}^{NO} \bullet \overline{\mathbf{n}} \\ \overline{\mathbf{V}}^{NO} \bullet \overline{\mathbf{o}} \end{bmatrix}$$

Rotation (around the Z-Axis)

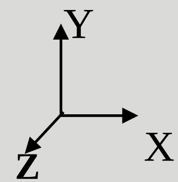


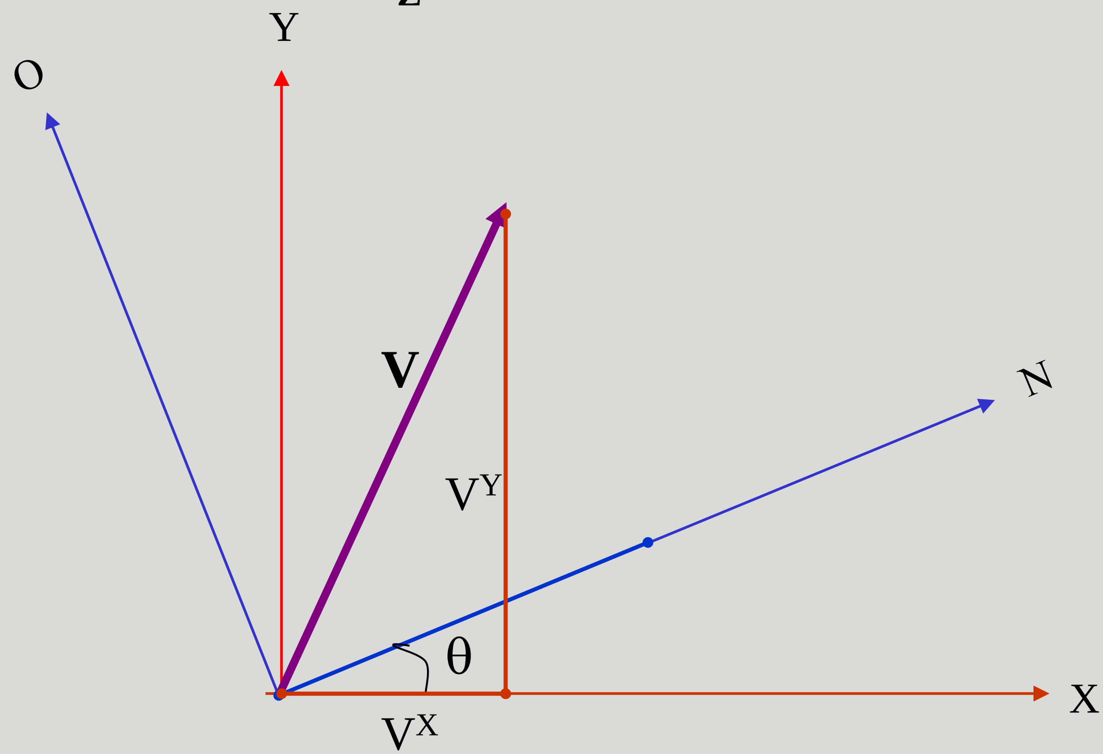


θ = Angle of rotation between the XY and NO coordinate axis

V Can be considered with respect to the XY coordinates or NO coordinates

Rotation (around the Z-Axis)





θ = Angle of rotation between the XY and NO coordinate axis

$$\overline{\mathbf{V}}^{\mathbf{XY}} = \begin{bmatrix} \mathbf{V}^{\mathbf{X}} \\ \mathbf{V}^{\mathbf{Y}} \end{bmatrix}$$

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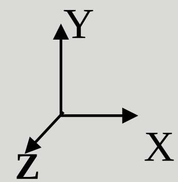
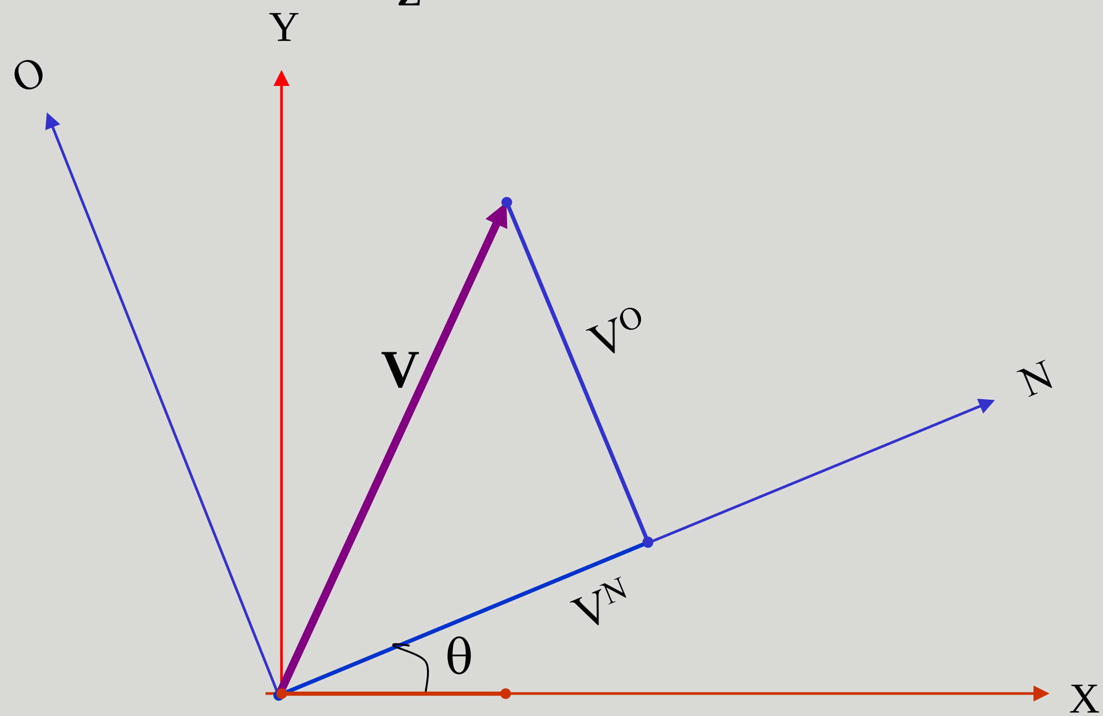
Rotation (around the Z-Axis)

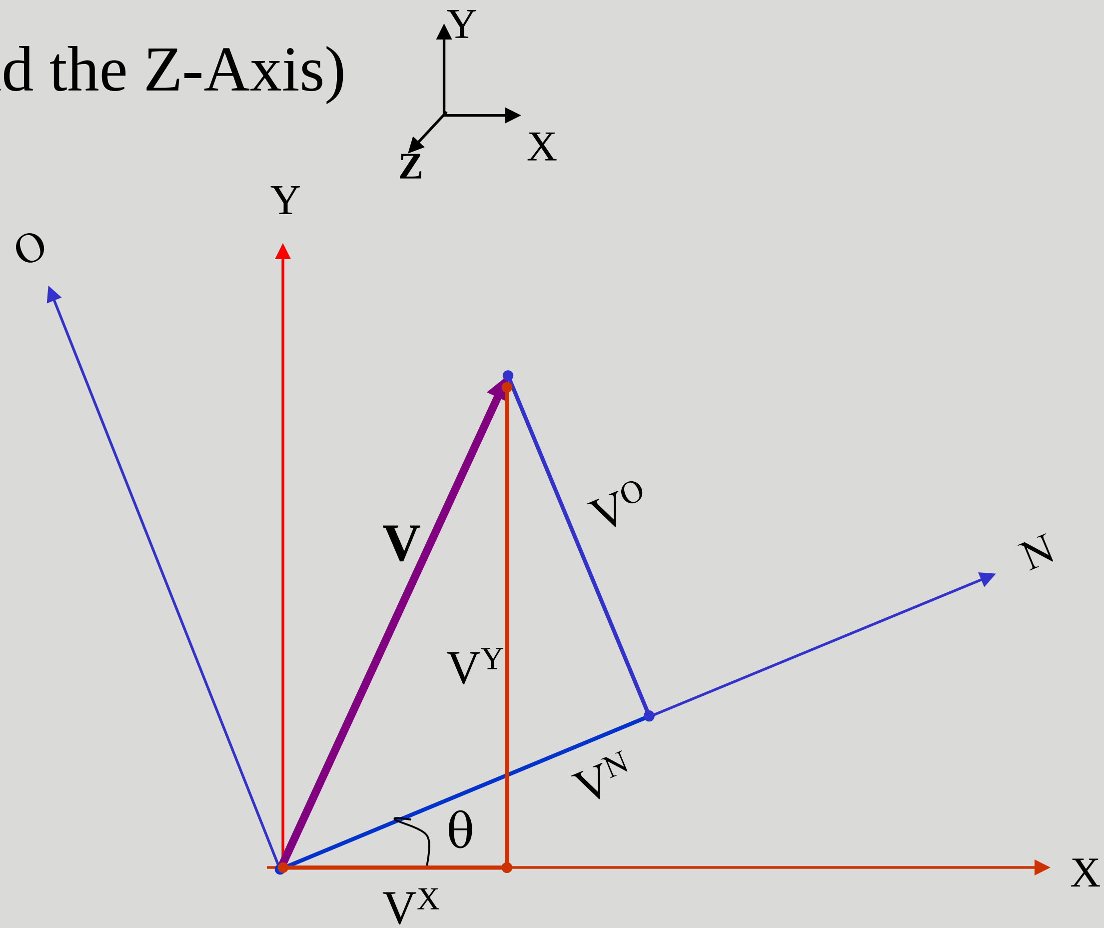




θ = Angle of rotation between the XY and NO coordinate axis

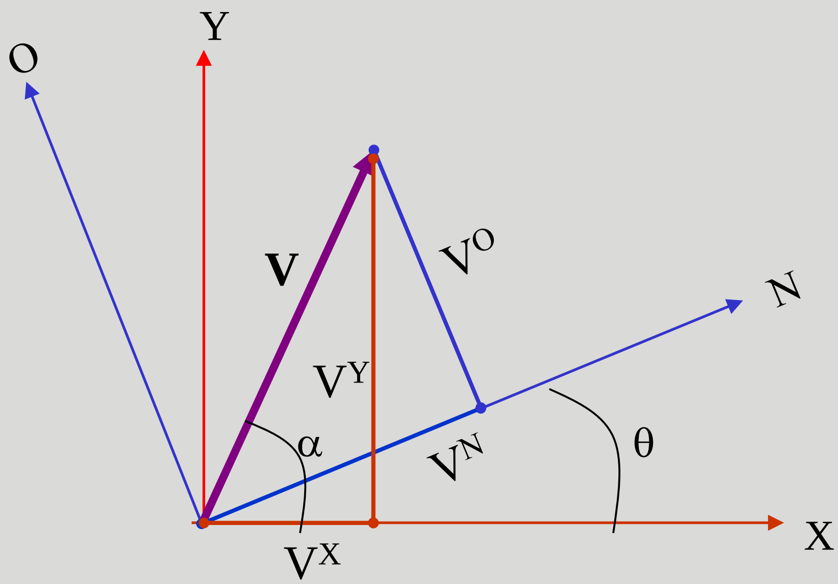
$$\overline{\mathbf{V}}^{\mathbf{NO}} = \begin{bmatrix} \mathbf{V}^{\mathbf{N}} \\ \mathbf{V}^{\mathbf{O}} \end{bmatrix}$$

