An Introduction to Robot Kinematics

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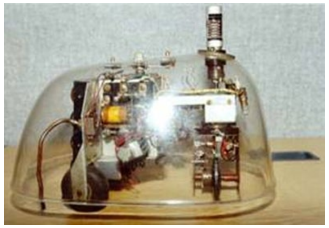
Robot kinematics refers to the geometry and movement of robotic mechanisms

Engineering/mechanics		Other (e.g. robotics)
Statics	Statics (equilibrium and relation to forces)	Dynamics
Dynamics	Kinetics (motion and relation to forces)	Dynamics
Dynamics	Kinematics (motion without forces)	Kinematics

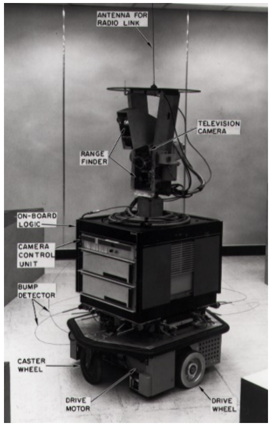
Classical mechanics

A select history of robotics

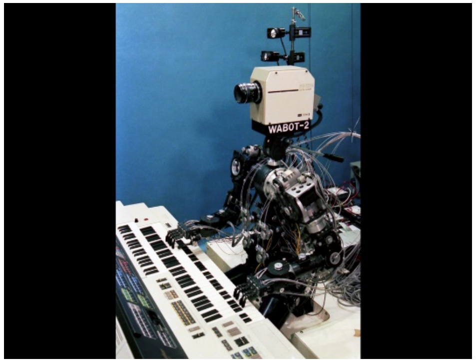


Elmer, 1948

Unimate, 1959



Shakey, 1966



Wabot 2, 1980

CyberKnife, 1991

Asimo, 2000

Big Dog, 2005

Baxter, 2011

Kuka KR AGILUS, 2014

https://en.wikipedia.org/wiki/Shakey_the_robot#/media/File:SRI_Shakey_with_callouts.jpg, http://roamerrobot.tumblr.com/post/23079345849/the-history-of-turtle-robots, https://www.robotics.org/joseph-engelberger/unimate.cfm, https://en.wikipedia.org/wiki/BigDog, https://www.techexplorist.com/wp-content/uploads/2016/03/Baxter_Robot-696x470.jpg

Goals

- Use robotics kinematics terms to explain real world situations.
- Express a point in one coordinate frame in a different coordinate frame.
- Represent complex translations and rotations using a homogenous transformation matrix.
- Determine the position and orientation of an end effector given link and joint information.

What does degrees of freedom mean?

Degrees of Freedom (DOF): the number of independent parameters that can fully define the configuration

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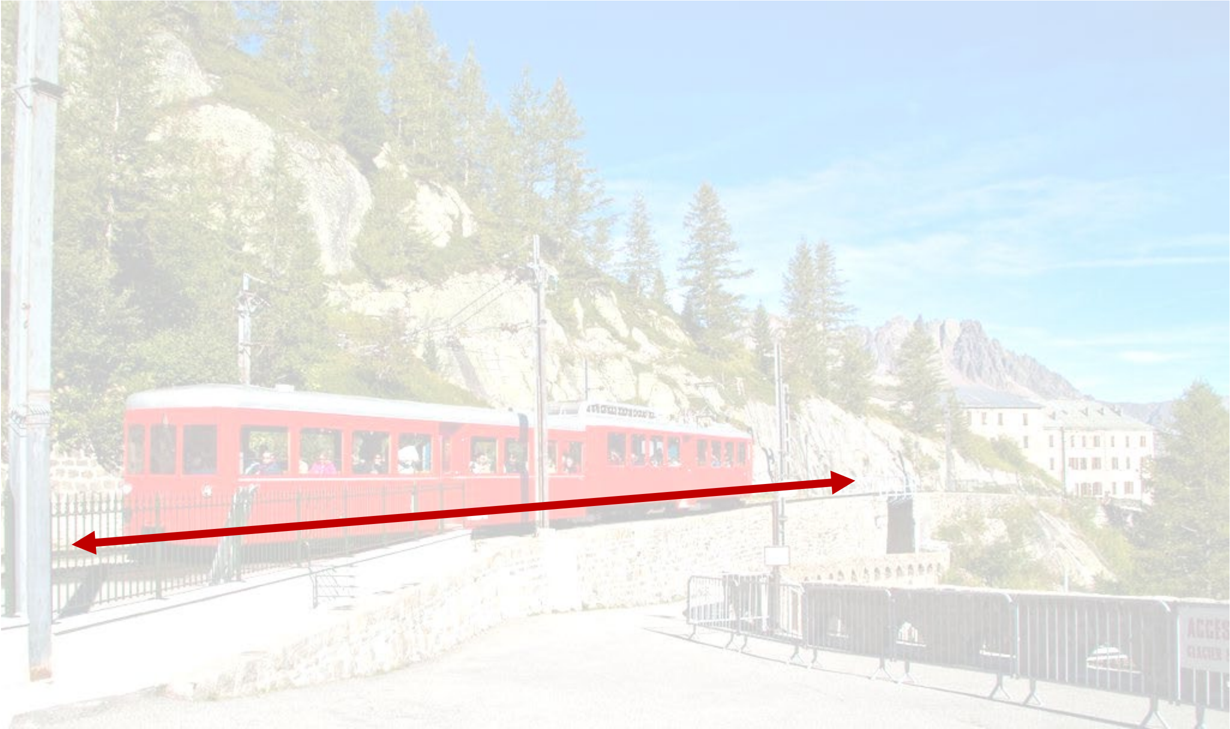
How many degrees of freedom does this have?

https://www.chamonix.net/english/leisure/sightseeing/mer-de-glace

1 DOF



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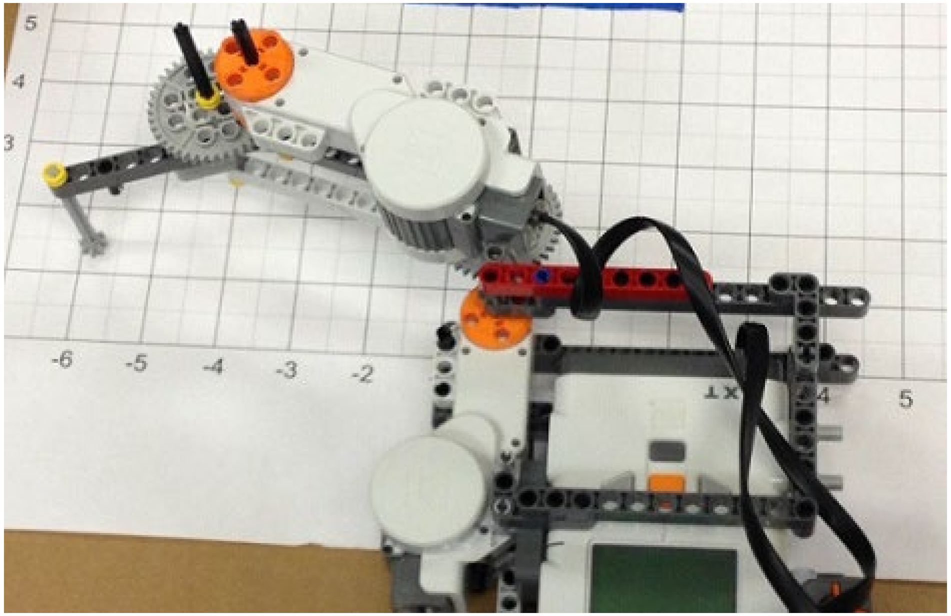
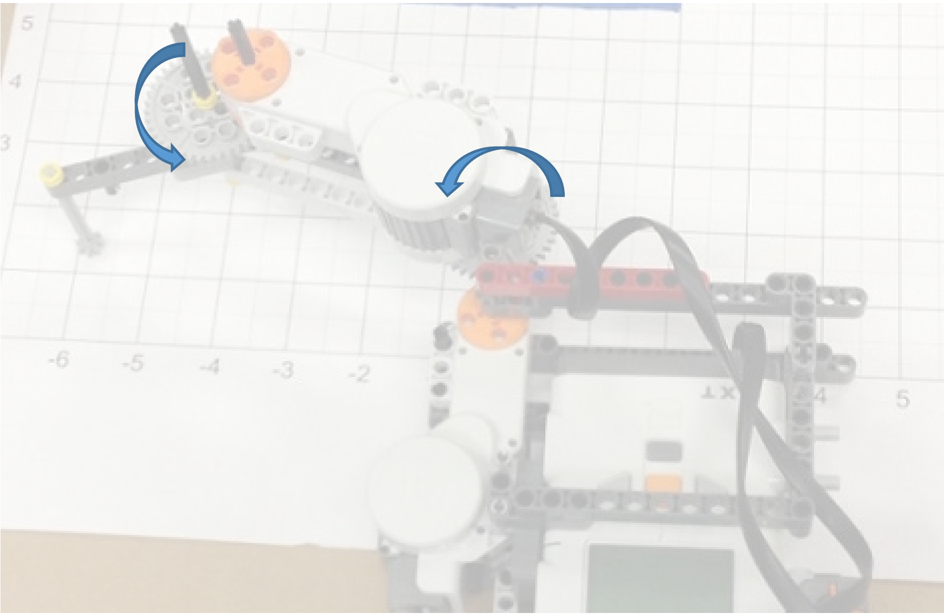
How many degrees of freedom does this have?



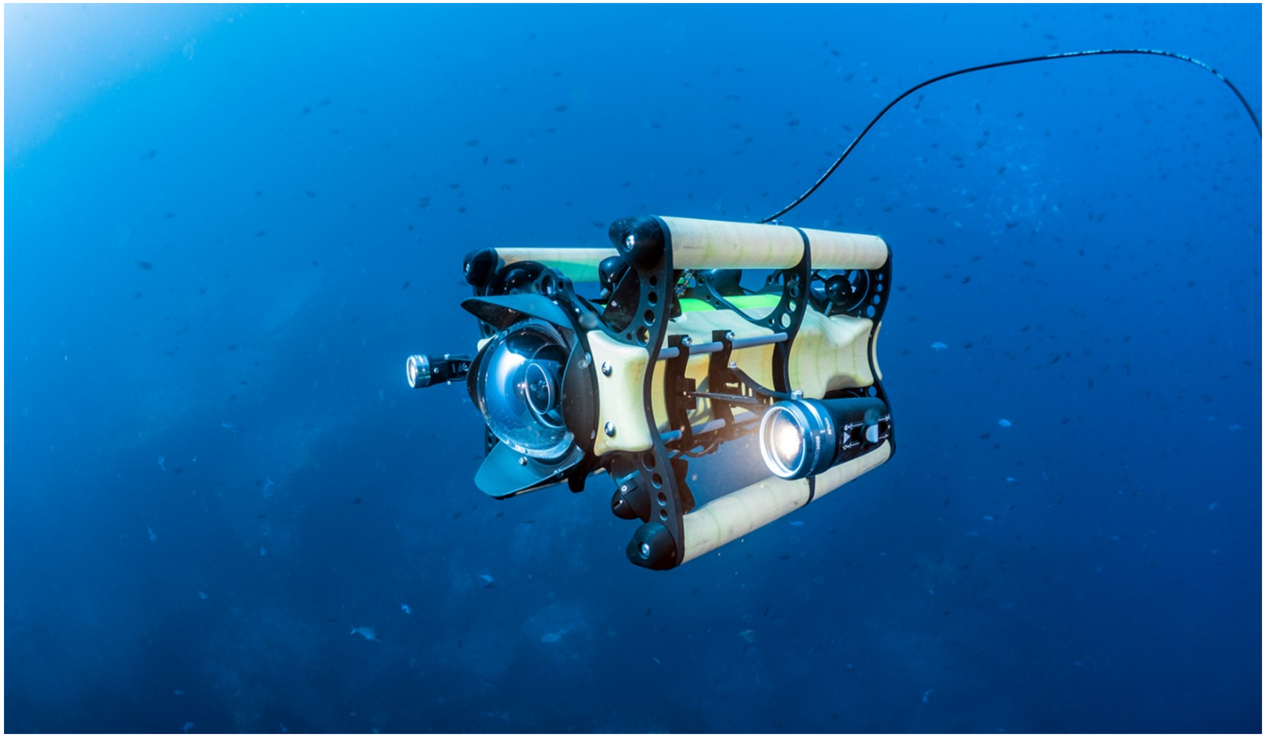
http://www.andrew.cmu.edu/user/kbrennan/TeamZ-Lab9.html

2 DOF



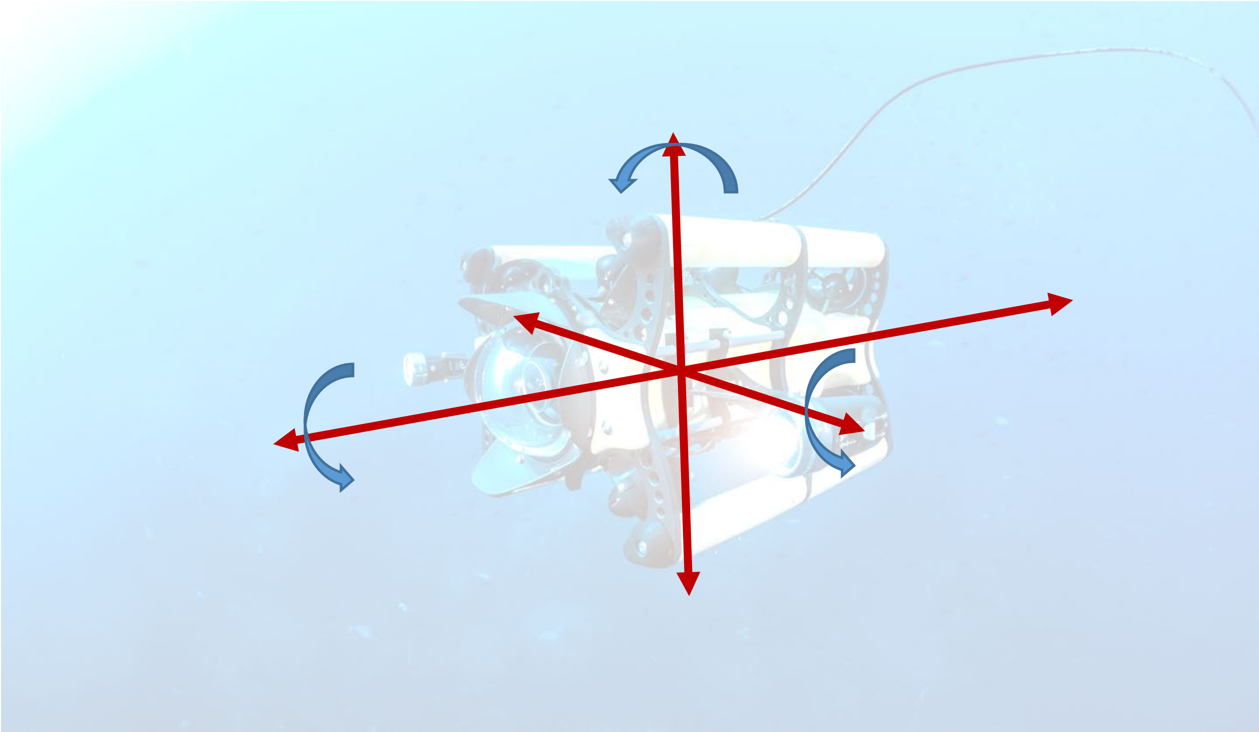
http://www.andrew.cmu.edu/user/kbrennan/TeamZ-Lab9.html

How many degrees of freedom does this have?