Rotation (around the Z-Axis)



θ = Angle of rotation between the XY and NO coordinate axis

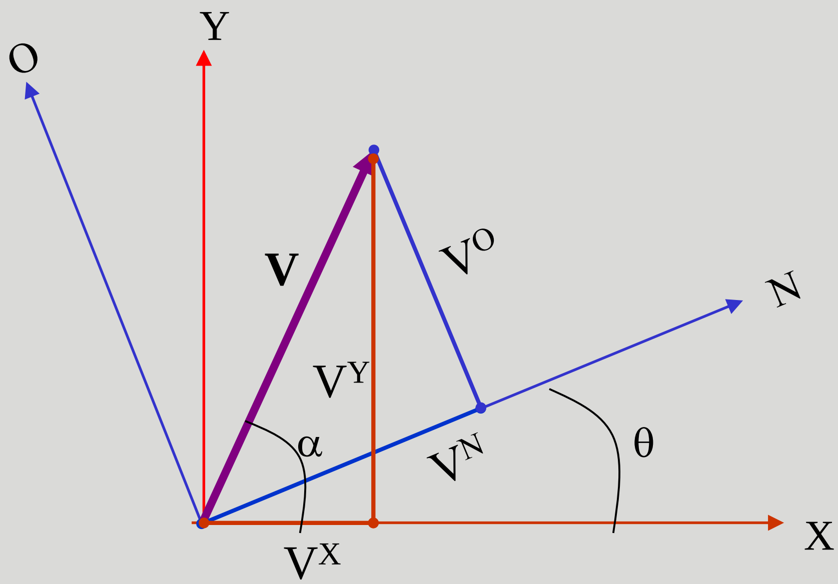
$$\overline{\mathbf{V}}^{XY} = \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \end{bmatrix} \qquad \overline{\mathbf{V}}^{NO} = \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \end{bmatrix}$$



$\overline{\mathbf{x}}$ Unit vector along X-Axis

$$\left\|\overline{\mathbf{V}}^{\mathbf{XY}}\right\| = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\|$$

$$\mathbf{V}^{\mathbf{X}} = \left\|\overline{\mathbf{V}}^{\mathbf{XY}}\right\|\mathbf{cos}\boldsymbol{\alpha} = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\|\mathbf{cos}\boldsymbol{\alpha} = \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{x}}$$



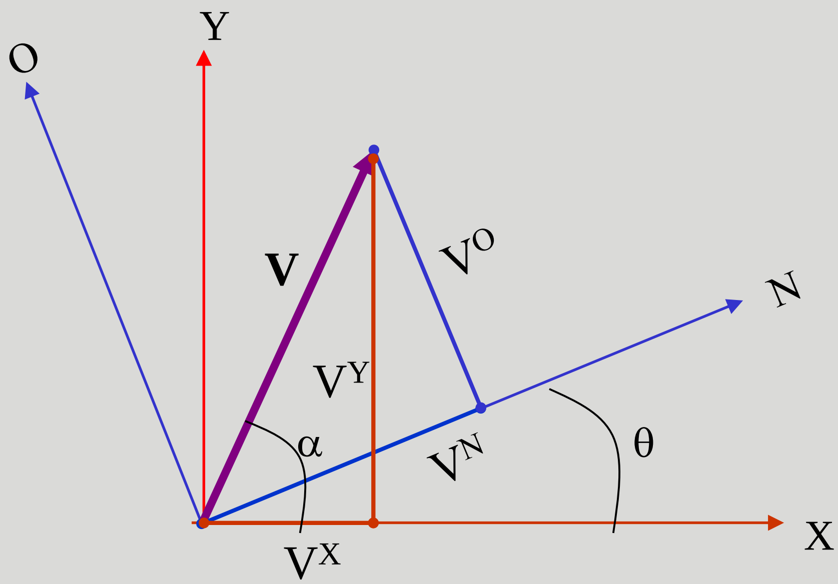
$\overline{\mathbf{x}}$ Unit vector along X-Axis

$$\left\|\overline{\mathbf{V}}^{\mathbf{XY}}\right\| = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\|$$

$$\mathbf{V}^{\mathbf{X}} = \left\|\overline{\mathbf{V}}^{\mathbf{XY}}\right\| \cos\alpha = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\| \cos\alpha = \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{x}}$$

$$\mathbf{V}^{\mathbf{X}} = (\mathbf{V}^{\mathbf{N}} * \overline{\mathbf{n}} + \mathbf{V}^{\mathbf{O}} * \overline{\mathbf{o}}) \bullet \overline{\mathbf{x}}$$

(Substituting for V^{NO} using the N and O components of the vector)



$\overline{\mathbf{x}}$ Unit vector along X-Axis

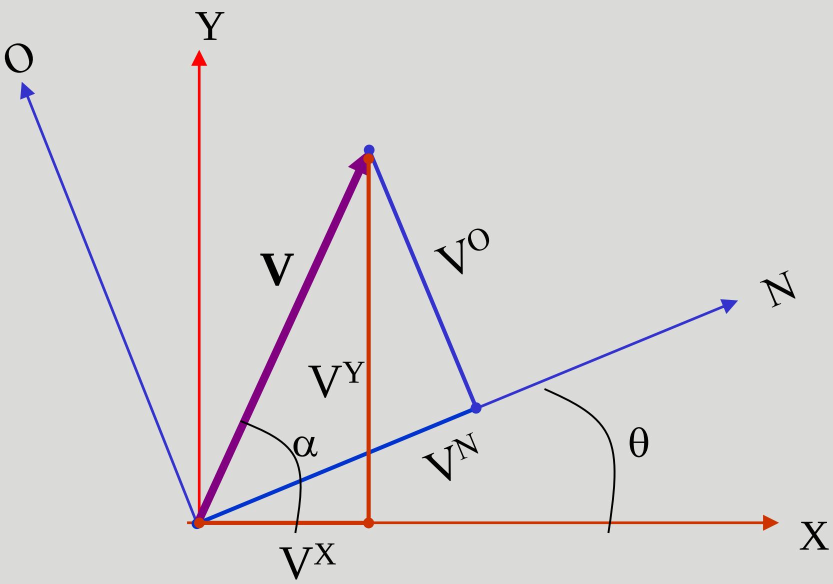
$$\left\|\overline{\mathbf{V}}^{\mathbf{XY}}\right\| = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\|$$

$$\mathbf{V}^{\mathbf{X}} = \left\|\overline{\mathbf{V}}^{\mathbf{XY}}\right\| \cos\alpha = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\| \cos\alpha = \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{x}}$$

$$\mathbf{V}^{\mathbf{X}} = (\mathbf{V}^{\mathbf{N}} * \overline{\mathbf{n}} + \mathbf{V}^{\mathbf{O}} * \overline{\mathbf{o}}) \bullet \overline{\mathbf{x}}$$

(Substituting for V^{NO} using the N and O components of the vector)

$$\mathbf{V}^{\mathbf{X}} = \mathbf{V}^{\mathbf{N}}(\overline{\mathbf{x}} \bullet \overline{\mathbf{n}}) + \mathbf{V}^{\mathbf{O}}(\overline{\mathbf{x}} \bullet \overline{\mathbf{o}})$$



$\overline{\mathbf{x}}$ Unit vector along X-Axis

$$\left\|\overline{\mathbf{V}}^{XY}\right\| = \left\|\overline{\mathbf{V}}^{NO}\right\|$$

$$\mathbf{V}^{X} = \left\|\overline{\mathbf{V}}^{XY}\right\|\cos\alpha = \left\|\overline{\mathbf{V}}^{NO}\right\|\cos\alpha = \overline{\mathbf{V}}^{NO} \bullet \overline{\mathbf{x}}$$

$$\mathbf{V}^{X} = (\mathbf{V}^{N} * \overline{\mathbf{n}} + \mathbf{V}^{O} * \overline{\mathbf{o}}) \bullet \overline{\mathbf{x}}$$

(Substituting for V^{NO} using the N and O components of the vector)

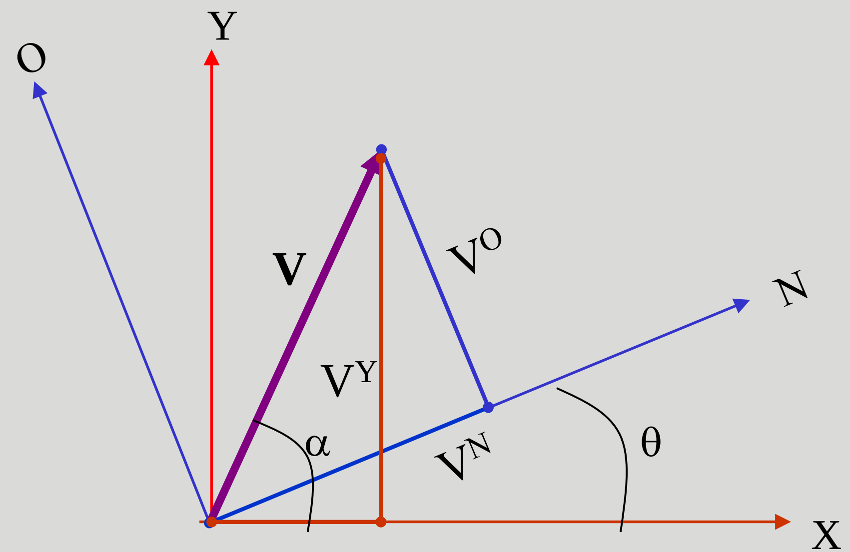
$$\mathbf{V}^{X} = \mathbf{V}^{N}(\overline{\mathbf{x}} \bullet \overline{\mathbf{n}}) + \mathbf{V}^{O}(\overline{\mathbf{x}} \bullet \overline{\mathbf{o}})$$

$$= \mathbf{V}^{N}(\cos\theta) + \mathbf{V}^{O}(\cos(\theta + 90))$$

$$= \mathbf{V}^{N}(\cos\theta) - \mathbf{V}^{O}(\sin\theta)$$

Similarly….

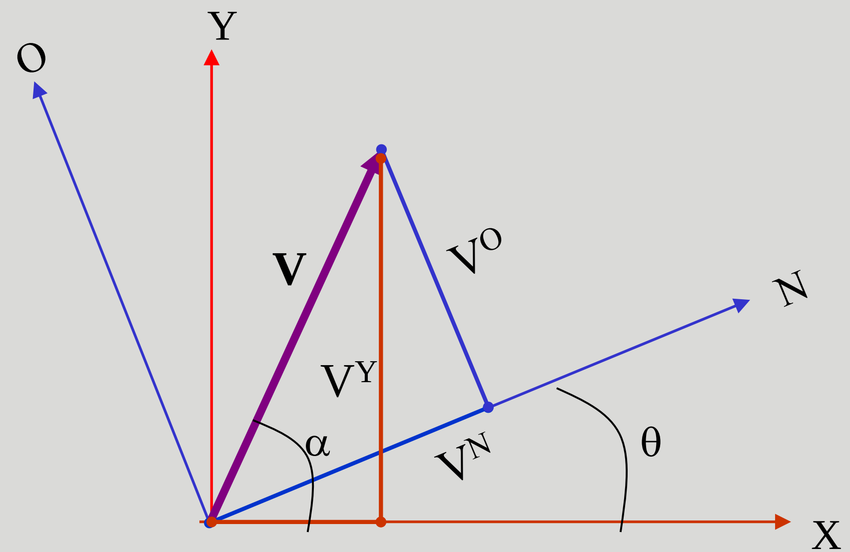
$$\mathbf{V}^{\mathbf{Y}} = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\| \sin\alpha = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\| \cos(90-\alpha) = \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{y}}$$



Similarly….

$$\mathbf{V}^{\mathbf{Y}} = \left\| \overline{\mathbf{V}}^{\mathbf{NO}} \right\| \sin\alpha = \left\| \overline{\mathbf{V}}^{\mathbf{NO}} \right\| \cos(90-\alpha) = \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{y}}$$

$$\mathbf{V}^{\mathbf{Y}} = (\mathbf{V}^{\mathbf{N}} * \overline{\mathbf{n}} + \mathbf{V}^{\mathbf{O}} * \overline{\mathbf{o}}) \bullet \overline{\mathbf{y}}$$

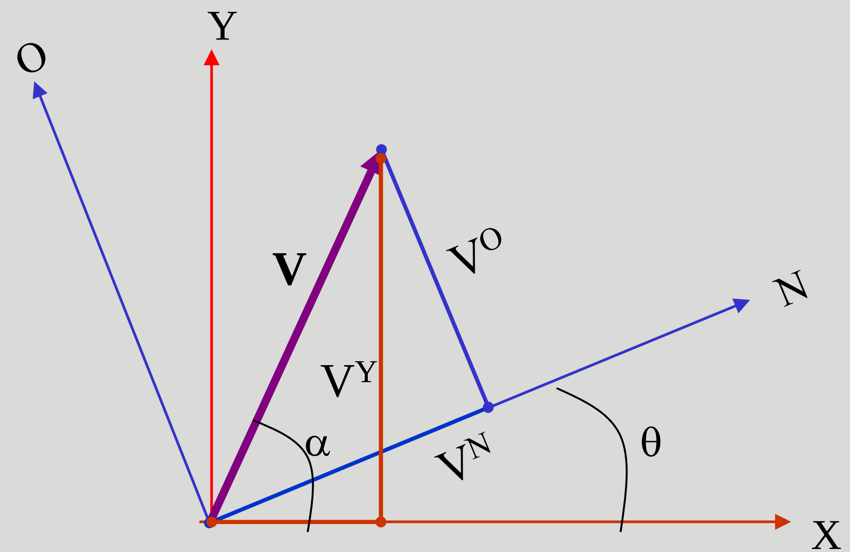


Similarly….

$$\mathbf{V}^{\mathbf{Y}} = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\| \sin\boldsymbol{\alpha} = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\| \cos(\mathbf{90} - \boldsymbol{\alpha}) = \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{y}}$$

$$\mathbf{V}^{\mathbf{Y}} = (\mathbf{V}^{\mathbf{N}} * \overline{\mathbf{n}} + \mathbf{V}^{\mathbf{O}} * \overline{\mathbf{o}}) \bullet \overline{\mathbf{y}}$$

$$\mathbf{V}^{\mathbf{Y}} = \mathbf{V}^{\mathbf{N}}(\overline{\mathbf{y}} \bullet \overline{\mathbf{n}}) + \mathbf{V}^{\mathbf{O}}(\overline{\mathbf{y}} \bullet \overline{\mathbf{o}})$$



Similarly….

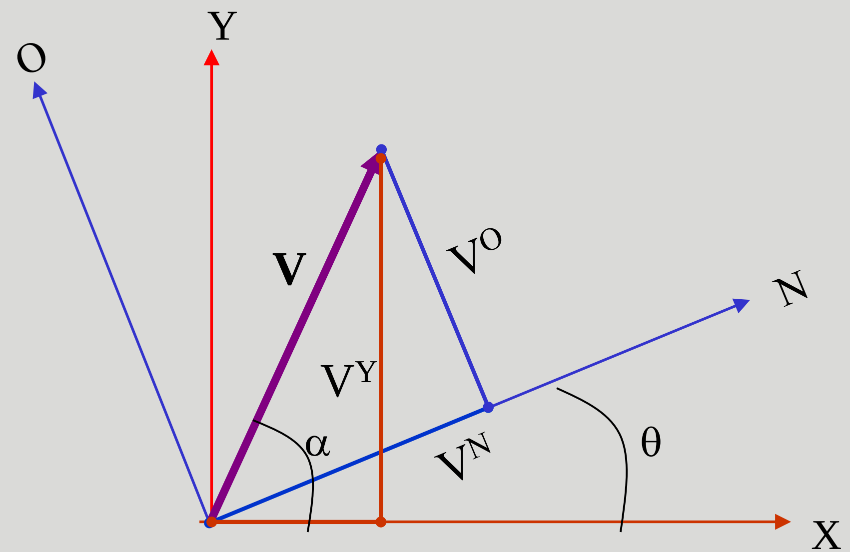
$$\mathbf{V}^{Y} = \left\|\overline{\mathbf{V}}^{NO}\right\| \sin\alpha = \left\|\overline{\mathbf{V}}^{NO}\right\| \cos(90-\alpha) = \overline{\mathbf{V}}^{NO} \bullet \overline{\mathbf{y}}$$

$$\mathbf{V}^{Y} = (\mathbf{V}^{N} * \overline{\mathbf{n}} + \mathbf{V}^{O} * \overline{\mathbf{o}}) \bullet \overline{\mathbf{y}}$$

$$\mathbf{V}^{Y} = \mathbf{V}^{N}(\overline{\mathbf{y}} \bullet \overline{\mathbf{n}}) + \mathbf{V}^{O}(\overline{\mathbf{y}} \bullet \overline{\mathbf{o}})$$

$$= \mathbf{V}^{N}(\cos(90-\theta)) + \mathbf{V}^{O}(\cos\theta)$$

$$= \mathbf{V}^{N}(\sin\theta) + \mathbf{V}^{O}(\cos\theta)$$



Similarly….

$$V^Y = \left\| \overline{V}^{NO} \right\| \sin\alpha = \left\| \overline{V}^{NO} \right\| \cos(90-\alpha) = \overline{V}^{NO} \bullet \overline{y}$$

$$V^Y = (V^N * \overline{n} + V^O * \overline{o}) \bullet \overline{y}$$

$$V^Y = V^N(\overline{y} \bullet \overline{n}) + V^O(\overline{y} \bullet \overline{o})$$

$$= V^N(\cos(90-\theta)) + V^O(\cos\theta)$$

$$= V^N(\sin\theta) + V^O(\cos\theta)$$

So….

$$V^X = V^N(\cos\theta) - V^O(\sin\theta)$$

$$V^Y = V^N(\sin\theta) + V^O(\cos\theta)$$

$$\overline{V}^{XY} = \begin{bmatrix} V^X \\ V^Y \end{bmatrix}$$

Similarly….

$$\mathbf{V^Y} = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\| \sin\alpha = \left\|\overline{\mathbf{V}}^{\mathbf{NO}}\right\| \cos(90-\alpha) = \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{y}}$$

$$\mathbf{V^Y} = (\mathbf{V^N} * \overline{\mathbf{n}} + \mathbf{V^O} * \overline{\mathbf{o}}) \bullet \overline{\mathbf{y}}$$

$$\begin{aligned}\mathbf{V^Y} &= \mathbf{V^N}(\overline{\mathbf{y}} \bullet \overline{\mathbf{n}}) + \mathbf{V^O}(\overline{\mathbf{y}} \bullet \overline{\mathbf{o}}) \\ &= \mathbf{V^N}(\cos(90-\theta)) + \mathbf{V^O}(\cos\theta) \\ &= \mathbf{V^N}(\sin\theta) + \mathbf{V^O}(\cos\theta)\end{aligned}$$

So….

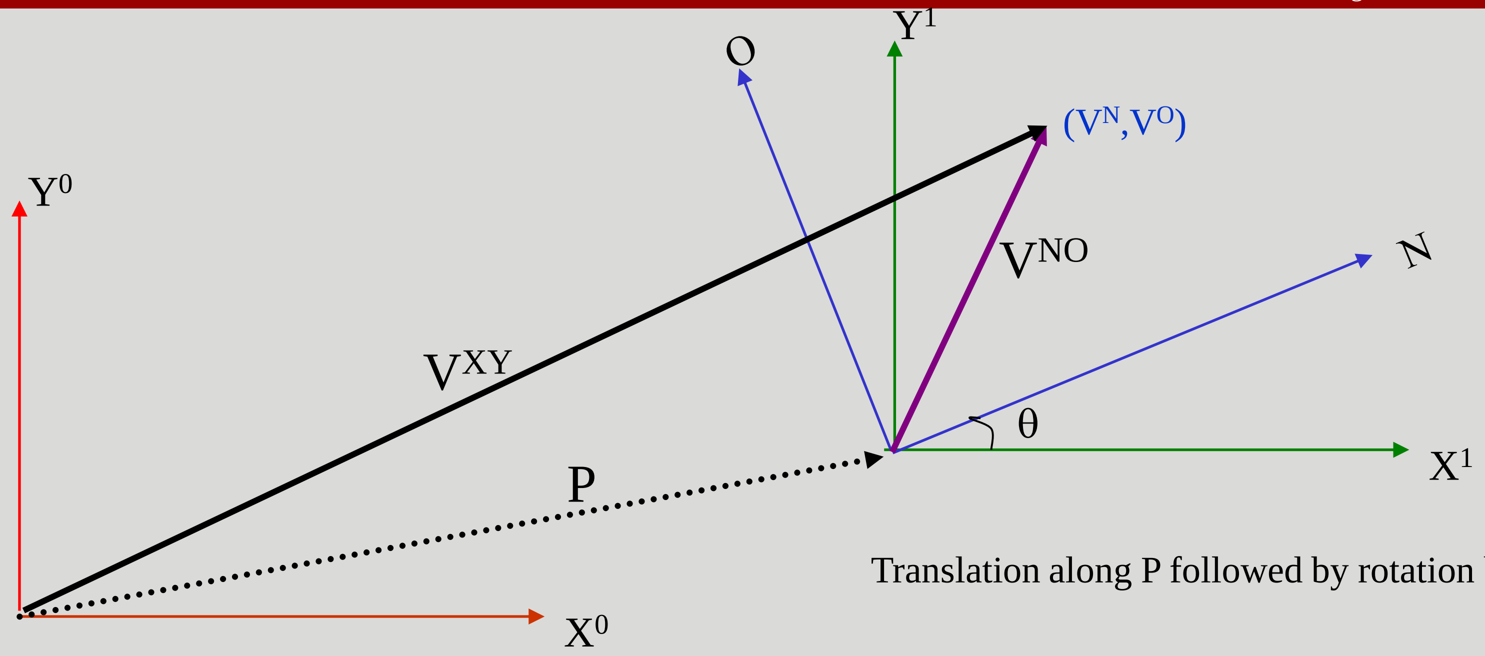
$$\mathbf{V^X} = \mathbf{V^N}(\cos\theta) - \mathbf{V^O}(\sin\theta)$$
$$\mathbf{V^Y} = \mathbf{V^N}(\sin\theta) + \mathbf{V^O}(\cos\theta)$$

$$\overline{\mathbf{V}}^{\mathbf{XY}} = \begin{bmatrix} \mathbf{V^X} \\ \mathbf{V^Y} \end{bmatrix}$$