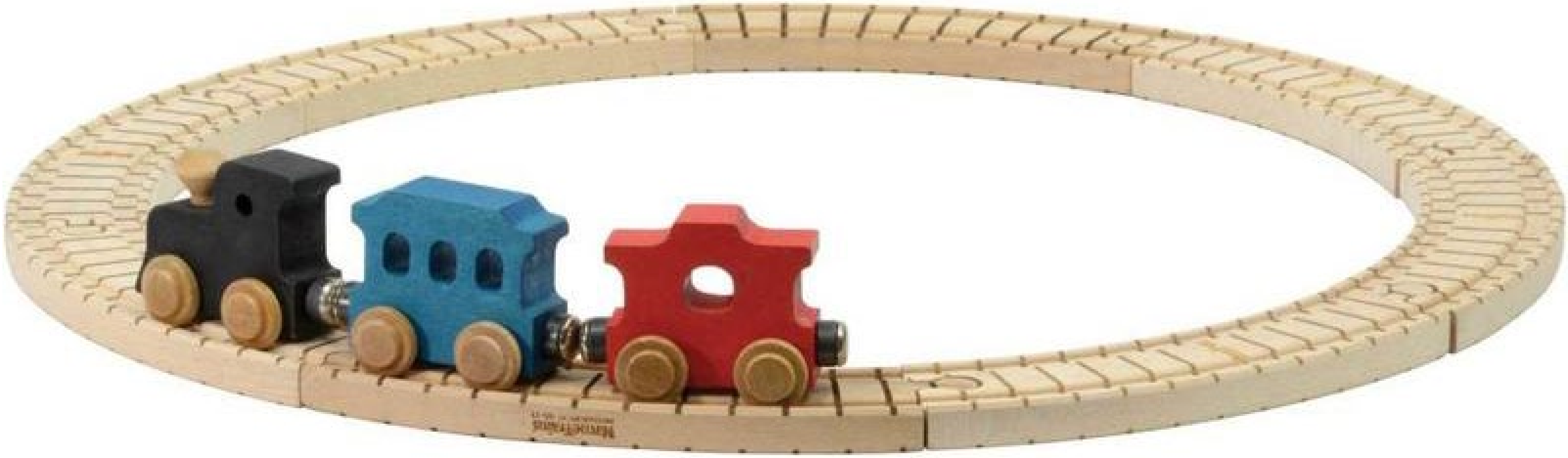
https://pureadvantage.org/news/2016/11/15/underwater-robots/

6 DOF



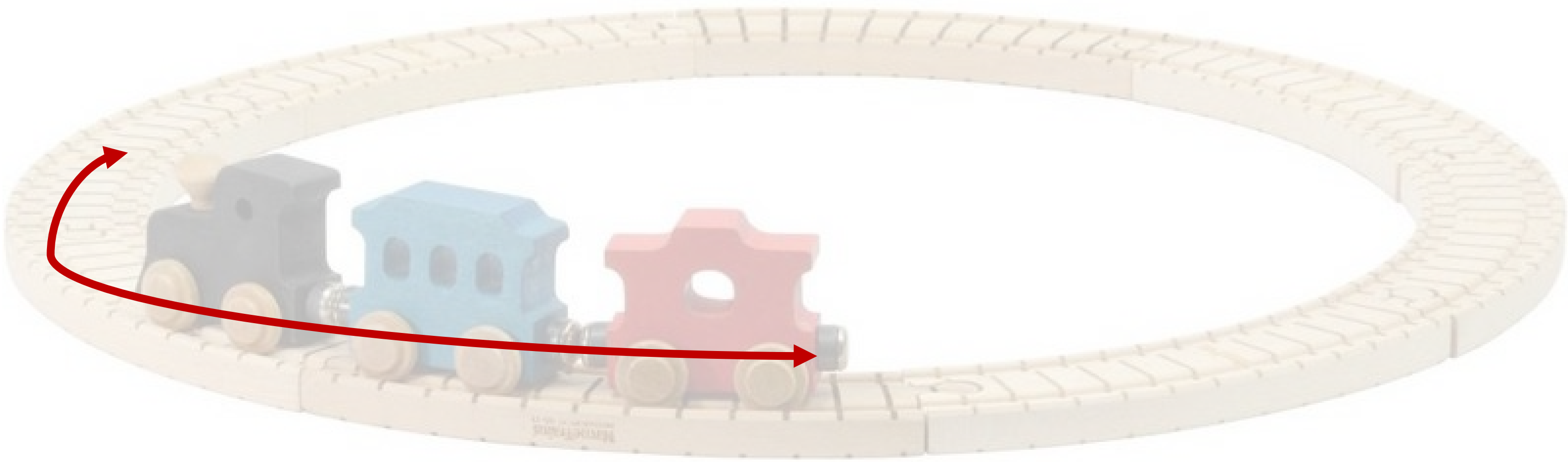
https://pureadvantage.org/news/2016/11/15/underwater-robots/

How many degrees of freedom does this have?



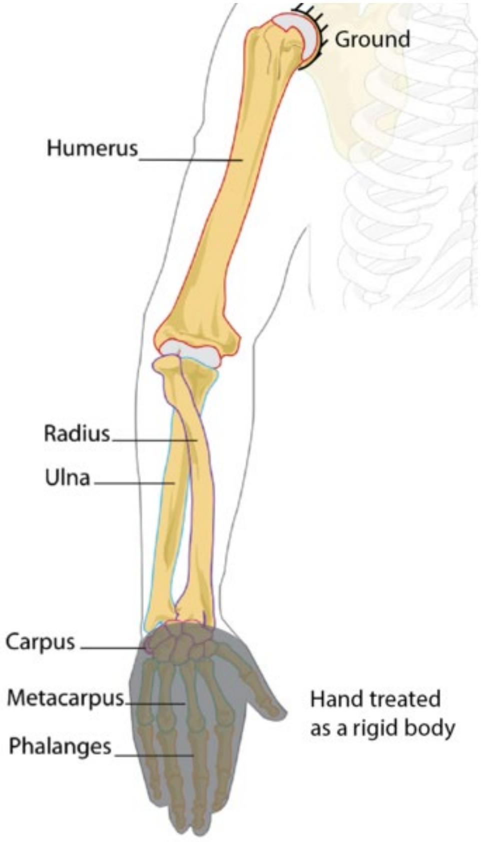
https://www.shopecobambino.com/maple-landmark-starter-train-set.html

1 DOF



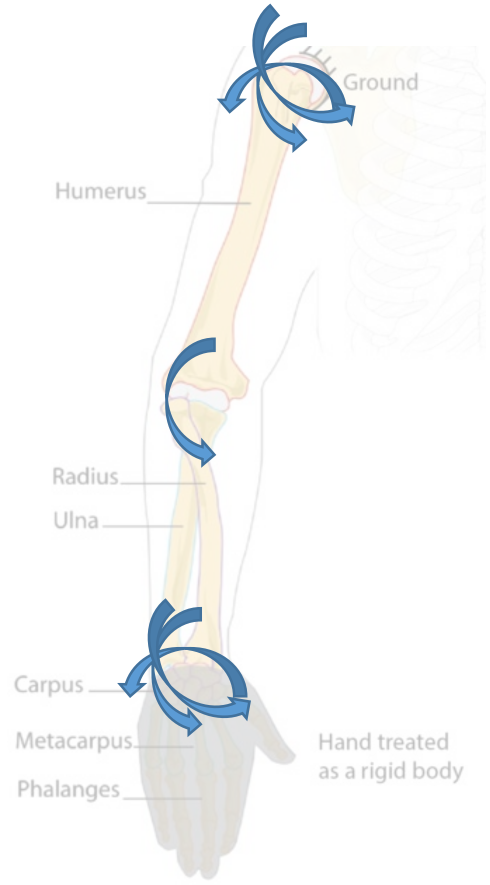
https://www.shopecobambino.com/maple-landmark-starter-train-set.html

How many degrees of freedom does this have?



http://hades.mech.northwestern.edu/index.php/File:Human-arm.png

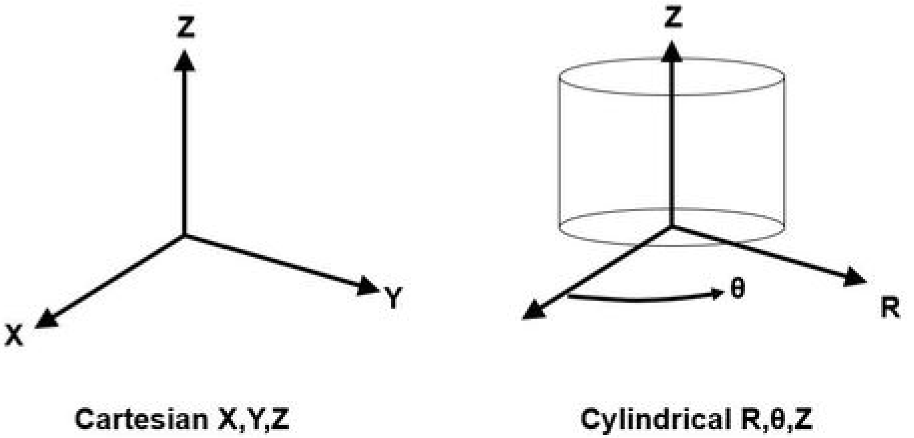
7 DOF



http://hades.mech.northwestern.edu/index.php/File:Human-arm.png

Definitions

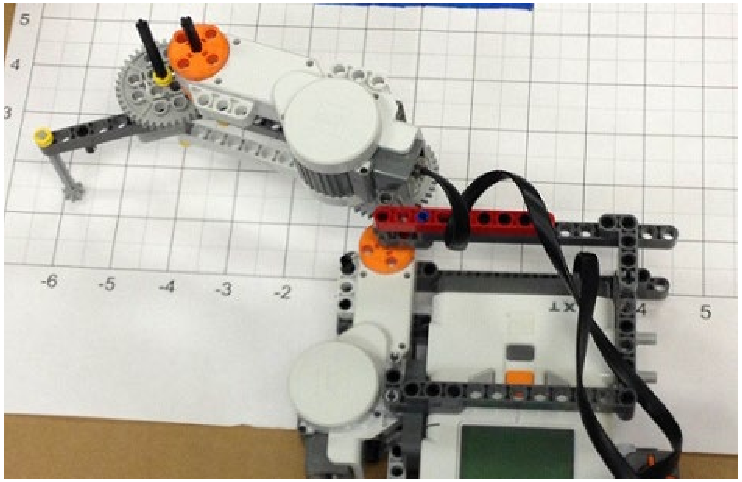
Reference Frame: Static coordinate system from which translations and rotations are based



Link: Single rigid body

Joint: Connection between links

Constraints: Limitations on movement



http://www.value-design-consulting.co.uk/co-ordinate-systems.html
http://www.andrew.cmu.edu/user/kbrennan/TeamZ-Lab9.html

Grübler’s Formula to find degrees of freedom

Basic Idea:

DOF of mechanism = Link DOFs – Joint Constraints

Grübler’s Formula to find degrees of freedom

$$M = 6n - \sum_{i=1}^{j} (6 - f_i)$$

M is the degrees of freedom

n is the number of moving links

j is the number of joints

f_i is the degrees of freedom of the ith joint

Grübler’s Formula – Simple Open Chain

$$M = \sum_{i=1}^{j} f_i$$

M is the degrees of freedom

n is the number of moving links

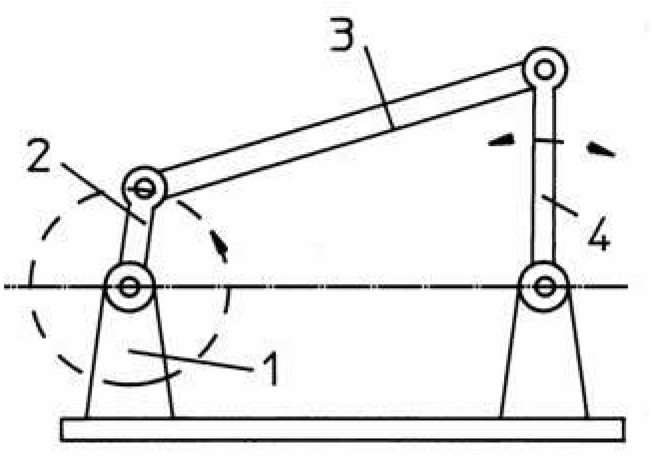
j is the number of joints

f_i is the degrees of freedom of the ith joint

http://hades.mech.northwestern.edu/index.php/File:Human-arm.png

Grübler's Formula – Simple Closed Chain

$$M = \sum_{i=1}^{j} f_i - d$$



M is the degrees of freedom

n is the number of moving links

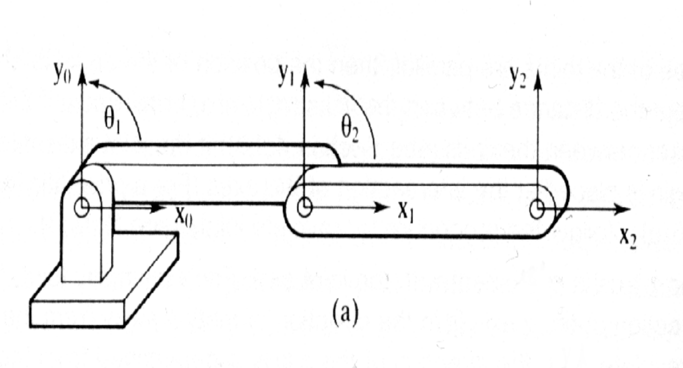
j is the number of joints

f_i is the degrees of freedom of the ith joint

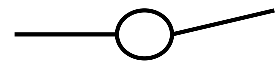
d is the dimension, 3 for planar, 6 for spatial

http://ocw.upm.es/ingenieria-mecanica/mechanical-devices-for-industry/contenidos/lectura-obligatoria/lesson-1/four-bar-linkages