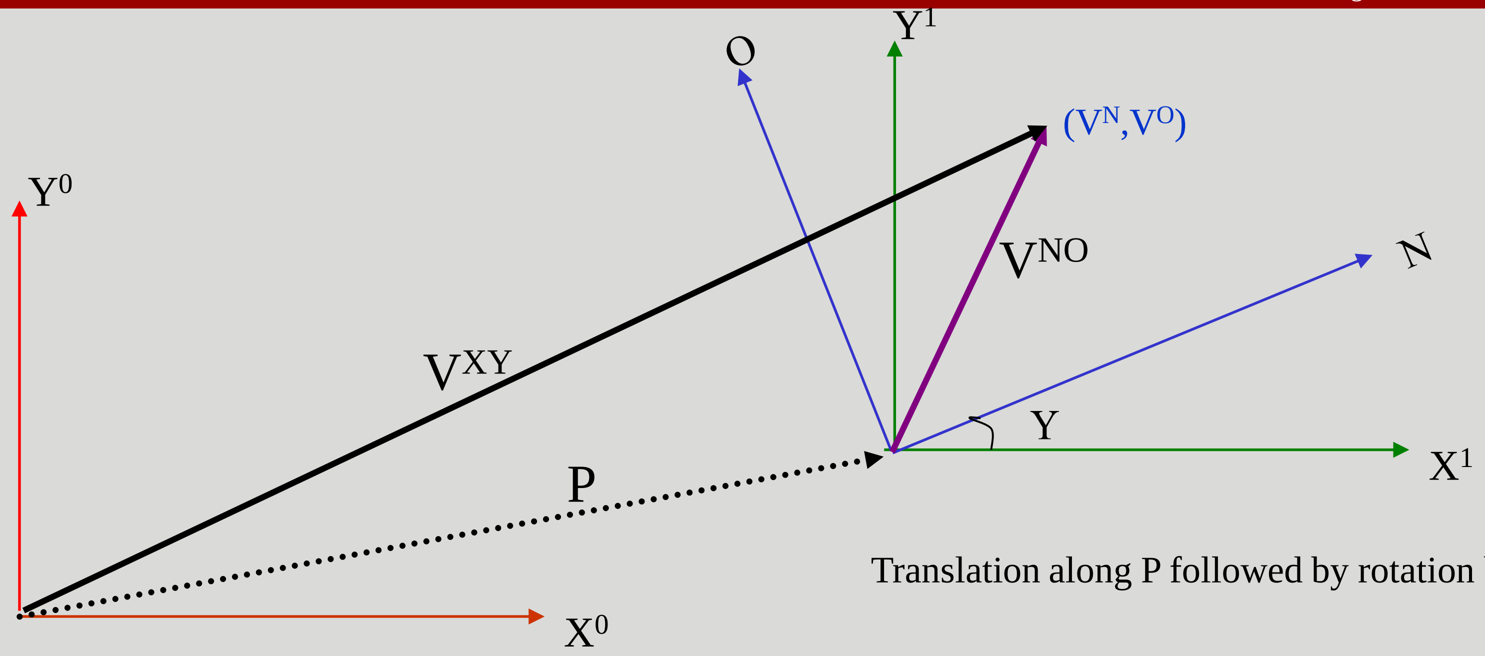
Written in Matrix Form

$$\overline{\mathbf{V}}^{\mathbf{XY}} = \begin{bmatrix} \mathbf{V^X} \\ \mathbf{V^Y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V^N} \\ \mathbf{V^O} \end{bmatrix}$$

Rotation Matrix about the z-axis

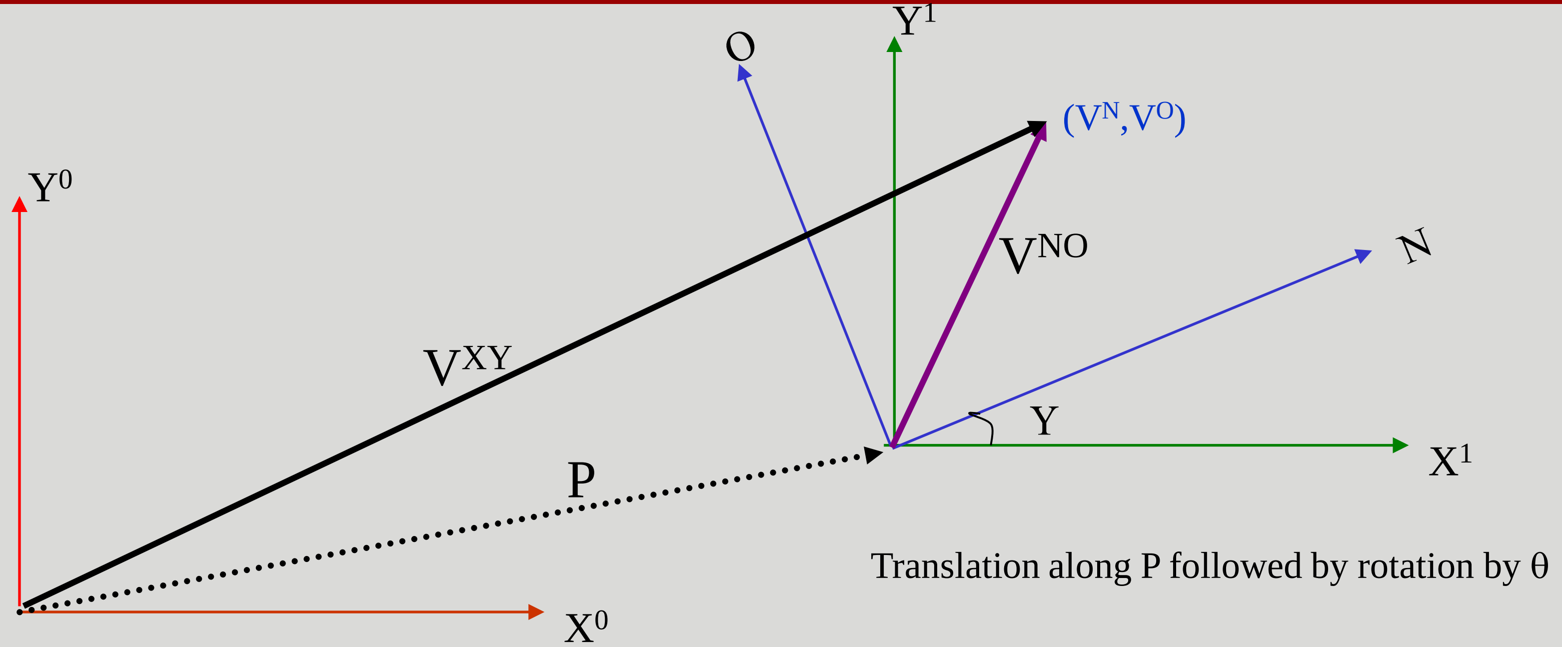


Translation along P followed by rotation by θ



Translation along P followed by rotation by θ

$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{V}^{X} \\ \mathbf{V}^{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V}^{N} \\ \mathbf{V}^{O} \end{bmatrix}$$



Translation along P followed by rotation by θ

$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{V}^{X} \\ \mathbf{V}^{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V}^{N} \\ \mathbf{V}^{O} \end{bmatrix}$$

(Note : P_x, P_y are relative to the original coordinate frame. Translation followed by rotation is different than rotation followed by translation.)

In other words, knowing the coordinates of a point (V^N, V^O) in some coordinate frame (NO) you can find the position of that point relative to your original coordinate frame (X^0Y^0).

HOMOGENEOUS REPRESENTATION

Putting it all into a Matrix

$$\mathbf{V}^{\mathbf{XY}} = \begin{bmatrix} \mathbf{V}^{\mathbf{X}} \\ \mathbf{V}^{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\mathbf{x}} \\ \mathbf{P}_{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \mathbf{cos\theta} & -\mathbf{sin\theta} \\ \mathbf{sin\theta} & \mathbf{cos\theta} \end{bmatrix} \begin{bmatrix} \mathbf{V}^{\mathbf{N}} \\ \mathbf{V}^{\mathbf{O}} \end{bmatrix}$$

What we found by doing a translation and a rotation

HOMOGENEOUS REPRESENTATION

Putting it all into a Matrix

$$\mathbf{V}^{\mathbf{XY}} = \begin{bmatrix} \mathbf{V}^{\mathbf{X}} \\ \mathbf{V}^{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\mathbf{x}} \\ \mathbf{P}_{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \mathbf{cos\theta} & -\mathbf{sin\theta} \\ \mathbf{sin\theta} & \mathbf{cos\theta} \end{bmatrix} \begin{bmatrix} \mathbf{V}^{\mathbf{N}} \\ \mathbf{V}^{\mathbf{O}} \end{bmatrix}$$

What we found by doing a translation and a rotation

$$= \begin{bmatrix} \mathbf{V}^{\mathbf{X}} \\ \mathbf{V}^{\mathbf{Y}} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\mathbf{x}} \\ \mathbf{P}_{\mathbf{y}} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{cos\theta} & -\mathbf{sin\theta} & \mathbf{0} \\ \mathbf{sin\theta} & \mathbf{cos\theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{V}^{\mathbf{N}} \\ \mathbf{V}^{\mathbf{O}} \\ \mathbf{1} \end{bmatrix}$$

Padding with 0’s and 1’s

HOMOGENEOUS REPRESENTATION

Putting it all into a Matrix

$$\mathbf{V}^{\mathbf{XY}} = \begin{bmatrix} \mathbf{V}^{\mathbf{X}} \\ \mathbf{V}^{\mathbf{Y}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\mathbf{x}} \\ \mathbf{P}_{\mathbf{y}} \end{bmatrix} + \begin{bmatrix} \mathbf{cos\theta} & -\mathbf{sin\theta} \\ \mathbf{sin\theta} & \mathbf{cos\theta} \end{bmatrix} \begin{bmatrix} \mathbf{V}^{\mathbf{N}} \\ \mathbf{V}^{\mathbf{O}} \end{bmatrix}$$

What we found by doing a translation and a rotation

$$= \begin{bmatrix} \mathbf{V}^{\mathbf{X}} \\ \mathbf{V}^{\mathbf{Y}} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{\mathbf{x}} \\ \mathbf{P}_{\mathbf{y}} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{cos\theta} & -\mathbf{sin\theta} & \mathbf{0} \\ \mathbf{sin\theta} & \mathbf{cos\theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{V}^{\mathbf{N}} \\ \mathbf{V}^{\mathbf{O}} \\ \mathbf{1} \end{bmatrix}$$

Padding with 0's and 1's

$$= \begin{bmatrix} \mathbf{V}^{\mathbf{X}} \\ \mathbf{V}^{\mathbf{Y}} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{cos\theta} & -\mathbf{sin\theta} & \mathbf{P}_{\mathbf{x}} \\ \mathbf{sin\theta} & \mathbf{cos\theta} & \mathbf{P}_{\mathbf{y}} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{V}^{\mathbf{N}} \\ \mathbf{V}^{\mathbf{O}} \\ \mathbf{1} \end{bmatrix}$$