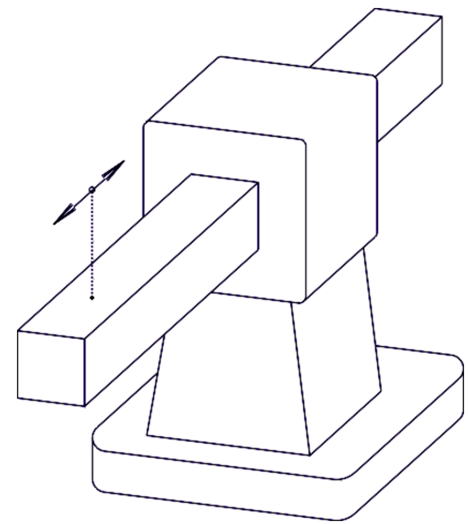
Types of Joints – Lower Pairs



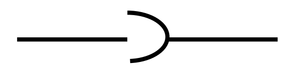
Revolute Joint



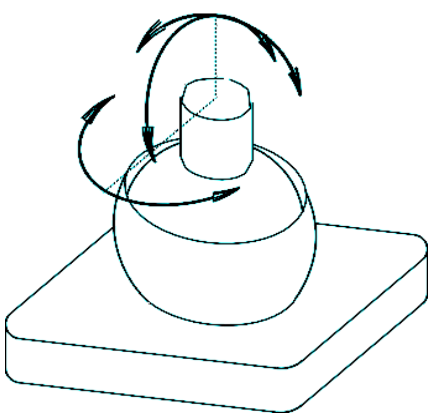
1 DOF (Variable - Θ)

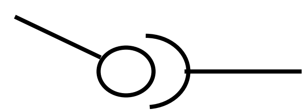


Prismatic Joint



1 DOF (linear) (Variables - d)

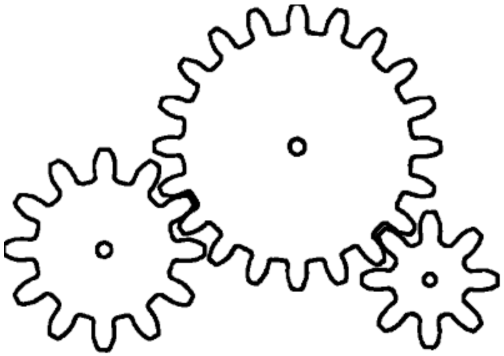




Spherical Joint

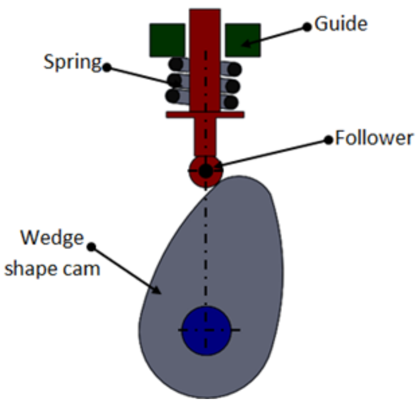
3 DOF (Variables - Θ_1, Θ_2, Θ_3)

Types of Joints – Higher Pairs



Gears
1 DOF (Variable - Θ)

Cam and Follower
1 DOF (linear) (Variables - d)



https://nptel.ac.in/courses/112103174/module4/lec3/1.html

Grübler’s Formula to find degrees of freedom

$$M = 3n - 2l - h$$

M is degrees of freedom
n is the number of moving links
j is the number of joints
l is the number of lower pairs
h is the number of higher pairs
f_i is the degrees of freedom of the i^{th} joint

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We are interested in two kinematics topics

Forward Kinematics (angles to position)

What you are given:
The length of each link
The angle of each joint

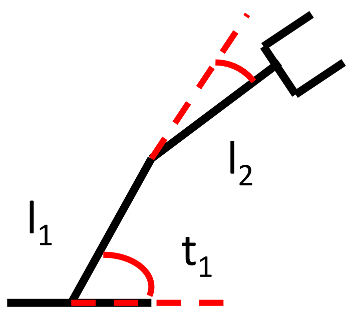
What you can find:
The position of any point
(i.e. it's (x, y, z) coordinates)

Inverse Kinematics (position to angles)

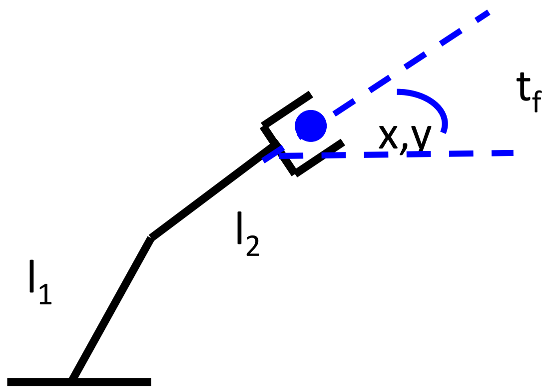
What you are given:
The length of each link
The position of some point on the robot

What you can find:
The angles of each joint needed to obtain that position

Forward Kinematics
(angles to position)

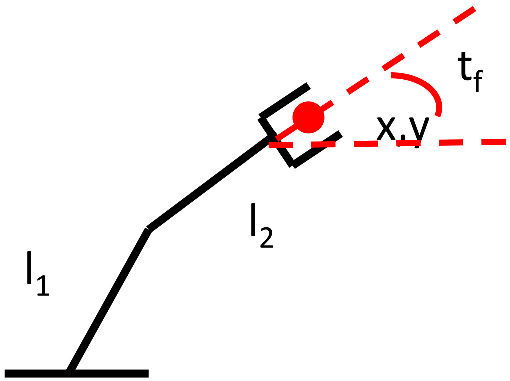


Given l_1, l_2, $t_{1,}$ t_2

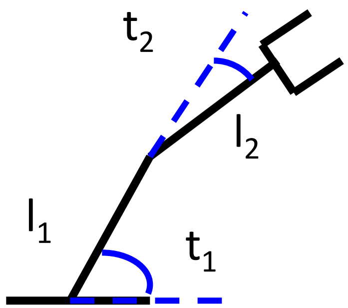


Find x, y, t_f

Inverse Kinematics
(angles to position)



Given l_1, l_2, x, y, t_f

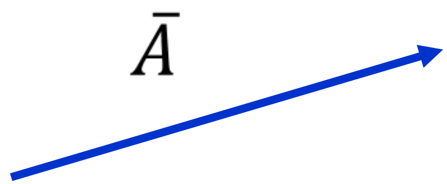


Find t_1, t_2

Quick Math Review

Vector:

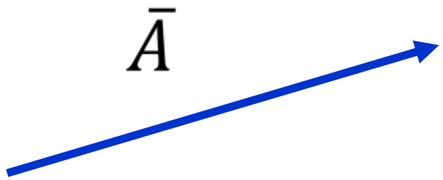
A geometric object with magnitude and direction



Quick Math Review

Vector:

A geometric object with magnitude and direction



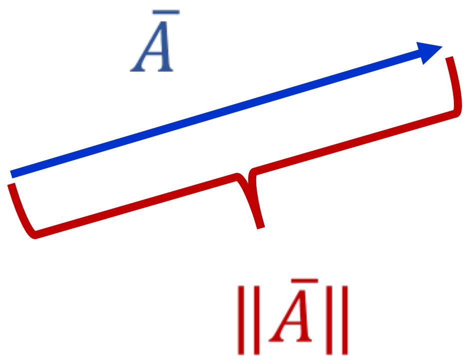
Examples of vector quantities:

Velocity, displacement, acceleration, force

Quick Math Review

Vector Magnitude:

Just the vector quantity without direction



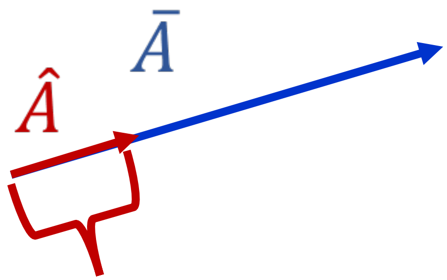
Examples:

Magnitude of velocity is speed, magnitude of displacement is distance, etc.

Quick Math Review

Unit Vector:

Vector with magnitude of 1



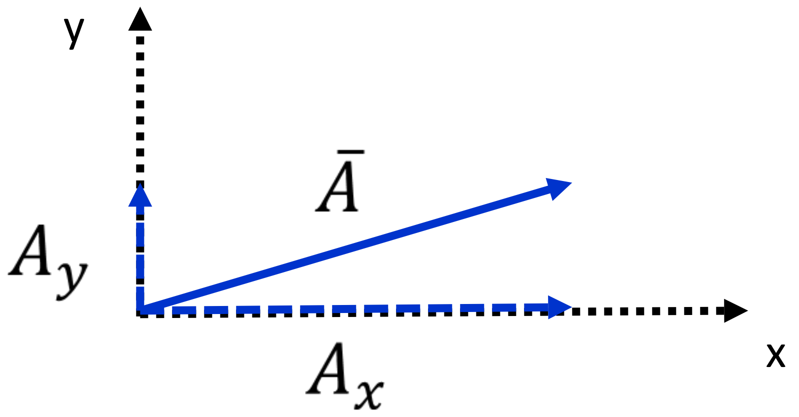
$$\|\hat{A}\| = 1$$

Used to indicate direction

Quick Math Review

Vector:

A geometric object with magnitude and direction



Can be written in matrix form as a column vector

$$\bar{A} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

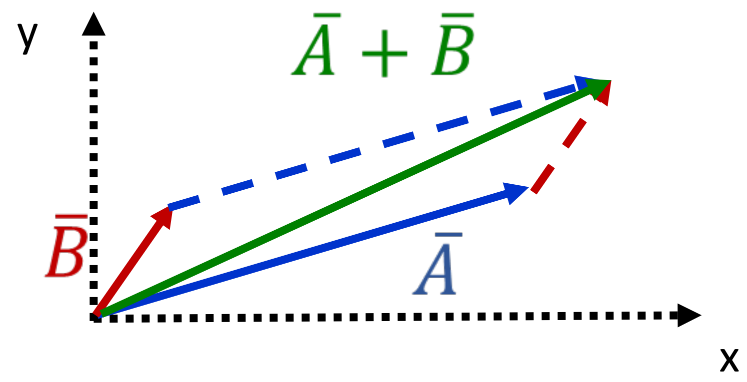
Quick Math Review

Vector Addition

- Sum each component of the vector

$$\bar{A} + \bar{B} = (A_1 + B_{1,} A_2 + B_{2,\ldots,} A_n + B_{n)}$$

$$\bar{A} + \bar{B} = \begin{bmatrix} A_x + B_x \\ A_y + B_y \end{bmatrix}$$



Yields a new vector
Commutative

Quick Math Review

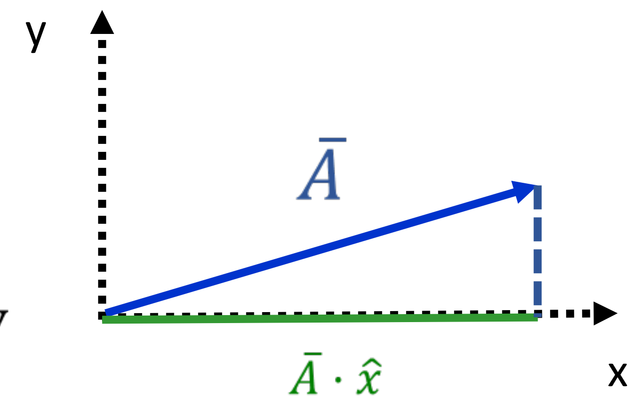
Dot Product

- Geometric Representation:

$$\bar{A} \cdot \bar{B} = \|A\| \|B\| \cos(\theta)$$

- Matrix Representation

$$\bar{A} \cdot \bar{B} = \begin{bmatrix} A_x \\ A_y \end{bmatrix} \cdot \begin{bmatrix} B_x \\ B_y \end{bmatrix} = A_x B_x + A_y B_y$$



$$\bar{A} \cdot \hat{x}$$
$$= \|A\| * 1 * \cos(\theta)$$
$$= A_x * 1 + A_y * 0$$

Yields a scalar

Commutative

Quick Math Review

Cross Product

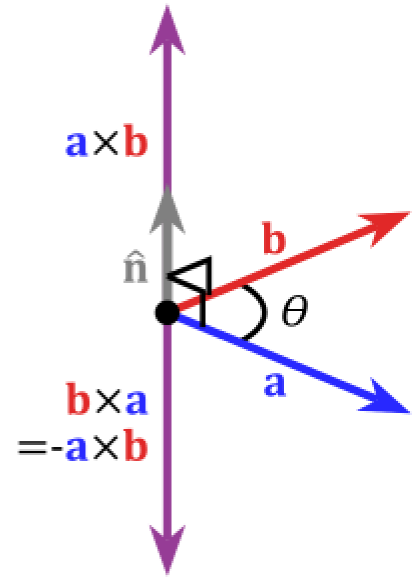
- Geometric Representation:

  $$\bar{A} \times \bar{B} = ||A||\,||B||\, sin(\theta)\, \hat{n}$$

  where $\hat{n}$ is perpendicular to both $\bar{A}$ and $\bar{B}$

- Matrix Representation

  $$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$



Yields a vector perpendicular to both original vectors

Not commutative

https://en.wikipedia.org/wiki/Cross_product

Quick Math Review

Matrix Addition

- Sum matching elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (a+e) & (b+f) \\ (c+g) & (d+h) \end{bmatrix}$$

Matrices must be of same size
Yields a new matrix of the same size
Commutative

Quick Math Review

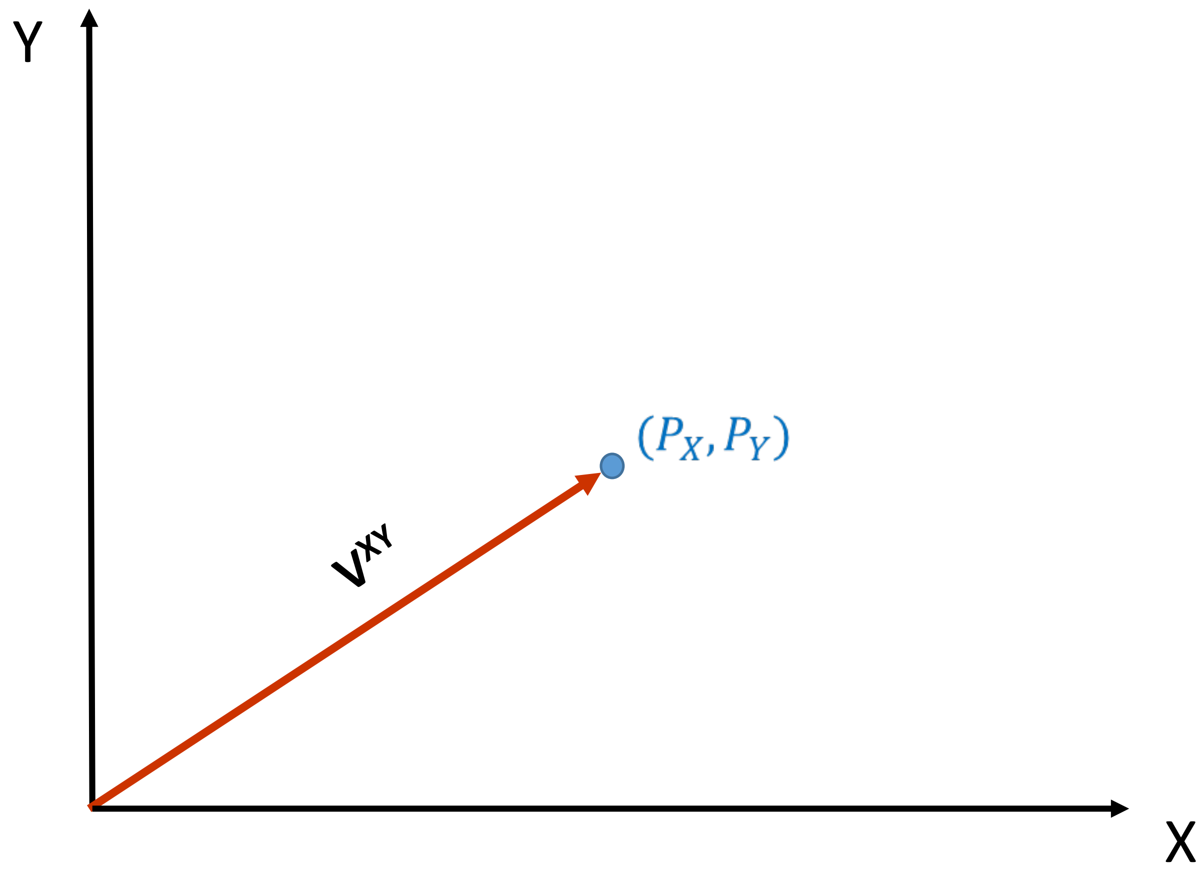
Matrix Multiplication

- Multiply rows and columns and sum products

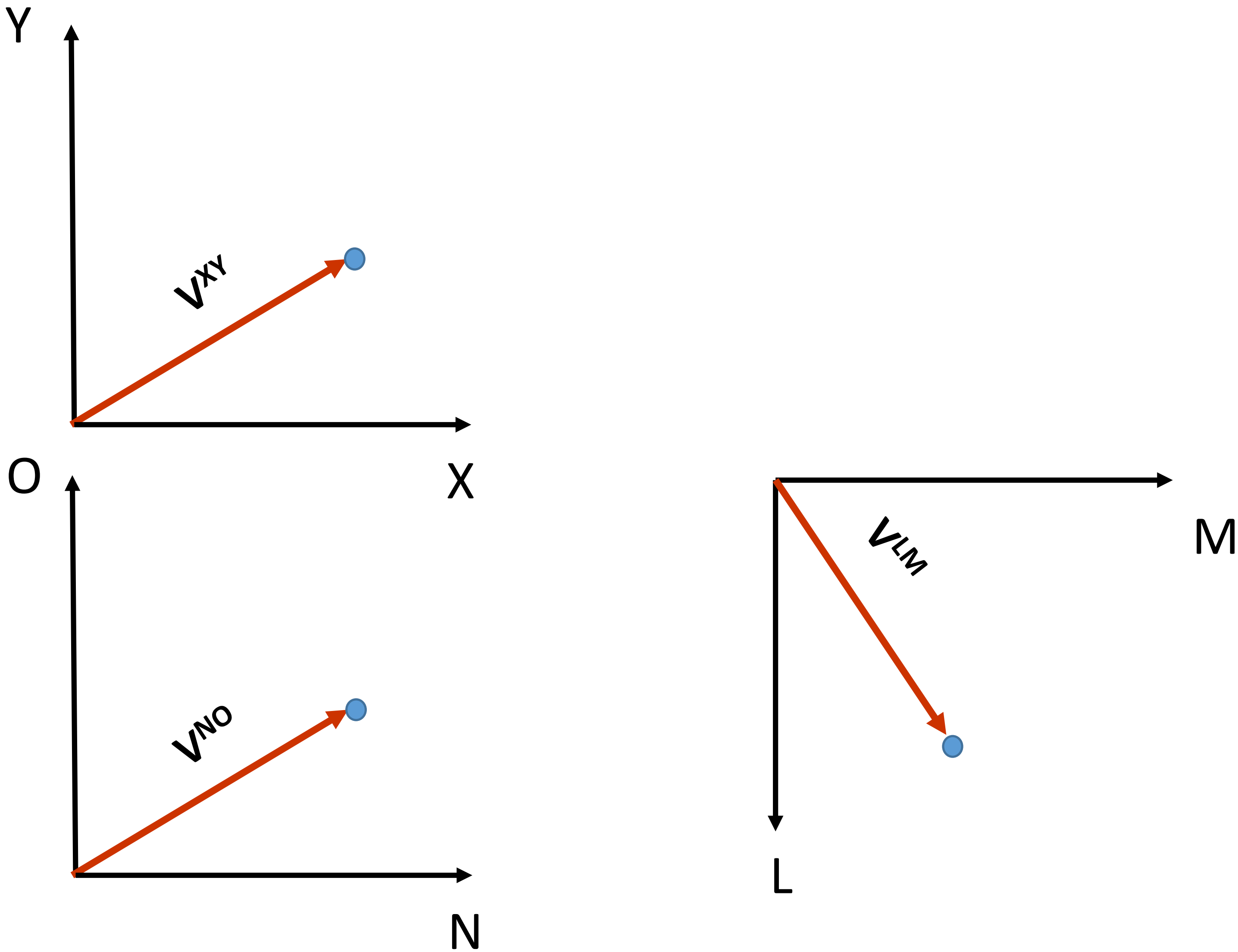
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (ae+bg) & (af+bh) \\ (ce+dg) & (cf+dh) \end{bmatrix}$$

Matrices must have the same inner dimension
Yields a new matrix of the same size
Not commutative

We can use vectors to succinctly represent a point with respect to a certain reference frame

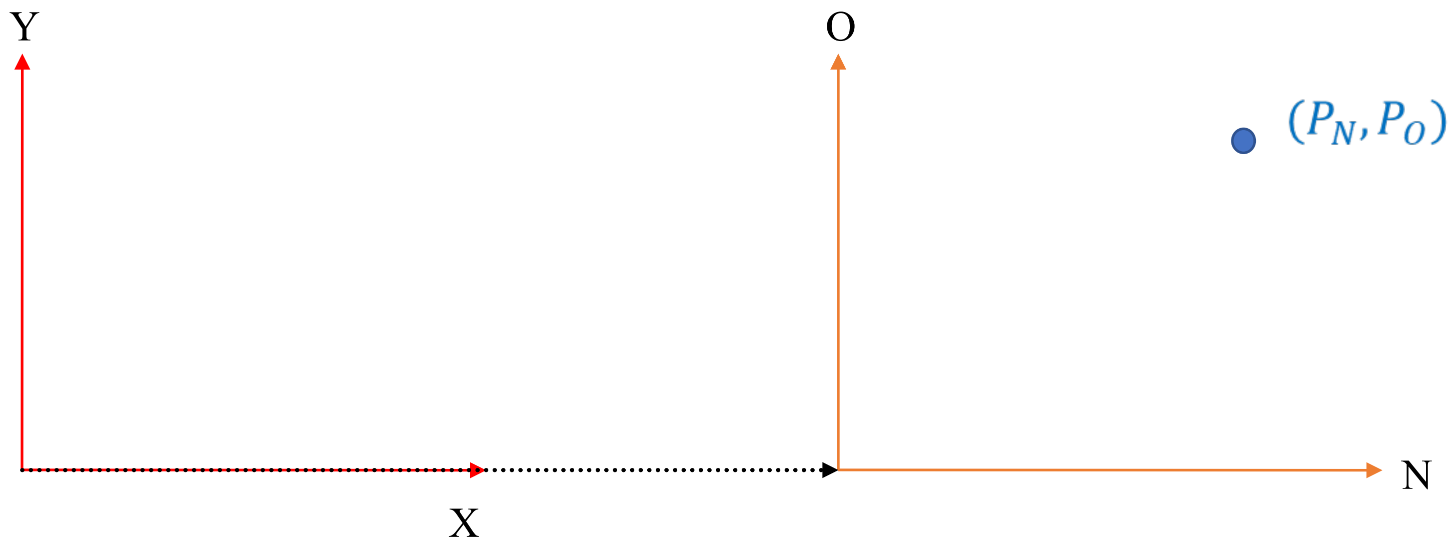


We will use superscripts to indicate our reference frame



Basic Transformations

Representing a point in a different frame: Translation along the x-axis



Basic Transformations

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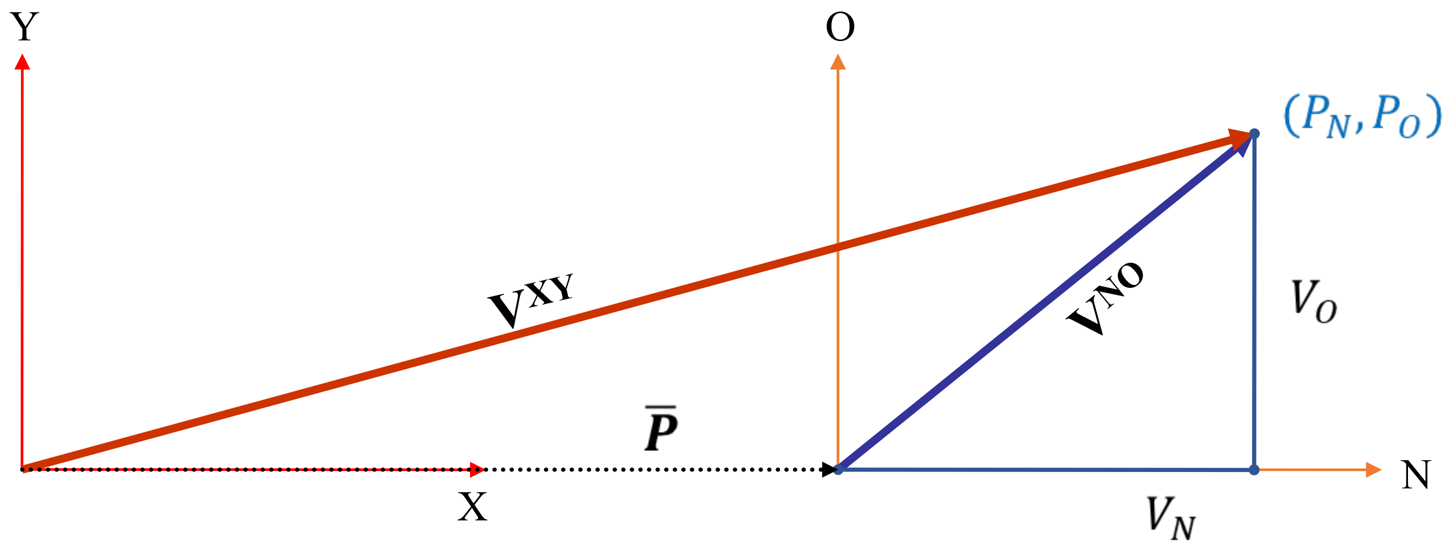