Simplifying into a matrix form

HOMOGENEOUS REPRESENTATION

Putting it all into a Matrix

$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \end{bmatrix}$$

What we found by doing a translation and a rotation

$$= \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \\ 1 \end{bmatrix}$$

Padding with 0's and 1's

$$= \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{P}_x \\ \sin\theta & \cos\theta & \mathbf{P}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \\ 1 \end{bmatrix}$$

Simplifying into a matrix form

$$\mathbf{H} = \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{P}_x \\ \sin\theta & \cos\theta & \mathbf{P}_y \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous Matrix for a Translation in XY plane, followed by a Rotation around the z-axis

Rotation Matrices in 3D – OK,lets return from homogenous repn

$$\mathbf{R_z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

← Rotation around the Z-Axis

$$\mathbf{R_y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

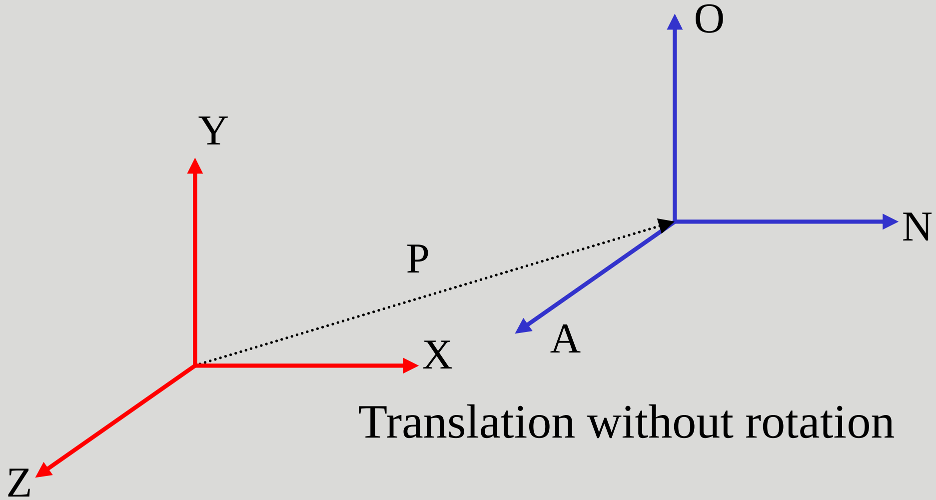
← Rotation around the Y-Axis

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

← Rotation around the X-Axis

Homogeneous Matrices in 3D

H is a 4x4 matrix that can describe a translation, rotation, or both in one matrix

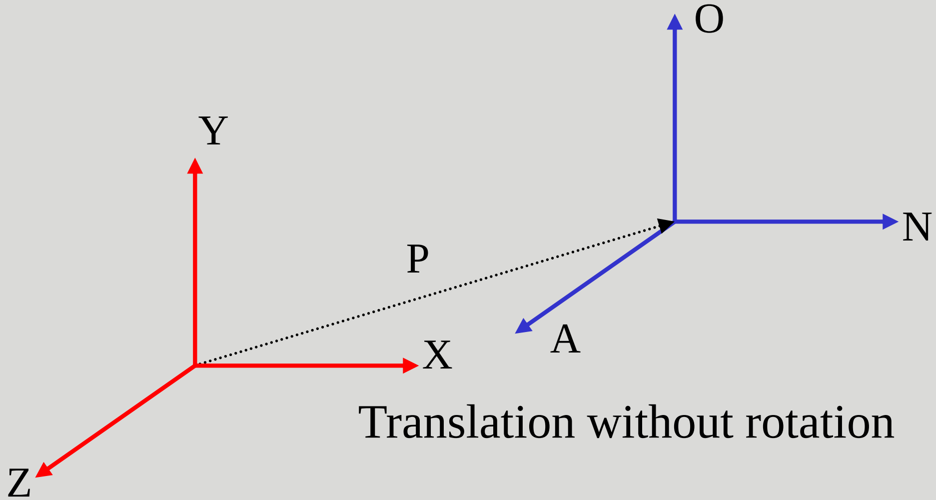


Translation without rotation

$$H = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{P_x} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{P_y} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{P_z} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

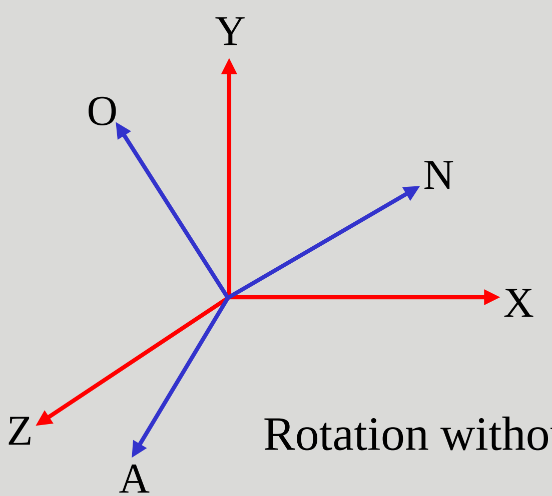
Homogeneous Matrices in 3D

H is a 4x4 matrix that can describe a translation, rotation, or both in one matrix



Translation without rotation

$$H = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Rotation without translation

$$H = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation part:

Could be rotation around z-axis, x-axis, y-axis or a combination of the three.

Homogeneous Continued….

$$\mathbf{V}^{XY} = \mathbf{H}\begin{bmatrix} \mathbf{V}^{N} \\ \mathbf{V}^{O} \\ \mathbf{V}^{A} \\ 1 \end{bmatrix}$$

The (n,o,a) position of a point relative to the current coordinate frame you are in.

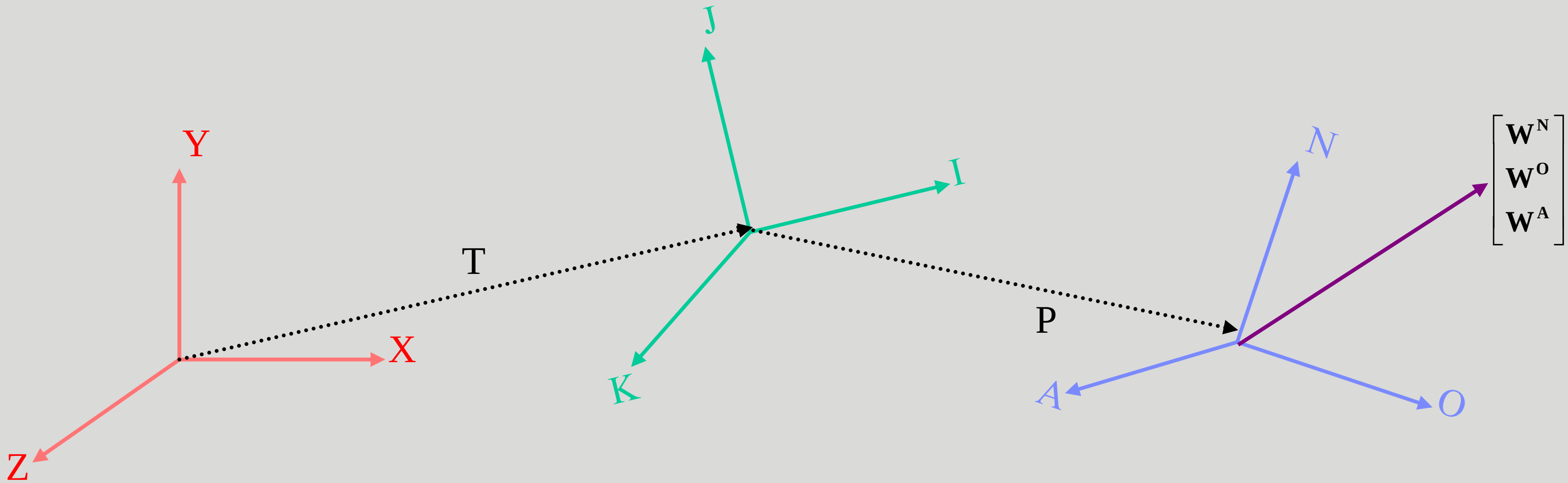
$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{n}_x & \mathbf{o}_x & \mathbf{a}_x & \mathbf{P}_x \\ \mathbf{n}_y & \mathbf{o}_y & \mathbf{a}_y & \mathbf{P}_y \\ \mathbf{n}_z & \mathbf{o}_z & \mathbf{a}_z & \mathbf{P}_z \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \mathbf{V}^{N} \\ \mathbf{V}^{O} \\ \mathbf{V}^{A} \\ 1 \end{bmatrix}$$

$$\mathbf{V}^{X} = \mathbf{n}_x\mathbf{V}^{N} + \mathbf{o}_x\mathbf{V}^{O} + \mathbf{a}_x\mathbf{V}^{A} + \mathbf{P}_x$$

The rotation and translation part can be combined into a single homogeneous matrix IF and ONLY IF both are relative to the same coordinate frame.

Finding the Homogeneous Matrix

EX.



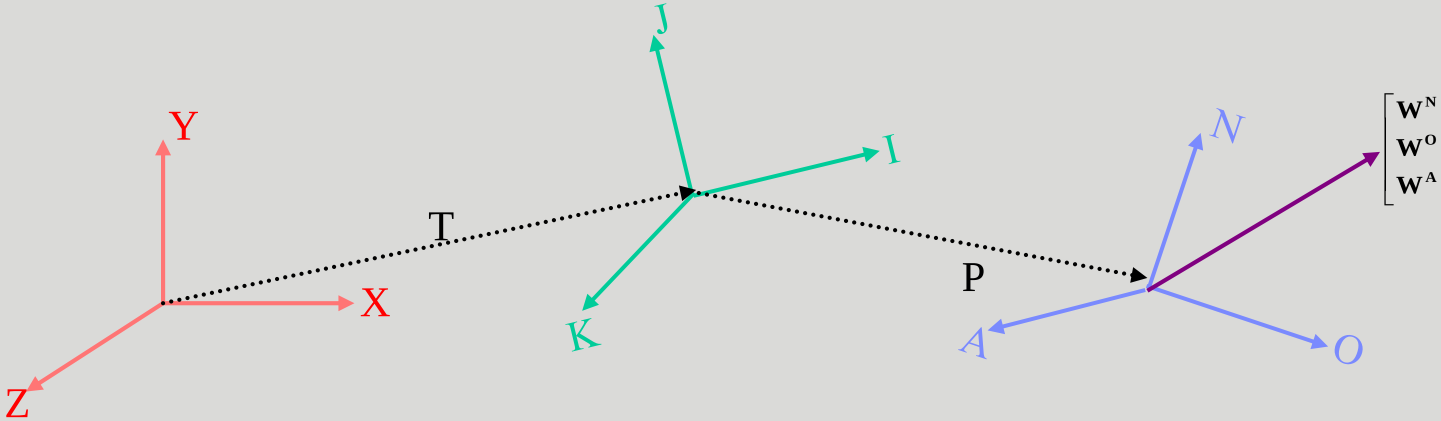
$$\begin{bmatrix} W^X \\ W^Y \\ W^Z \end{bmatrix}$$ Point relative to the X-Y-Z frame

$$\begin{bmatrix} W^I \\ W^J \\ W^K \end{bmatrix}$$ Point relative to the I-J-K frame

$$\begin{bmatrix} W^N \\ W^O \\ W^A \end{bmatrix}$$ Point relative to the N-O-A frame

$$\begin{bmatrix} W^I \\ W^J \\ W^K \end{bmatrix} = \begin{bmatrix} P_i \\ P_j \\ P_k \end{bmatrix} + \begin{bmatrix} n_i & o_i & a_i \\ n_j & o_j & a_j \\ n_k & o_k & a_k \end{bmatrix} \begin{bmatrix} W^N \\ W^O \\ W^A \end{bmatrix}$$

$$\begin{bmatrix} W^I \\ W^J \\ W^K \\ 1 \end{bmatrix} = \begin{bmatrix} n_i & o_i & a_i & P_i \\ n_j & o_j & a_j & P_j \\ n_k & o_k & a_k & P_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} W^N \\ W^O \\ W^A \\ 1 \end{bmatrix}$$



$$\begin{bmatrix}W^X\\W^Y\\W^Z\end{bmatrix}=\begin{bmatrix}T_x\\T_y\\T_z\end{bmatrix}+\begin{bmatrix}i_x&j_x&k_x\\i_y&j_y&k_y\\i_z&j_z&k_z\end{bmatrix}\begin{bmatrix}W^I\\W^J\\W^k\end{bmatrix}\longrightarrow\begin{bmatrix}W^X\\W^Y\\W^Z\\1\end{bmatrix}=\begin{bmatrix}i_x&j_x&k_x&T_x\\i_y&j_y&k_y&T_y\\i_z&j_z&k_z&T_z\\0&0&0&1\end{bmatrix}\begin{bmatrix}W^I\\W^J\\W^K\\1\end{bmatrix}$$