Basic Transformations

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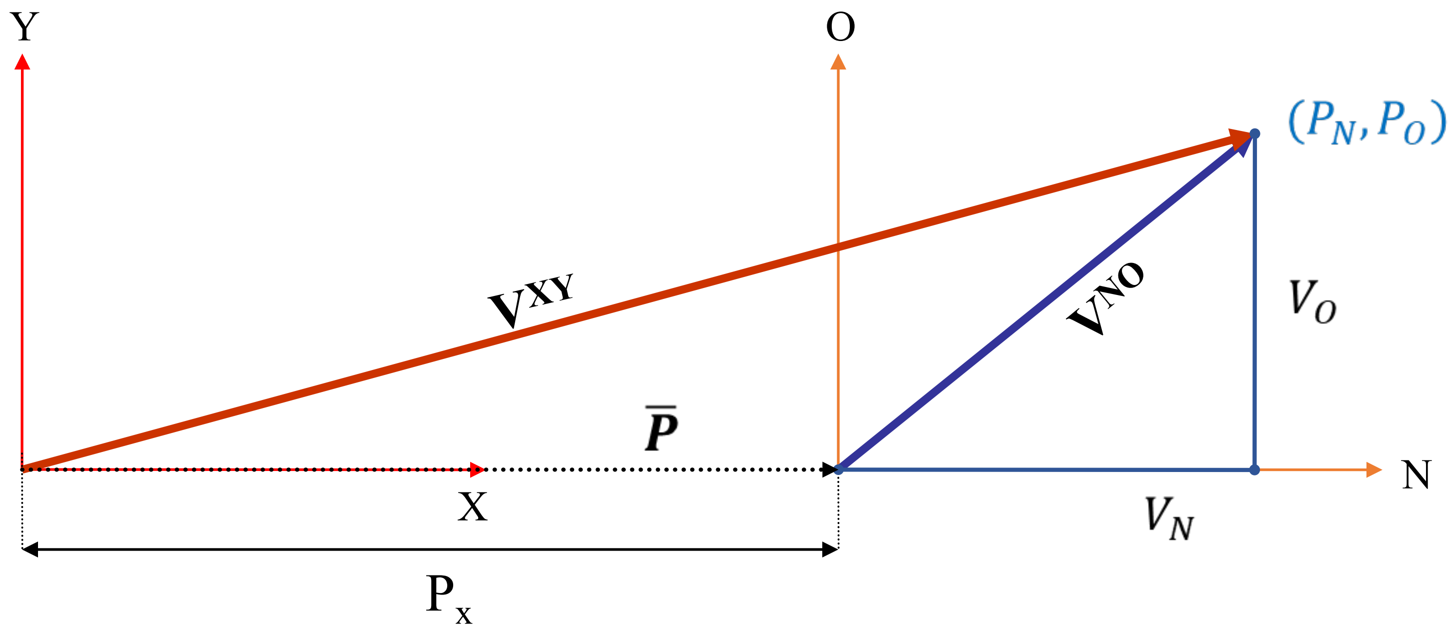
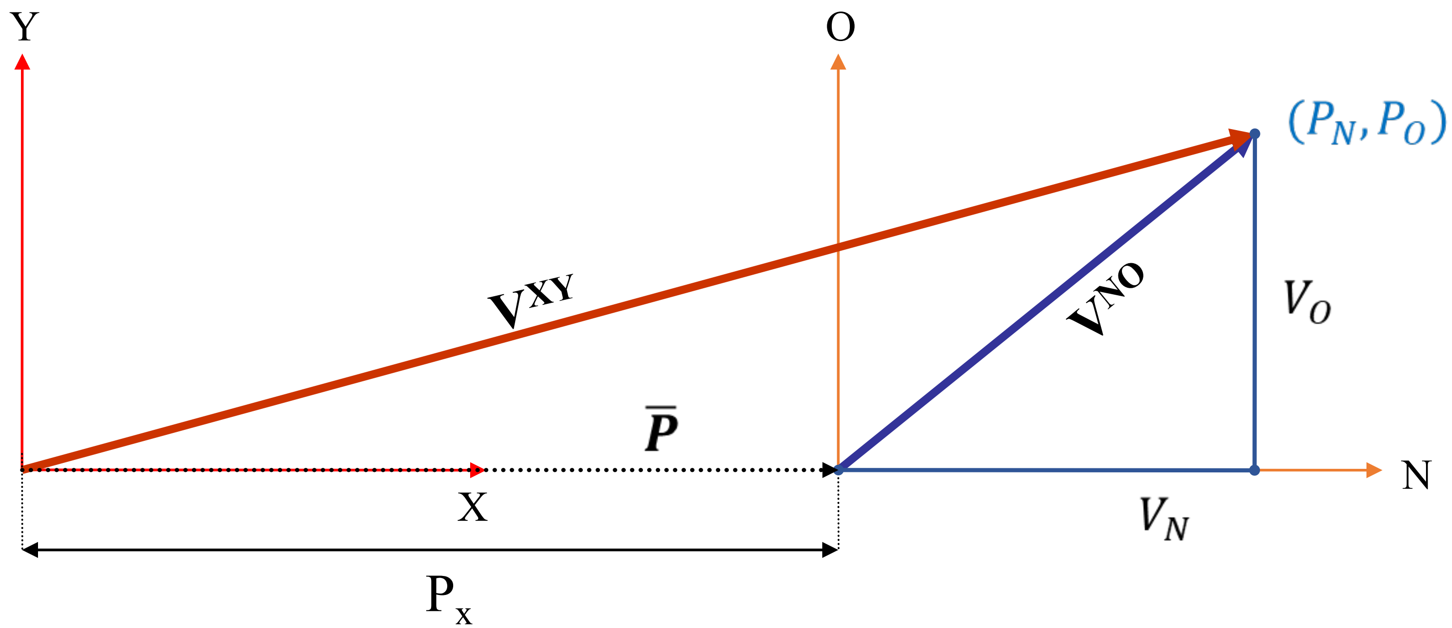


P_x = distance between the XY and NO coordinate planes

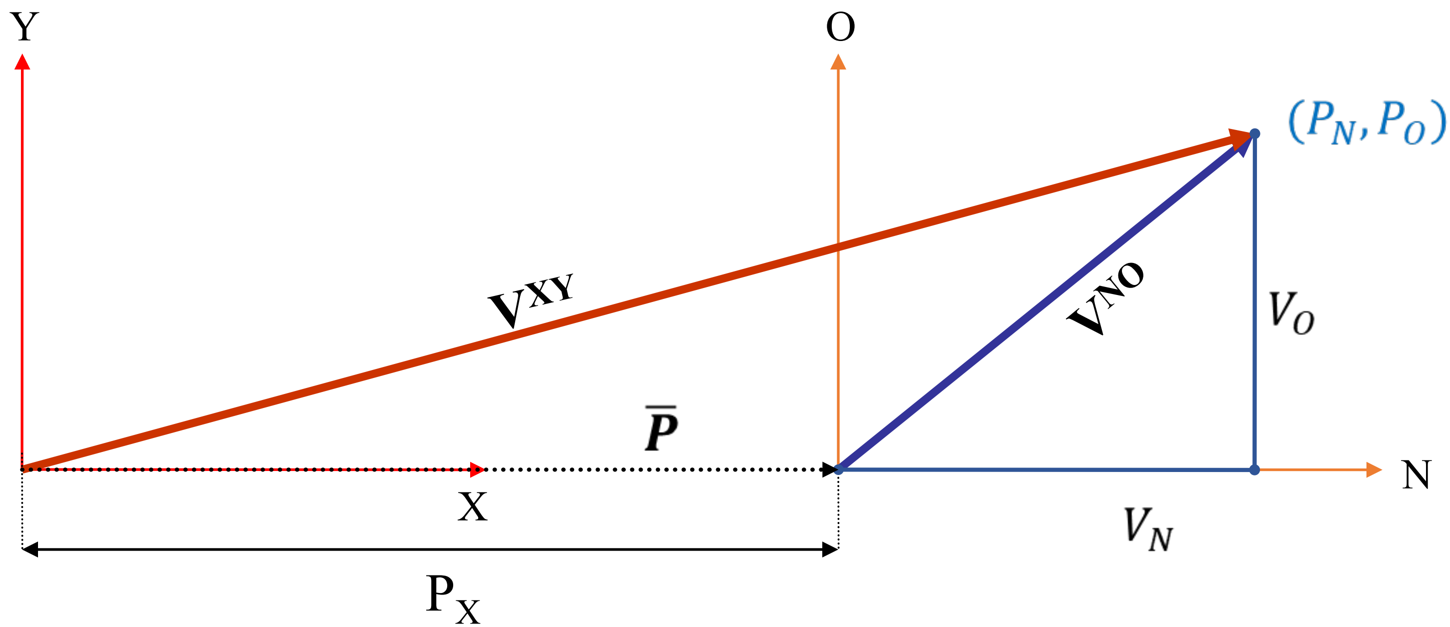
Notation: $\bar{V}^{XY} = \begin{bmatrix} V_X \\ V_Y \end{bmatrix}$ $\bar{V}^{NO} = \begin{bmatrix} V_N \\ V_O \end{bmatrix}$ $\bar{P} = \begin{bmatrix} P_X \\ 0 \end{bmatrix}$

Writing $\overline{\mathbf{V}}^{\mathbf{XY}}$ in terms of $\overline{\mathbf{V}}^{\mathbf{NO}}$



$$\bar{V}^{XY} = \bar{P} + \bar{V}^{NO} = \begin{bmatrix} P_X + V_N \\ V_O \end{bmatrix}$$

Writing $\overline{\mathbf{V}}^{\mathbf{XY}}$ in terms of $\overline{\mathbf{V}}^{\mathbf{NO}}$



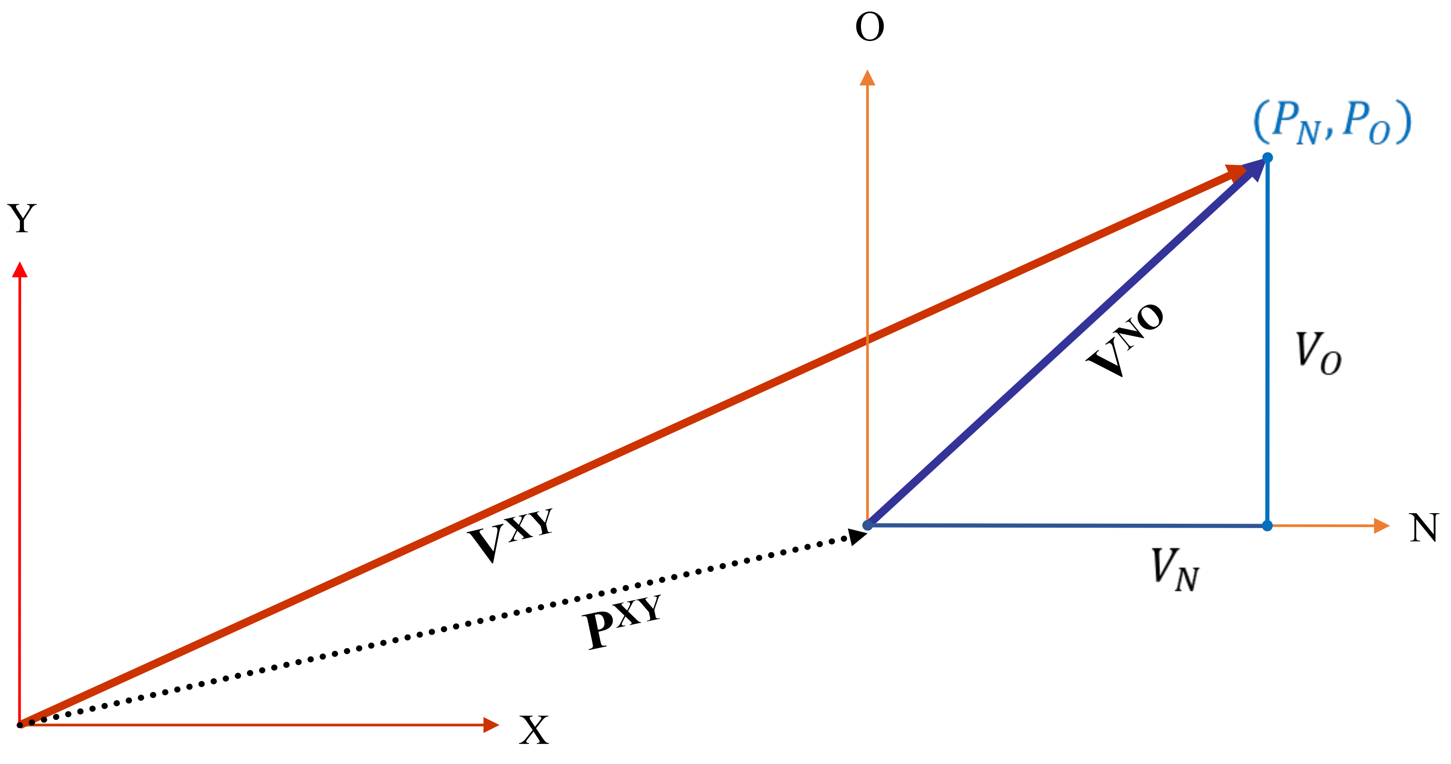
$$\bar{V}^{XY} = \bar{P} + \bar{V}^{NO} = \begin{bmatrix} P_X + V_N \\ V_O \end{bmatrix}$$

$$V_X^{XY} = P_X + V_N$$

$$V_Y^{XY} = V_O$$

Basic Transformations

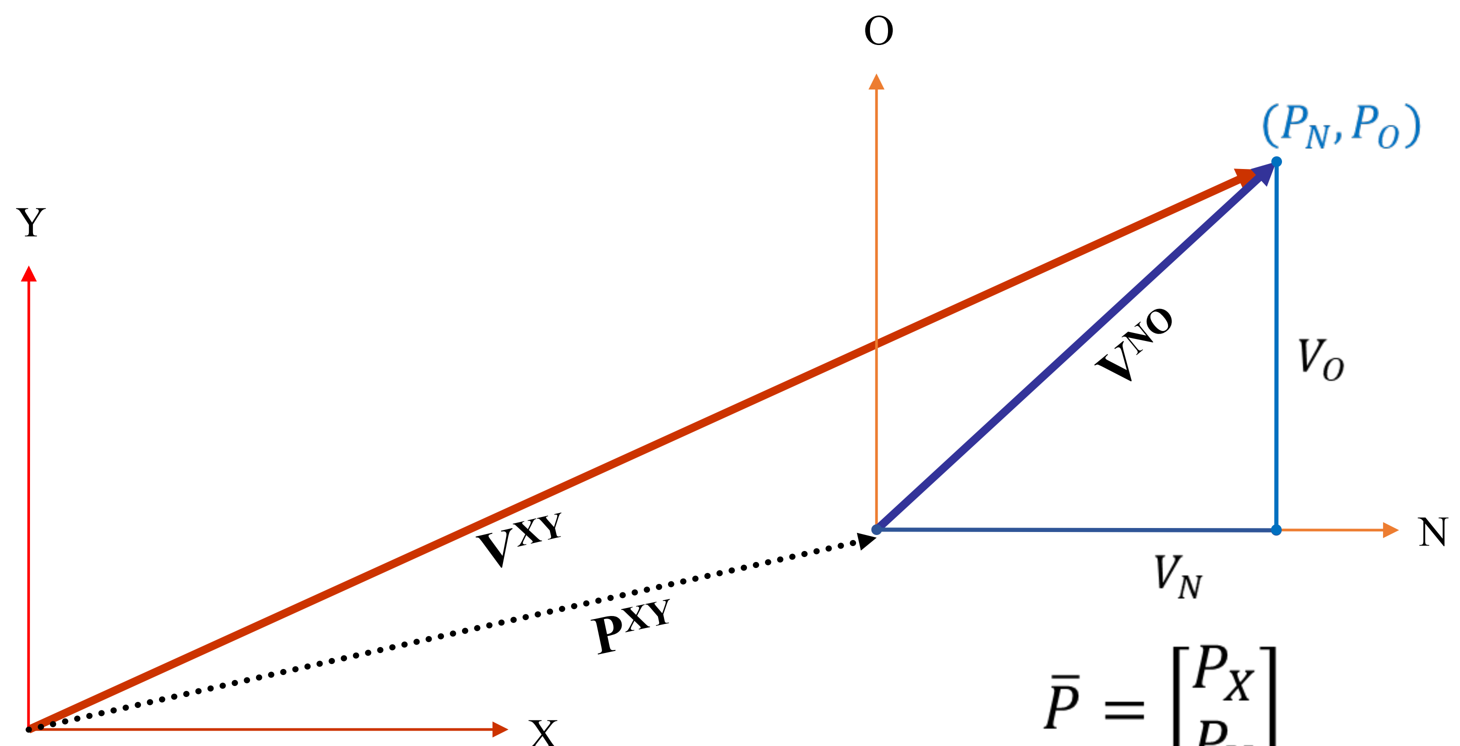
Representing a point in a different frame: Translation along the x- and y-axes



$$\bar{V}^{XY} =$$

Basic Transformations

Representing a point in a different frame: Translation along the x- and y-axes



$$\bar{P} = \begin{bmatrix} P_X \\ P_Y \end{bmatrix}$$

$$\bar{V}^{XY} = \bar{P} + \bar{V}^{NO} = \begin{bmatrix} P_X + V_N \\ P_Y + V_O \end{bmatrix}$$

$$V_X^{XY} = P_X + V_N$$

$$V_Y^{XY} = P_Y + V_O$$

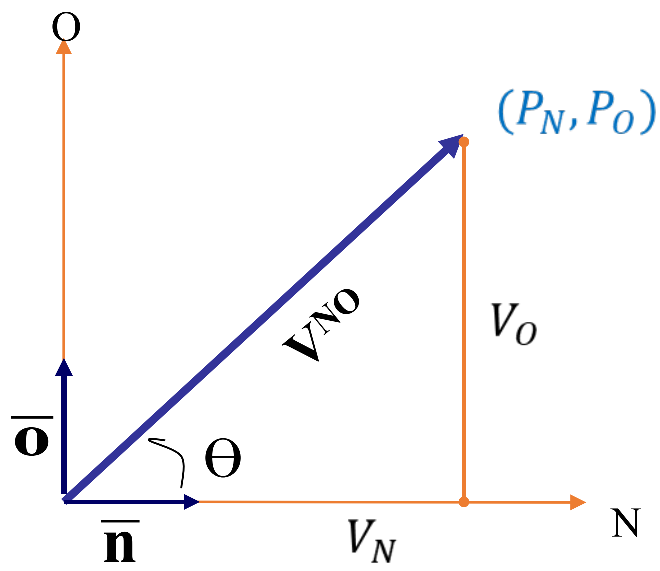
Using Basis Vectors

Basis vectors are unit vectors that point along a coordinate axis

$\overline{\mathbf{n}}$ Unit vector along the N-Axis

$\overline{\mathbf{o}}$ Unit vector along the N-Axis

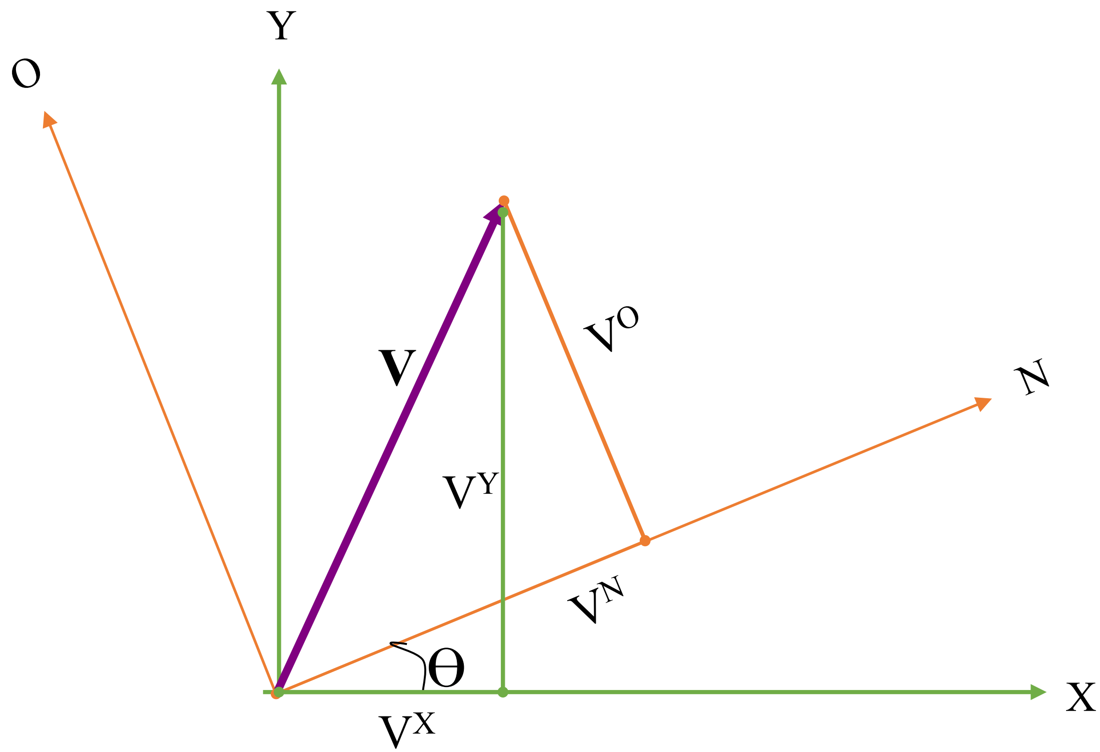
$\left\| \mathbf{V}^{\mathbf{NO}} \right\|$ Magnitude of the V^{NO} vector



$$\overline{\mathbf{V}}^{\mathbf{NO}} = \begin{bmatrix} \mathbf{V}^{\mathbf{N}} \\ \mathbf{V}^{\mathbf{O}} \end{bmatrix} = \begin{bmatrix} \left\| \mathbf{V}^{\mathbf{NO}} \right\| \cos\theta \\ \left\| \mathbf{V}^{\mathbf{NO}} \right\| \sin\theta \end{bmatrix} = \begin{bmatrix} \left\| \mathbf{V}^{\mathbf{NO}} \right\| \cos\theta \\ \left\| \mathbf{V}^{\mathbf{NO}} \right\| \cos(90-\theta) \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{n}} \\ \overline{\mathbf{V}}^{\mathbf{NO}} \bullet \overline{\mathbf{o}} \end{bmatrix}$$

Basic Transformations

Representing a point in a different frame: Rotation about z-axis (out of the board)



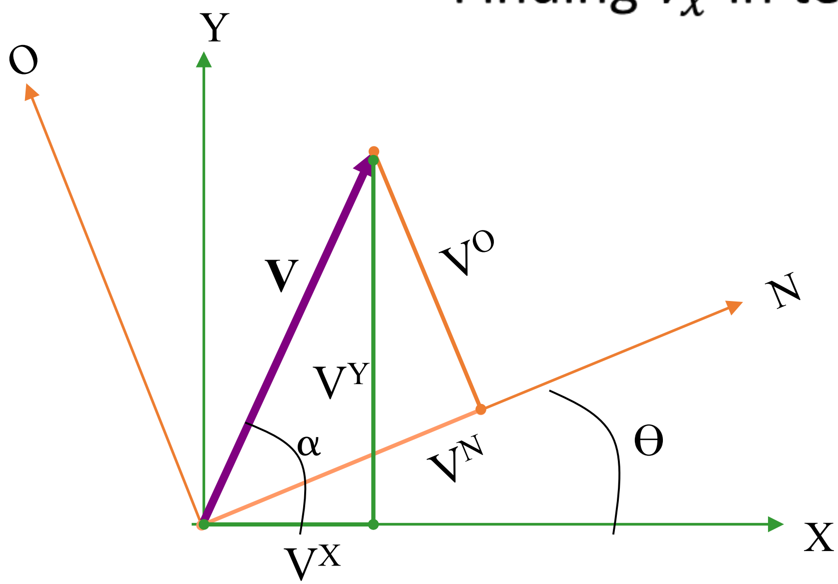
Θ = Angle of rotation between the XY and NO coordinate axis

$$\bar{V}^{XY} = \begin{bmatrix} V_X \\ V_y \end{bmatrix} \qquad \bar{V}^{NO} = \begin{bmatrix} V_N \\ V_O \end{bmatrix}$$

Basic Transformations

Rotation about z-axis (out of the board)

Finding V_x in terms of V_N and V_O



$\bar{V}$ can be considered with respect to the XY coordinates or NO coordinates

$$\|\bar{V}^{XY}\| = \|\bar{V}^{NO}\|$$

$$V_x = \|\bar{V}^{XY}\|\cos(\alpha) = \|\bar{V}^{NO}\|\cos(\alpha) = \bar{V}^{NO} \cdot \hat{X}$$

$$= (V_N\hat{N} + V_O\hat{O}) \cdot \hat{X}$$ (Substituting for V^{NO} using the N and O components of the vector)

$$= V_N(\hat{N} \cdot \hat{X}) + V_O(\hat{O} \cdot \hat{X})$$

$$= V_N(1 * 1 * \cos(\theta)) + V_O(1 * 1 * \cos(90 + \theta))$$

$$= V_N(\cos(\theta)) - V_O(sin(\theta))$$

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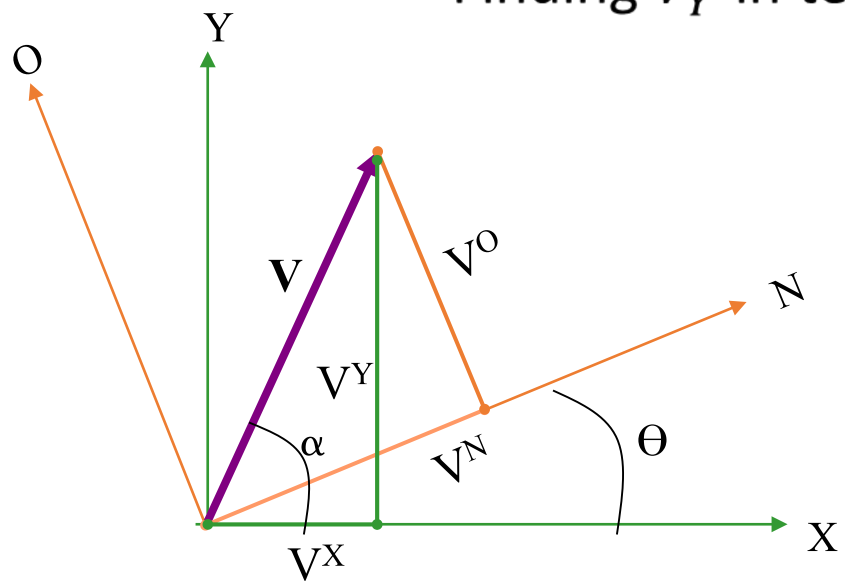
Basic Transformations

Rotation about z-axis (out of the board)

Finding V_Y in terms of V_N and V_O



$\bar{V}$ can be considered with respect to the XY coordinates or NO coordinates

$$\|\bar{V}^{XY}\| = \|\bar{V}^{NO}\|$$

$$V_Y = \|\bar{V}^{XY}\|\, sin(\alpha) = \|\bar{V}^{NO}\|\, sin(\alpha) = \|\bar{V}^{NO}\|\, cos(90-\alpha) = \bar{V}^{NO}\cdot\hat{Y}$$

$$= (V_N\hat{N} + V_O\hat{O})\cdot\hat{Y}$$

(Substituting for V^{NO} using the N and O components of the vector)

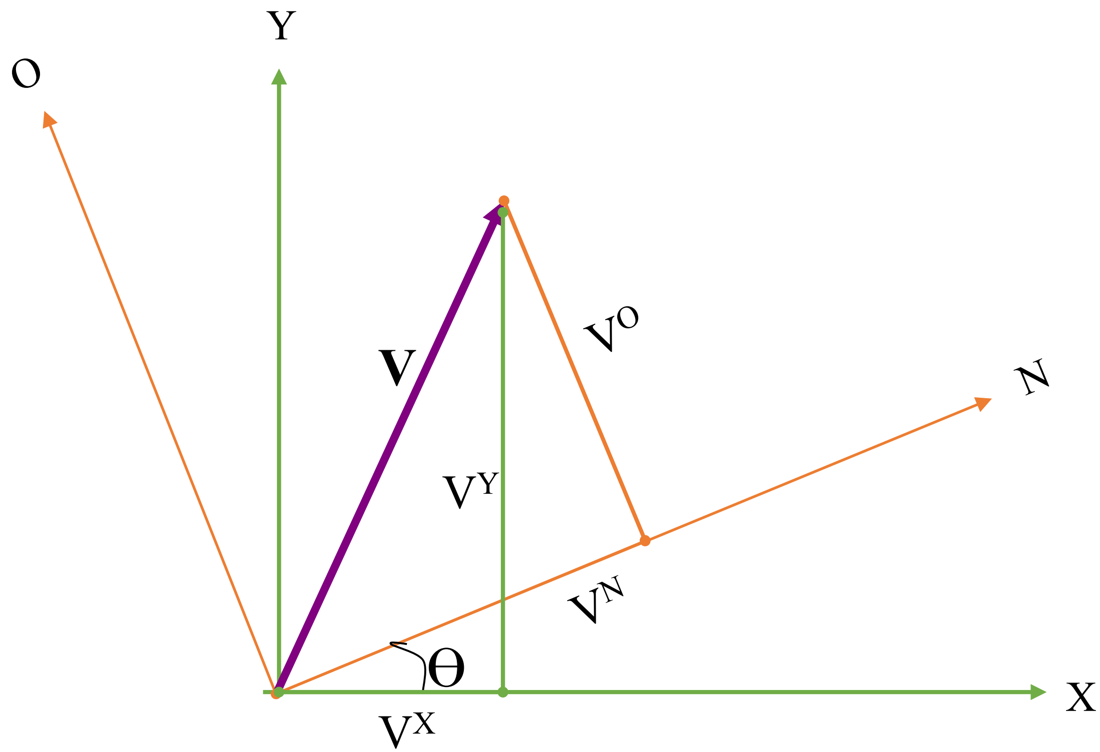
$$= V_N(\hat{N}\cdot\hat{Y}) + V_O(\hat{O}\cdot\hat{Y})$$

$$= V_N(1*1*\cos(90-\theta)) + V_O(1*1*\cos(\theta))$$

$$= V_N(\sin(\theta)) + V_O(cos(\theta))$$

Basic Transformations

Representing a point in a different frame: Rotation about z-axis (out of the board)



$$\bar{V}^{XY} = \begin{bmatrix} V_X \\ V_y \end{bmatrix} = \begin{bmatrix} V_N(\cos(\theta)) - V_O(sin(\theta)) \\ V_N(\sin(\theta)) + V_O(cos(\theta)) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} V_N \\ V_O \end{bmatrix}$$