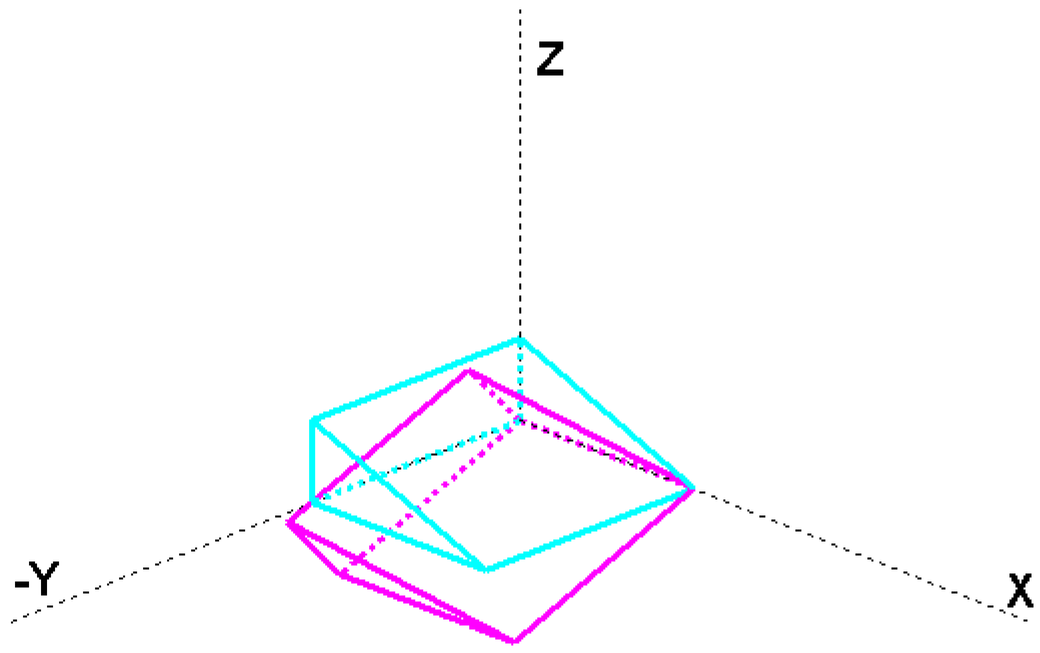
Substituting for $\begin{bmatrix}W^I\\W^J\\W^K\end{bmatrix}$

$$\begin{bmatrix}W^X\\W^Y\\W^Z\\1\end{bmatrix}=\begin{bmatrix}i_x&j_x&k_x&T_x\\i_y&j_y&k_y&T_y\\i_z&j_z&k_z&T_z\\0&0&0&1\end{bmatrix}\begin{bmatrix}n_i&o_i&a_i&P_i\\n_j&o_j&a_j&P_j\\n_k&o_k&a_k&P_k\\0&0&0&1\end{bmatrix}\begin{bmatrix}W^N\\W^O\\W^A\\1\end{bmatrix}$$

Three-Dimensional Example

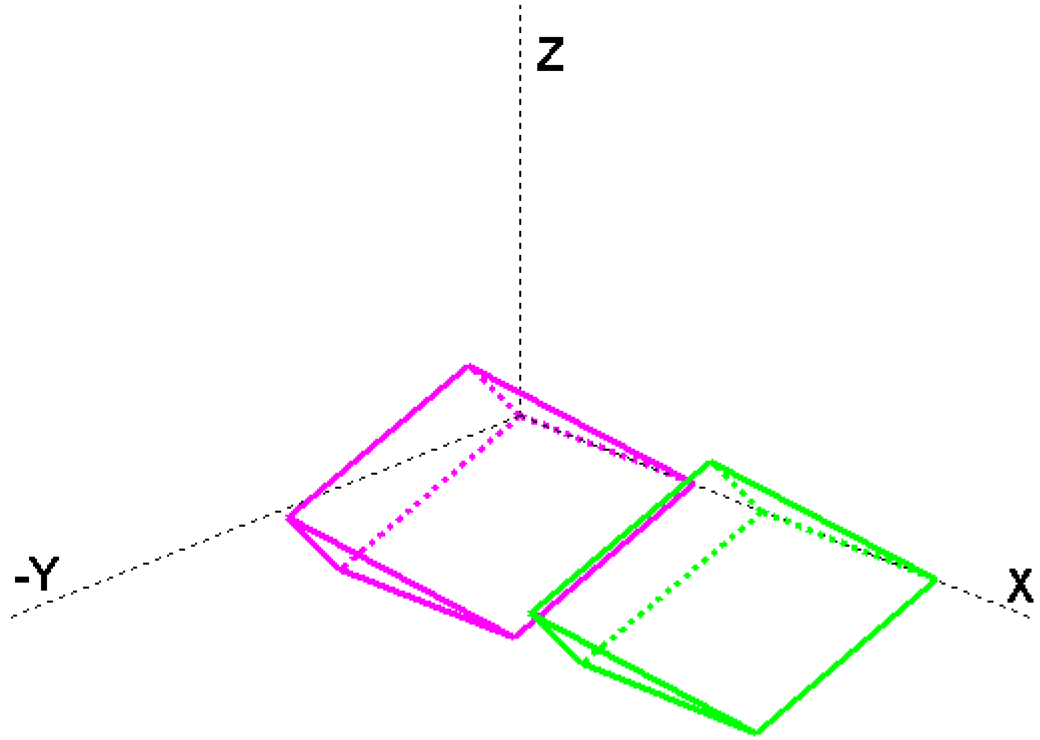
- Rotate X
- Translate X
- Rotate Z
- Translate Z

Rotation about X



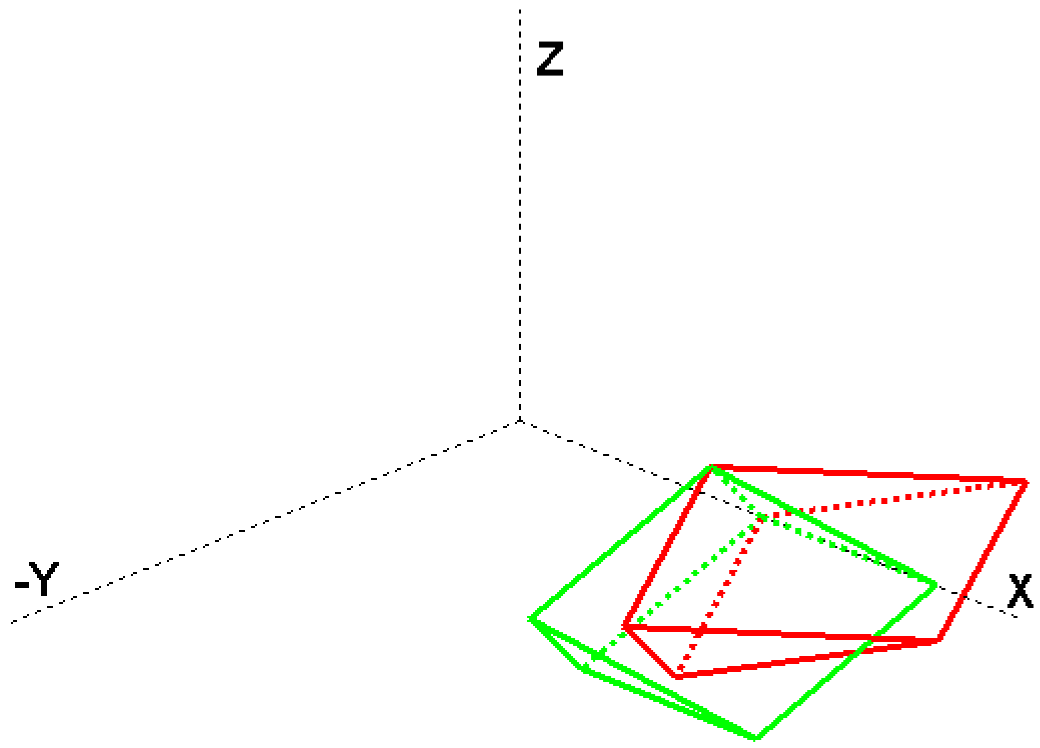
$$\text{Rotation about X - axis by } \theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \text{Cos}\,\theta & -\text{Sin}\,\theta & 0 \\ 0 & \text{Sin}\,\theta & \text{Cos}\,\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation about X



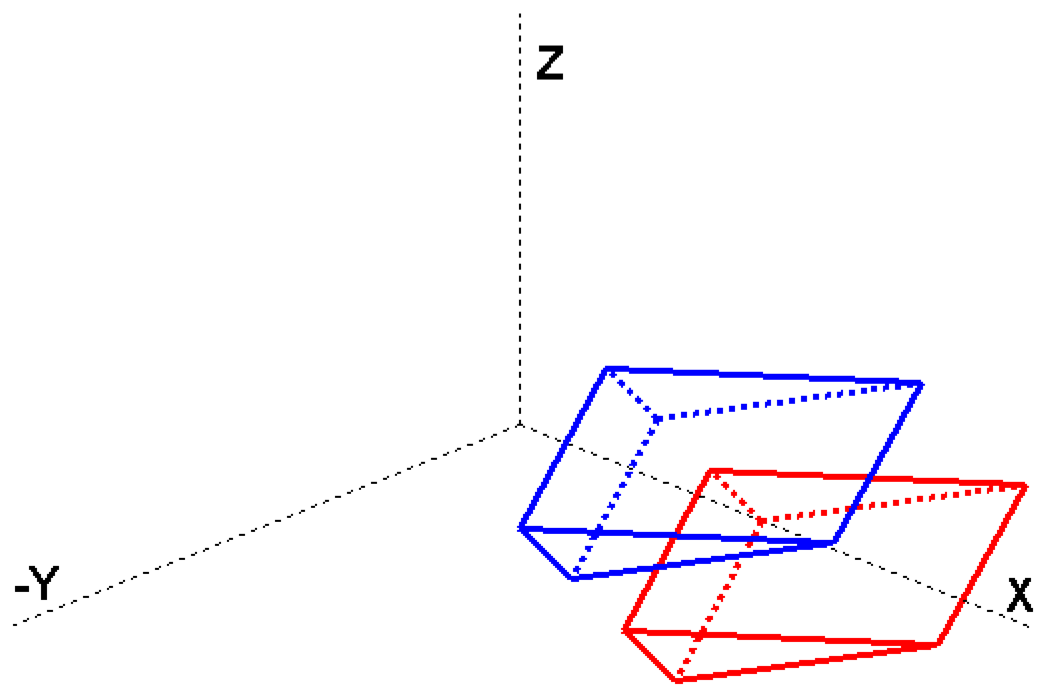
$$\text{Translation along X - axis} = \begin{pmatrix} 1 & 0 & 0 & p^{x} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation about Z



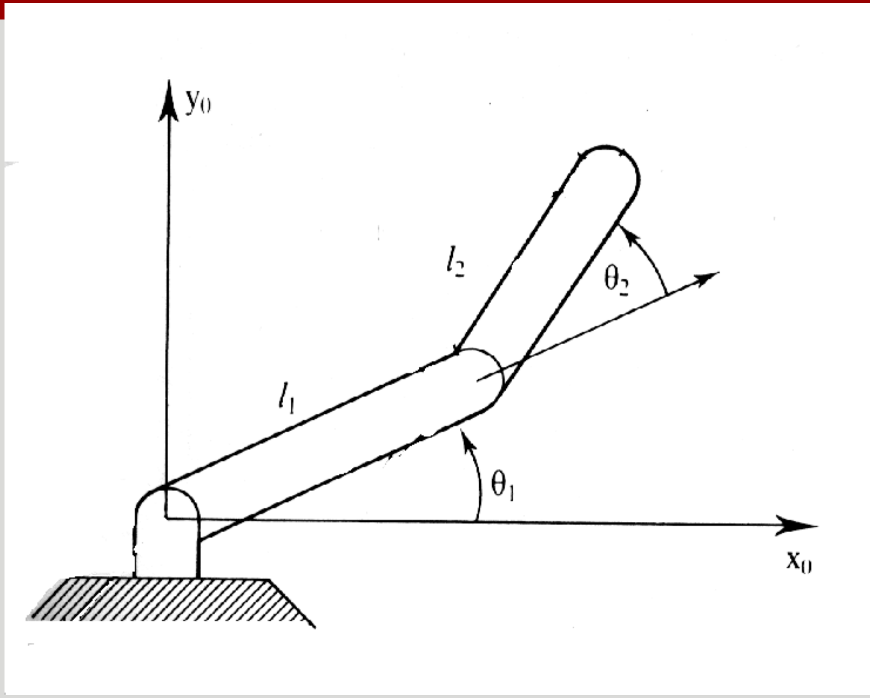
Rotation about local Z - axis = $\begin{pmatrix} \text{Cos}[\theta] & -\text{Sin}[\theta] & 0 & 0 \\ \text{Sin}[\theta] & \text{Cos}[\theta] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Translation in Z



$$\text{Translation along local Z-axis} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & p^z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Forward Kinematics



The Situation:

You have a robotic arm that starts out aligned with the x_o-axis. You tell the first link to move by Y_1 and the second link to move by Y_2.

The Quest:

What is the position of the end of the robotic arm?

Solution:

1. Geometric Approach

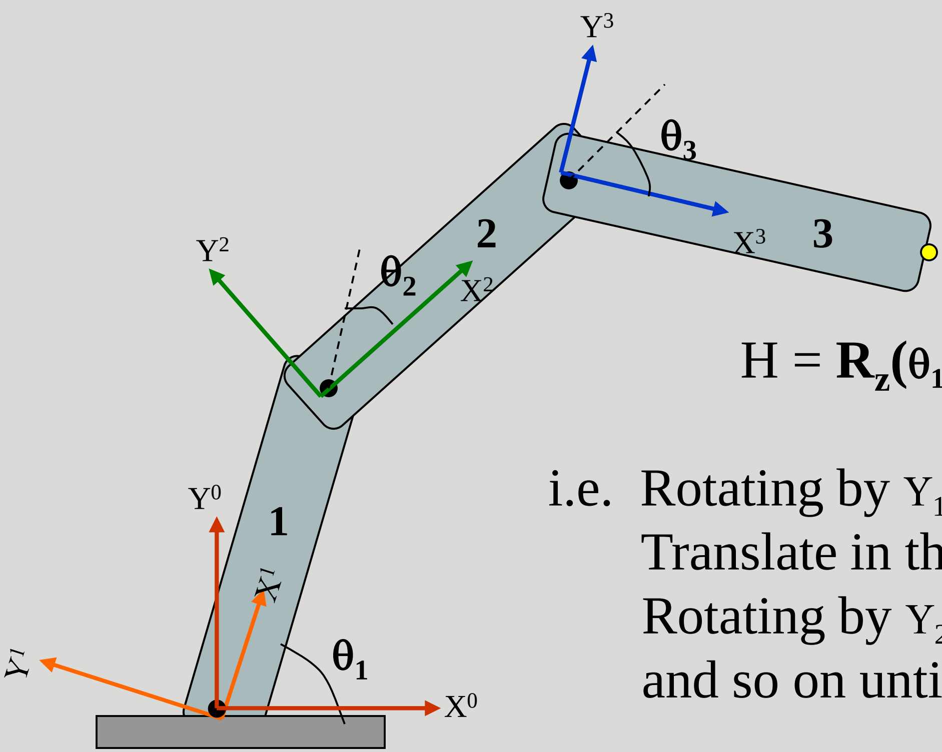
This might be the easiest solution for the simple situation. However, notice that the angles are measured relative to the direction of the previous link. (The first link is the exception. The angle is measured relative to it's initial position.) For robots with more links and whose arm extends into 3 dimensions the geometry gets much more tedious.

2. Algebraic Approach

Involves coordinate transformations.

Example Problem:

You are have a three link arm that starts out aligned in the x-axis. Each link has lengths l_1, l_2, l_3, respectively. You tell the first one to move by Y_1 , and so on as the diagram suggests. Find the Homogeneous matrix to get the position of the yellow dot in the X^0Y^0 frame.



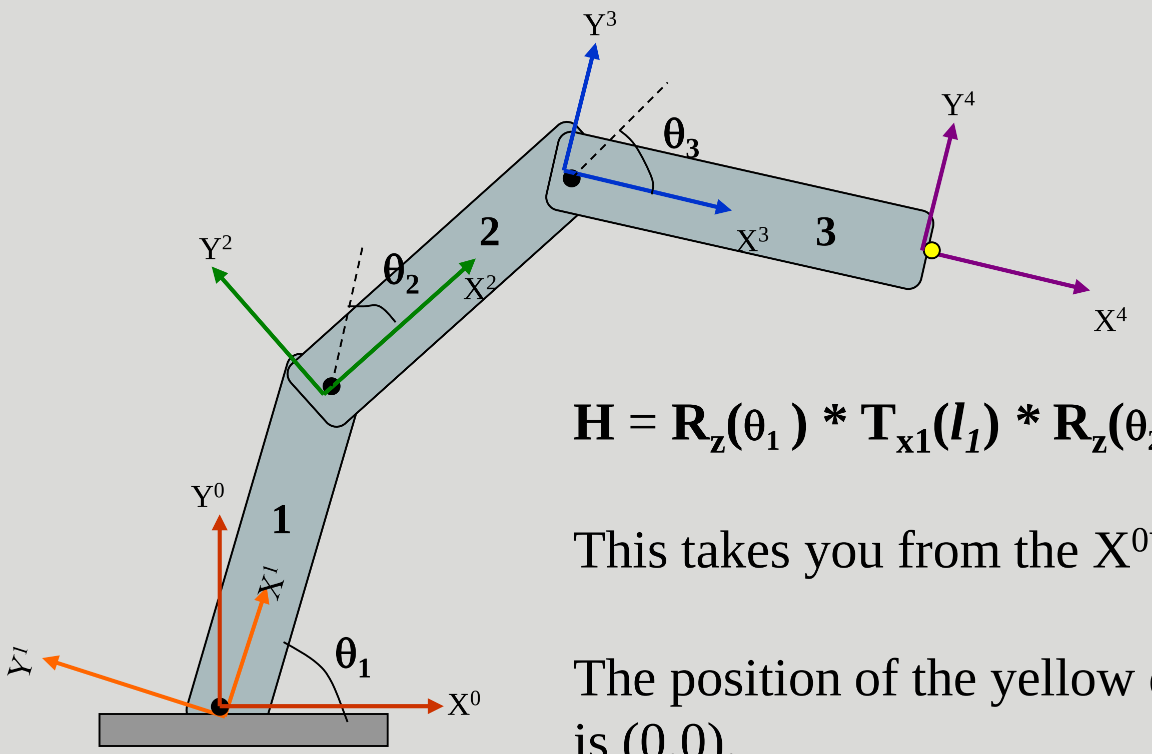
$$H = \mathbf{R_z}(\theta_1) * \mathbf{T_{x1}}(\boldsymbol{l_1}) * \mathbf{R_z}(\theta_2) * \mathbf{T_{x2}}(\boldsymbol{l_2}) * \mathbf{R_z}(\theta_3)$$

i.e. Rotating by Y_1 will put you in the X^1Y^1 frame.
Translate in the along the X^1 axis by l_1.
Rotating by Y_2 will put you in the X^2Y^2 frame.
and so on until you are in the X^3Y^3 frame.

The position of the yellow dot relative to the X^3Y^3 frame is (l_1, 0). Multiplying H by that position vector will give you the coordinates of the yellow point relative the the X^0Y^0 frame.

Slight variation on the last solution:

Make the yellow dot the origin of a new coordinate X^4Y^4 frame



$$\mathbf{H} = \mathbf{R_z}(\theta_1) * \mathbf{T_{x1}}(\boldsymbol{l_1}) * \mathbf{R_z}(\theta_2) * \mathbf{T_{x2}}(\boldsymbol{l_2}) * \mathbf{R_z}(\theta_3) * \mathbf{T_{x3}}(\boldsymbol{l_3})$$

This takes you from the X^0Y^0 frame to the X^4Y^4 frame.

The position of the yellow dot relative to the X^4Y^4 frame is (0,0).

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \mathbf{H} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

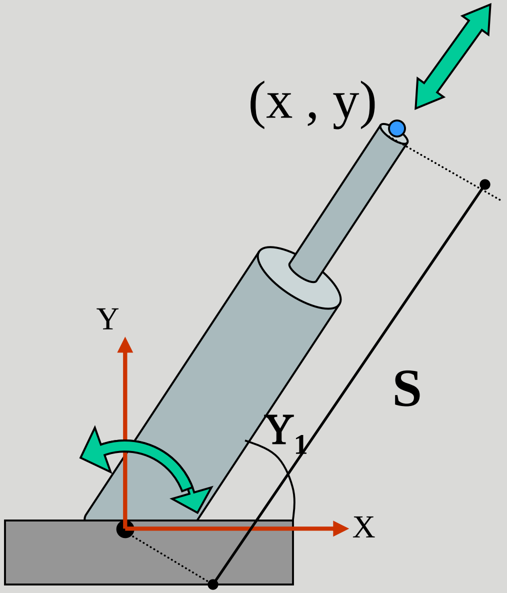
Notice that multiplying by the (0,0,0,1) vector will equal the last column of the H matrix.