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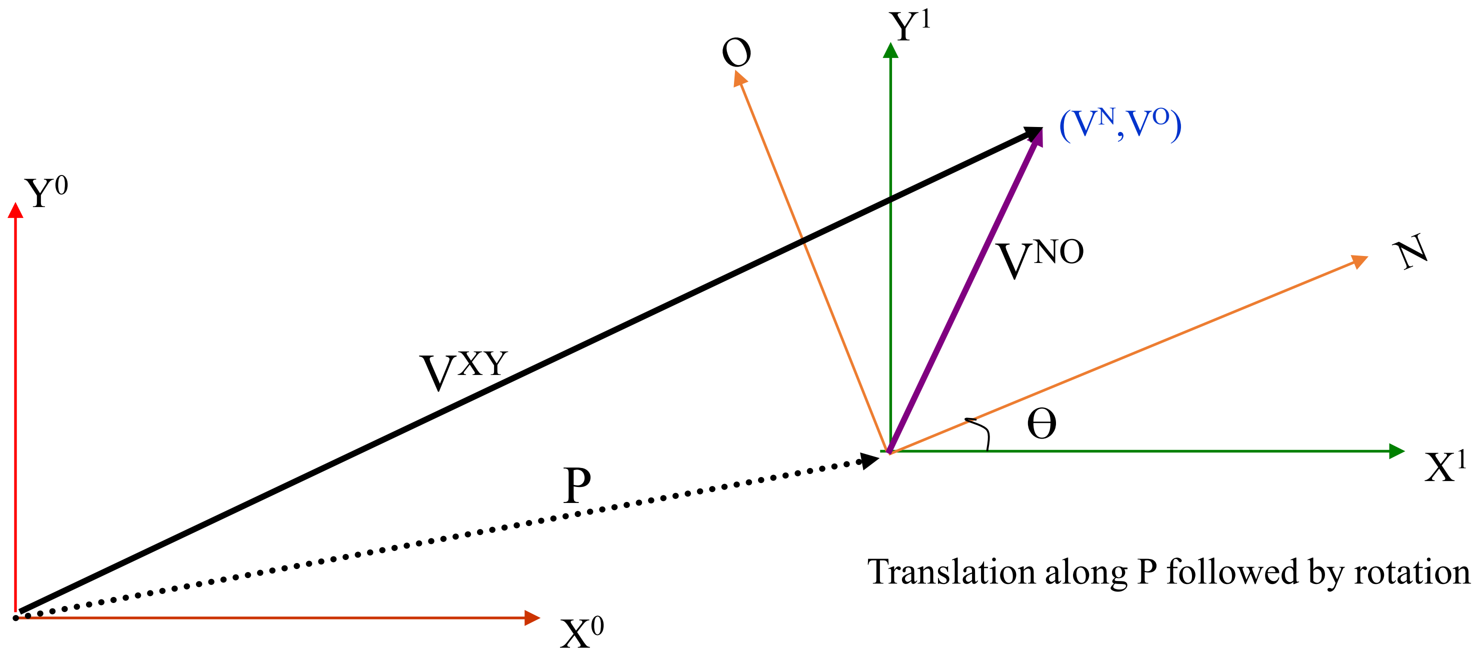
Compound Transformations

Representing a point in a different frame: Translation along the x- and y-axes and rotation

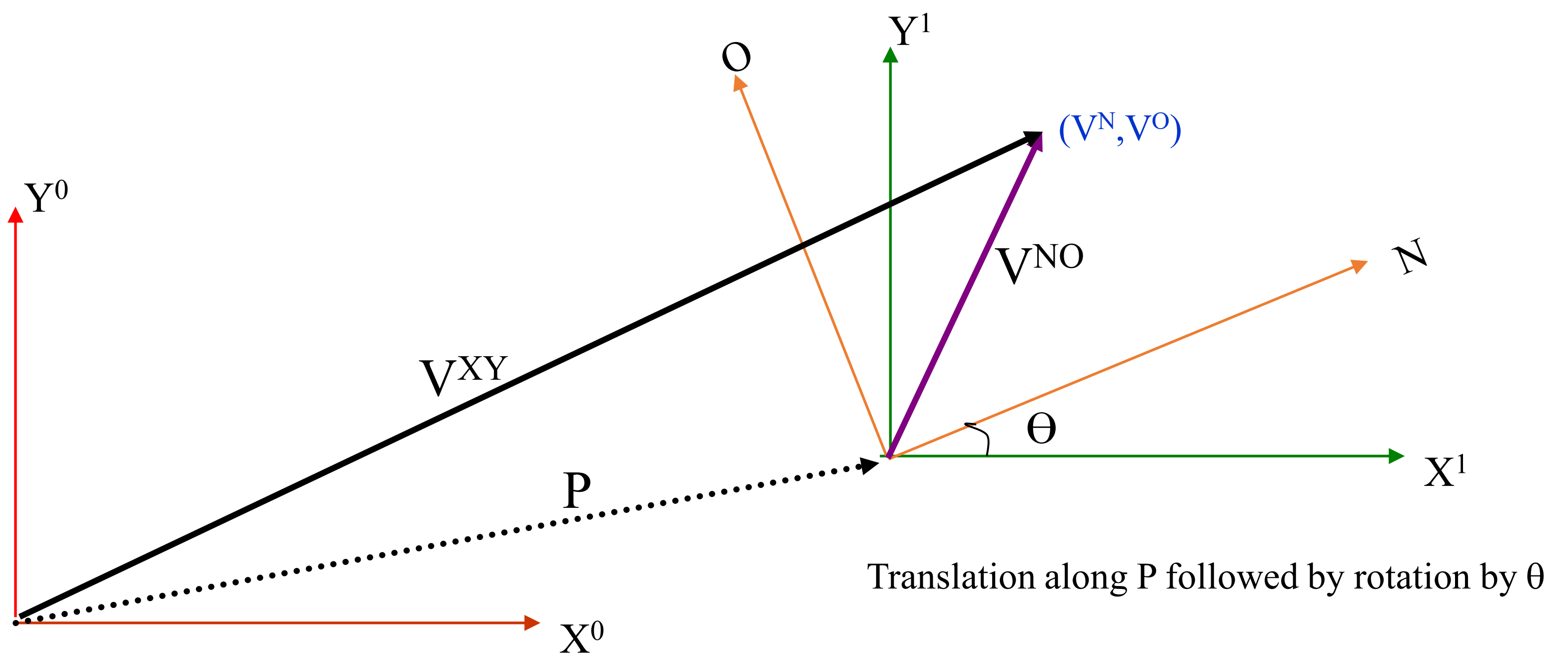


Translation along P followed by rotation by θ

$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{V}^{X} \\ \mathbf{V}^{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V}^{N} \\ \mathbf{V}^{O} \end{bmatrix}$$

Compound Transformations

Representing a point in a different frame: Translation along the x- and y-axes and rotation



$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \end{bmatrix}$$

(Note : P_x, P_y are relative to the original coordinate frame. Translation followed by rotation is different than rotation followed by translation.)

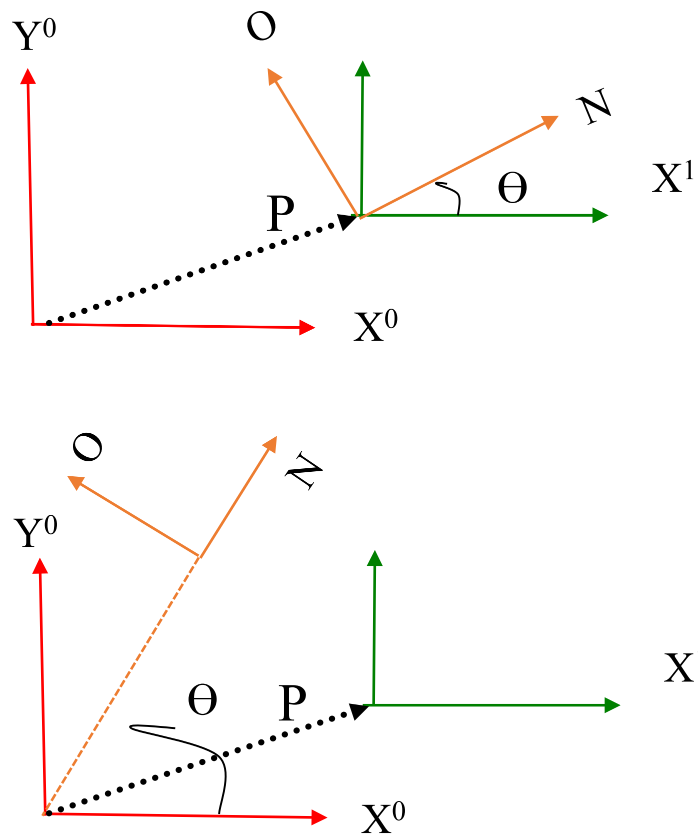
Relative versus absolute translation

Relative:

- Can be composed to create homogenous transformation matrix.
- Translations are with respect to a frame fixed to the robot or point.

Absolute:

- Translations are with respect to a fixed world frame.



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The Homogeneous Matrix can represent both translation and rotation

$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \end{bmatrix}$$

What we found by doing a translation and a rotation

$$= \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_x \\ \mathbf{P}_y \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \\ 1 \end{bmatrix}$$

Padding with 0's and 1's

$$= \begin{bmatrix} \mathbf{V}^X \\ \mathbf{V}^Y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{P}_x \\ \sin\theta & \cos\theta & \mathbf{P}_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{V}^N \\ \mathbf{V}^O \\ 1 \end{bmatrix}$$

Simplifying into a matrix form

$$\mathbf{H} = \begin{bmatrix} \cos\theta & -\sin\theta & \mathbf{P}_x \\ \sin\theta & \cos\theta & \mathbf{P}_y \\ 0 & 0 & 1 \end{bmatrix}$$

Homogenous Matrix for a Translation in XY plane, followed by a Rotation around the z-axis

Rotation Matrices in 3D

$$\mathbf{R_z} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

← Rotation around the Z-Axis

$$\mathbf{R_y} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

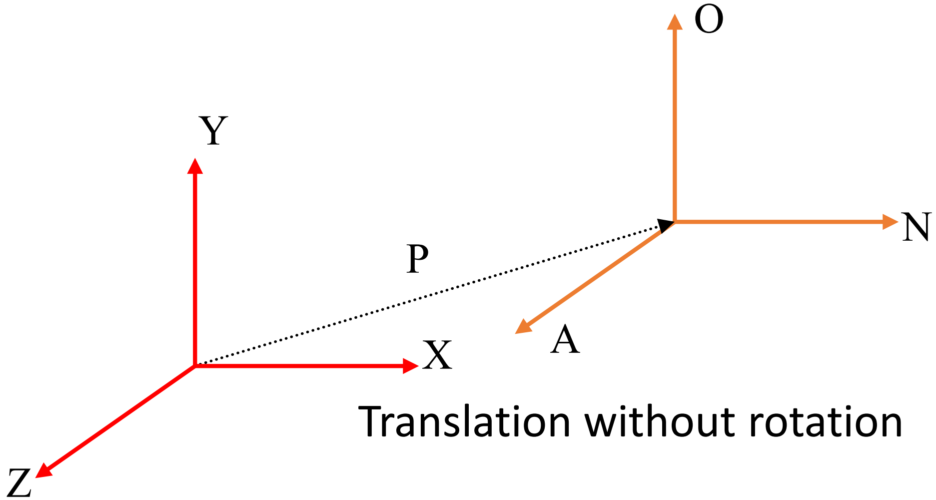
← Rotation around the Y-Axis

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

← Rotation around the X-Axis

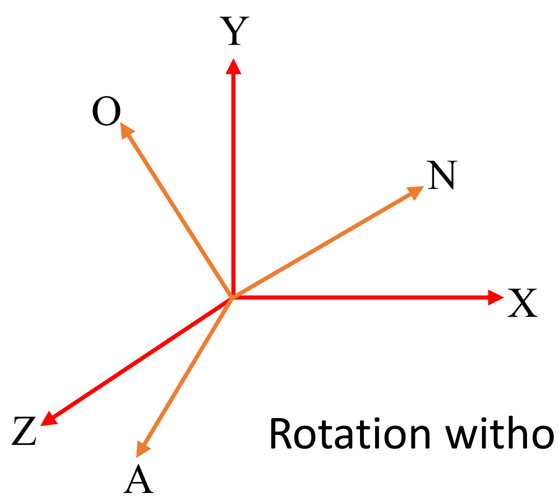
Homogeneous Matrices in 3D

H is a 4x4 matrix that can describe a translation, rotation, or both in one matrix



Translation without rotation

$$H = \begin{bmatrix} 1 & 0 & 0 & P_x \\ 0 & 1 & 0 & P_y \\ 0 & 0 & 1 & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Rotation without translation

$$H = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Could be rotation around z-axis, x-axis, y-axis or a combination of the three.