Inverse Kinematics

From Position to Angles

A Simple Example

Revolute and Prismatic Joints Combined



Finding **Y**:

$$\theta = \arctan(y, x)$$

More Specifically:

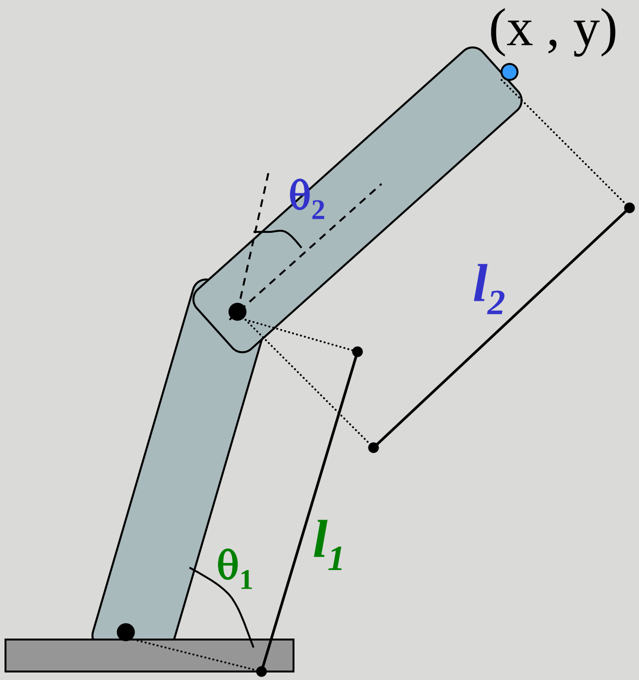
$$\theta = \arctan2(y, x)$$

arctan2() specifies that it's in the first quadrant

Finding **S**:

$$S = \sqrt{(x^2 + y^2)}$$

Inverse Kinematics of a Two Link Manipulator

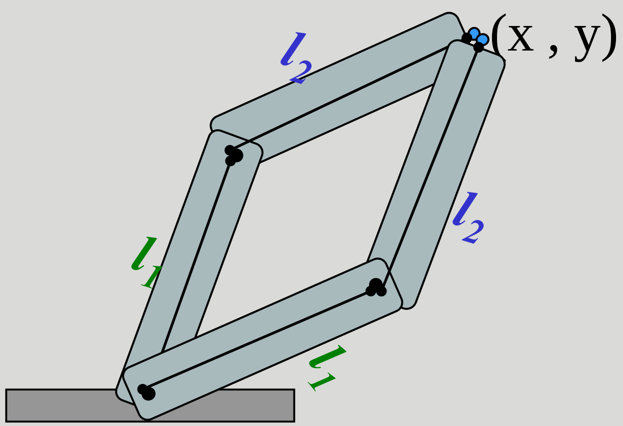


Given: l_1, l_2, x , y

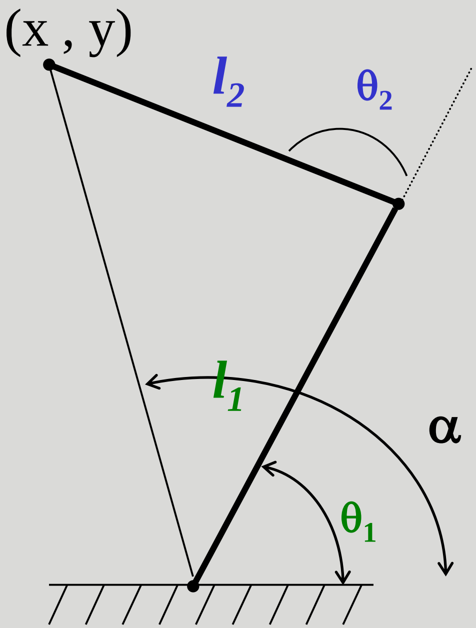
Find: Y_1, Y_2

Redundancy:

A unique solution to this problem does not exist. Notice, that using the "givens" two solutions are possible. Sometimes no solution is possible.



The Geometric Solution



Using the Law of Cosines:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin\bar{\theta}_1}{l_2} = \frac{\sin(180-\theta_2)}{\sqrt{x^2+y^2}} = \frac{\sin(\theta_2)}{\sqrt{x^2+y^2}}$$

$$\theta_1 = \alpha - \bar{\theta}_1$$

$$\alpha = \arctan 2\left(\frac{y}{x}\right)$$

Using the Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$(x^2+y^2) = l_1^{\,2} + l_2^{\,2} - 2l_1 l_2 \cos(180-\theta_2)$$

$$\cos(180-\theta_2) = -\cos(\theta_2)$$

$$\cos(\theta_2) = \frac{x^2+y^2-l_1^{\,2}-l_2^{\,2}}{2l_1 l_2}$$

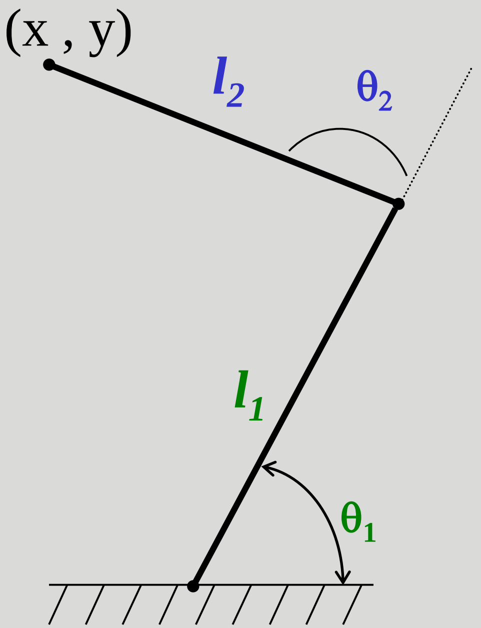
$$\theta_2 = \arccos\left(\frac{x^2+y^2-l_1^{\,2}-l_2^{\,2}}{2l_1 l_2}\right)$$

Redundant since θ_2 could be in the first or fourth quadrant.

Redundancy caused since θ_2 has two possible values

$$\theta_1 = \arctan 2(y, x) - \arcsin\left(\frac{l_2\sin(\theta_2)}{\sqrt{x^2+y^2}}\right)$$

The Algebraic Solution



$$c_1 = \cos\theta_1$$

$$c_{1+2} = \cos(\theta_2 + \theta_1)$$

$$(1)\ x = l_1\, c_1 + l_2\, c_{1+2}$$

$$(2)\ y = l_1\, s_1 + l_2\, \sin_{1+2}$$

$$(1)^2 + (2)^2 = x^2 + y^2 =$$

$$= \left(l_1^{\ 2}\, c_1^{\ 2} + l_2^{\ 2}(c_{1+2})^2 + 2l_1 l_2\, c_1(c_{1+2})\right) + \left(l_1^{\ 2}\, s_1^{\ 2} + l_2^{\ 2}(\sin_{1+2})^2 + 2l_1 l_2\, s_1(\sin_{1+2})\right)$$

$$= l_1^{\ 2} + l_2^{\ 2} + 2l_1 l_2\left(c_1(c_{1+2}) + s_1(\sin_{1+2})\right)$$

$$= l_1^{\ 2} + l_2^{\ 2} + 2l_1 l_2\, c_2$$

← Only Unknown

$$\therefore \theta_2 = \arccos\left(\frac{x^2 + y^2 - l_1^{\ 2} - l_2^{\ 2}}{2l_1 l_2}\right)$$

Note:

$$\cos(a \pm b) = (\cos a)(\cos b) \mp (\sin a)(\sin b)$$

$$\sin(a \pm b) = (\cos a)(\sin b) \pm (\cos b)(\sin a)$$

$$
\begin{aligned}
x &= l_1 c_1 + l_2 c_{1+2} \\
&= l_1 c_1 + l_2 c_1 c_2 - l_2 s_1 s_2 \\
&= c_1 (l_1 + l_2 c_2) - s_1 (l_2 s_2)
\end{aligned}
$$

Note:

$$\cos(a \pm b) = (\cos a)(\cos b) \mp (\sin a)(\sin b)$$

$$\sin(a \pm b) = (\cos a)(\sin b) \pm (\cos b)(\sin a)$$

$$
\begin{aligned}
y &= l_1 s_1 + l_2 \sin_{1+2} \\
&= l_1 s_1 + l_2 s_1 c_2 + l_2 s_2 c_1 \\
&= c_1 (l_2 s_2) + s_1 (l_1 + l_2 c_2)
\end{aligned}
$$

We know what θ_2 is from the previous slide. We need to solve for θ_1 . Now we have two equations and two unknowns (sin θ_1 and cos θ_1)

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (l_1 + l_2 c_2) & (-l_2 s_2) \\ (l_2 s_2) & (l_1 + l_2 c_2) \end{bmatrix} \begin{bmatrix} c_1 \\ s_1 \end{bmatrix}
$$

Substituting for c_1 and simplifying many times

Notice this is the law of cosines and can be replaced by $x^2 + y^2$

$$\theta_1 = \arctan2(s_1, c_1)$$

$$
\begin{aligned}
x &= l_1\, c_1 + l_2\, c_{1+2} \\
&= l_1\, c_1 + l_2\, c_1 c_2 - l_2 s_1 s_2 \\
&= c_1 (l_1 + l_2\, c_2) - s_1 (l_2 s_2)
\end{aligned}
$$

Note:

$$\cos(a \pm b) = (\cos a)(\cos b) \mp (\sin a)(\sin b)$$

$$\sin(a \pm b) = (\cos a)(\sin b) \pm (\cos b)(\sin a)$$

$$
\begin{aligned}
y &= l_1\, s_1 + l_2\, \sin_{1+2} \\
&= l_1\, s_1 + l_2\, s_1 c_2 + l_2\, s_2 c_1 \\
&= c_1 (l_2\, s_2) + s_1 (l_1 + l_2\, c_2)
\end{aligned}
$$

We know what θ_2 is from the previous slide. We need to solve for θ_1 . Now we have two equations and two unknowns (sin θ_1 and cos θ_1)

$$c_1 = \frac{x + s_1 (l_2 s_2)}{(l_1 + l_2\, c_2)}$$

$$y = \frac{x + s_1 (l_2 s_2)}{(l_1 + l_2\, c_2)} (l_2\, s_2) + s_1 (l_1 + l_2\, c_2)$$

Substituting for c_1 and simplifying many times

$$= \frac{1}{(l_1 + l_2\, c_2)} \left(x\, l_2 s_2 + s_1 (l_1^{\,2} + l_2^{\,2} + 2 l_1 l_2\, c_2) \right)$$

Notice this is the law of cosines and can be replaced by $x^2 + y^2$

$$s_1 = \frac{y(l_1 + l_2\, c_2) - x\, l_2 s_2}{x^2 + y^2}$$

$$\theta_1 = \text{arctan2}(s_1, c_1)$$

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