Homogeneous Continued....

$$\mathbf{V}^{XY} = \mathbf{H}\begin{bmatrix} \mathbf{V}^{N} \\ \mathbf{V}^{O} \\ \mathbf{V}^{A} \\ 1 \end{bmatrix}$$

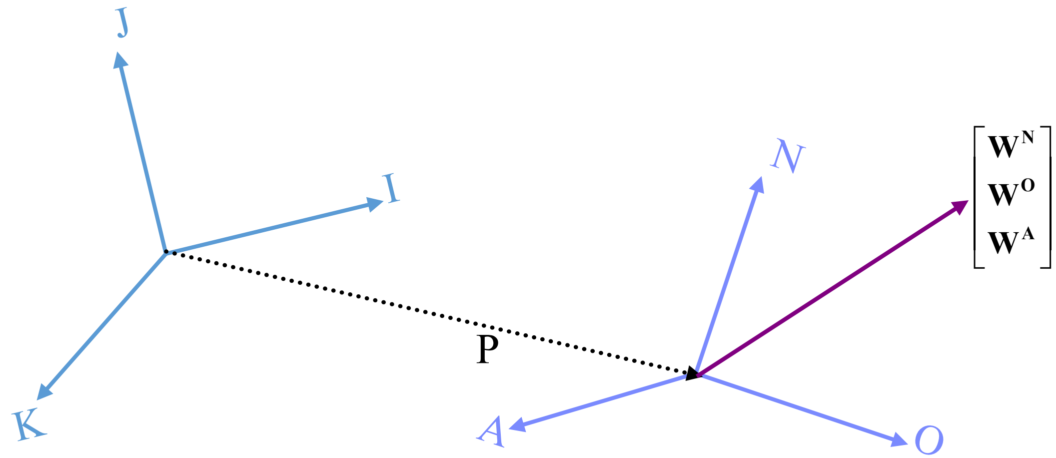
The (n,o,a) position of a point relative to the current coordinate frame you are in.

$$\mathbf{V}^{XY} = \begin{bmatrix} \mathbf{n}_x & \mathbf{o}_x & \mathbf{a}_x & \mathbf{P}_x \\ \mathbf{n}_y & \mathbf{o}_y & \mathbf{a}_y & \mathbf{P}_y \\ \mathbf{n}_z & \mathbf{o}_z & \mathbf{a}_z & \mathbf{P}_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{V}^{N} \\ \mathbf{V}^{O} \\ \mathbf{V}^{A} \\ 1 \end{bmatrix}$$

$$\mathbf{V}^{X} = \mathbf{n}_x\mathbf{V}^{N} + \mathbf{o}_x\mathbf{V}^{O} + \mathbf{a}_x\mathbf{V}^{A} + \mathbf{P}_x$$

The rotation and translation part can be combined into a single homogeneous matrix IF and ONLY IF both are relative to the same coordinate frame.

Finding the Homogeneous Matrix



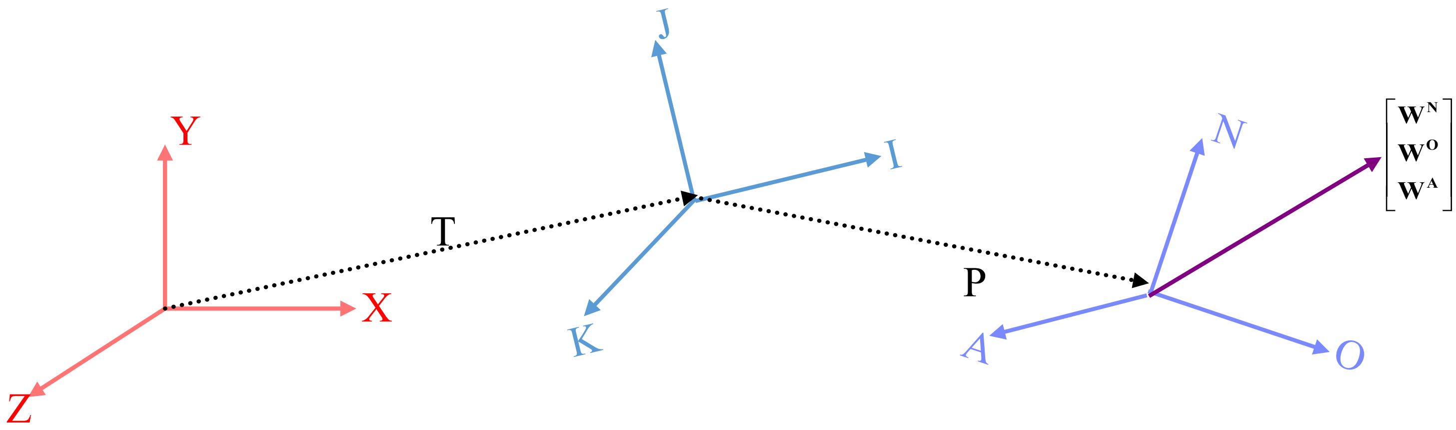
$$\begin{bmatrix} W^I \\ W^J \\ W^K \end{bmatrix}$$ Point relative to the I-J-K frame

$$\begin{bmatrix} W^N \\ W^O \\ W^A \end{bmatrix}$$ Point relative to the N-O-A frame

$$\begin{bmatrix} W^I \\ W^J \\ W^K \end{bmatrix} = \begin{bmatrix} P_i \\ P_j \\ P_k \end{bmatrix} + \begin{bmatrix} n_i & o_i & a_i \\ n_j & o_j & a_j \\ n_k & o_k & a_k \end{bmatrix} \begin{bmatrix} W^N \\ W^O \\ W^A \end{bmatrix}$$

$$\begin{bmatrix} W^I \\ W^J \\ W^K \\ 1 \end{bmatrix} = \begin{bmatrix} n_i & o_i & a_i & P_i \\ n_j & o_j & a_j & P_j \\ n_k & o_k & a_k & P_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} W^N \\ W^O \\ W^A \\ 1 \end{bmatrix}$$

Finding the Homogeneous Matrix



$$\begin{bmatrix} W^X \\ W^Y \\ W^Z \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + \begin{bmatrix} i_x & j_x & k_x \\ i_y & j_y & k_y \\ i_z & j_z & k_z \end{bmatrix} \begin{bmatrix} W^I \\ W^J \\ W^k \end{bmatrix} \longrightarrow \begin{bmatrix} W^X \\ W^Y \\ W^Z \\ 1 \end{bmatrix} = \begin{bmatrix} i_x & j_x & k_x & T_x \\ i_y & j_y & k_y & T_y \\ i_z & j_z & k_z & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} W^I \\ W^J \\ W^K \\ 1 \end{bmatrix}$$

Substituting for $\begin{bmatrix} W^I \\ W^J \\ W^K \end{bmatrix}$

$$\begin{bmatrix} W^X \\ W^Y \\ W^Z \\ 1 \end{bmatrix} = \begin{bmatrix} i_x & j_x & k_x & T_x \\ i_y & j_y & k_y & T_y \\ i_z & j_z & k_z & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_i & o_i & a_i & P_i \\ n_j & o_j & a_j & P_j \\ n_k & o_k & a_k & P_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} W^N \\ W^O \\ W^A \\ 1 \end{bmatrix}$$

The Homogeneous Matrix is a concatenation of numerous translations and rotations

$$\begin{bmatrix} W^X \\ W^Y \\ W^Z \\ 1 \end{bmatrix} = H \begin{bmatrix} W^N \\ W^O \\ W^A \\ 1 \end{bmatrix}$$

$$H = \begin{bmatrix} i_x & j_x & k_x & T_x \\ i_y & j_y & k_y & T_y \\ i_z & j_z & k_z & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_i & o_i & a_i & P_i \\ n_j & o_j & a_j & P_j \\ n_k & o_k & a_k & P_k \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

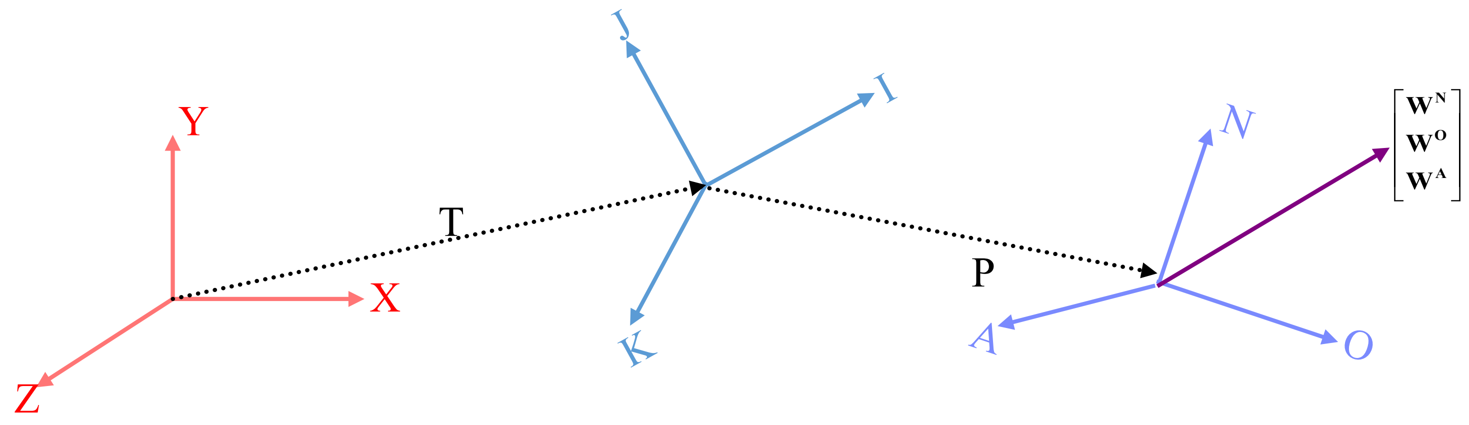
Product of the two matrices

Notice that H can also be written as:

$$H = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_x & j_x & k_x & 0 \\ i_y & j_y & k_y & 0 \\ i_z & j_z & k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & P_i \\ 0 & 1 & 0 & P_j \\ 0 & 0 & 1 & P_k \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_i & o_i & a_i & 0 \\ n_j & o_j & a_j & 0 \\ n_k & o_k & a_k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

H = (Translation relative to the XYZ frame) * (Rotation relative to the XYZ frame) * (Translation relative to the IJK frame) * (Rotation relative to the IJK frame)

One more variation on finding the homogeneous transformation matrix



H = (Rotate so that the X-axis is aligned with T)

* (Translate along the new t-axis by || T || (magnitude of T))

* (Rotate so that the t-axis is aligned with P)

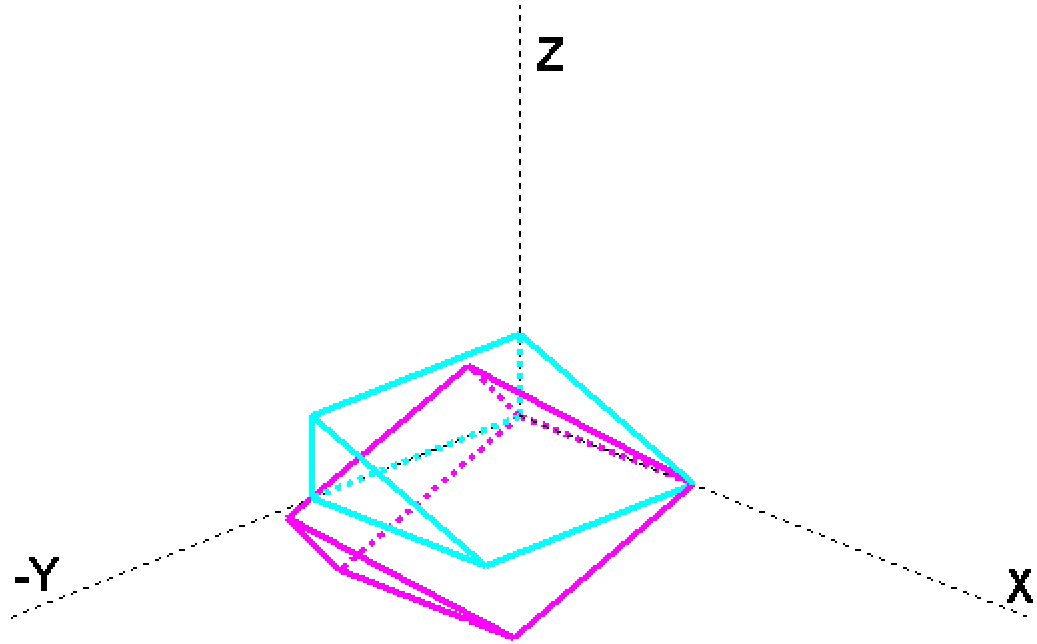
* (Translate along the p-axis by || P || (magnitude of P))

* (Rotate so that the p-axis is aligned with the O-axis)

Three-Dimensional Illustration

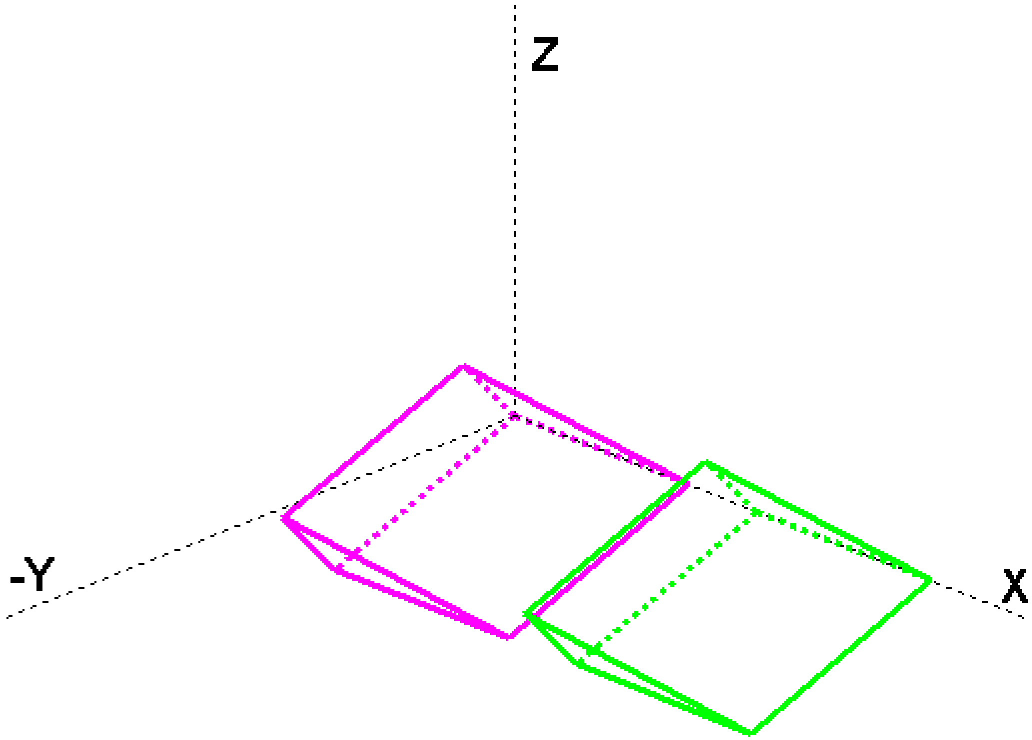
- Rotate X
- Translate X
- Rotate Z
- Translate Z

Rotation about X



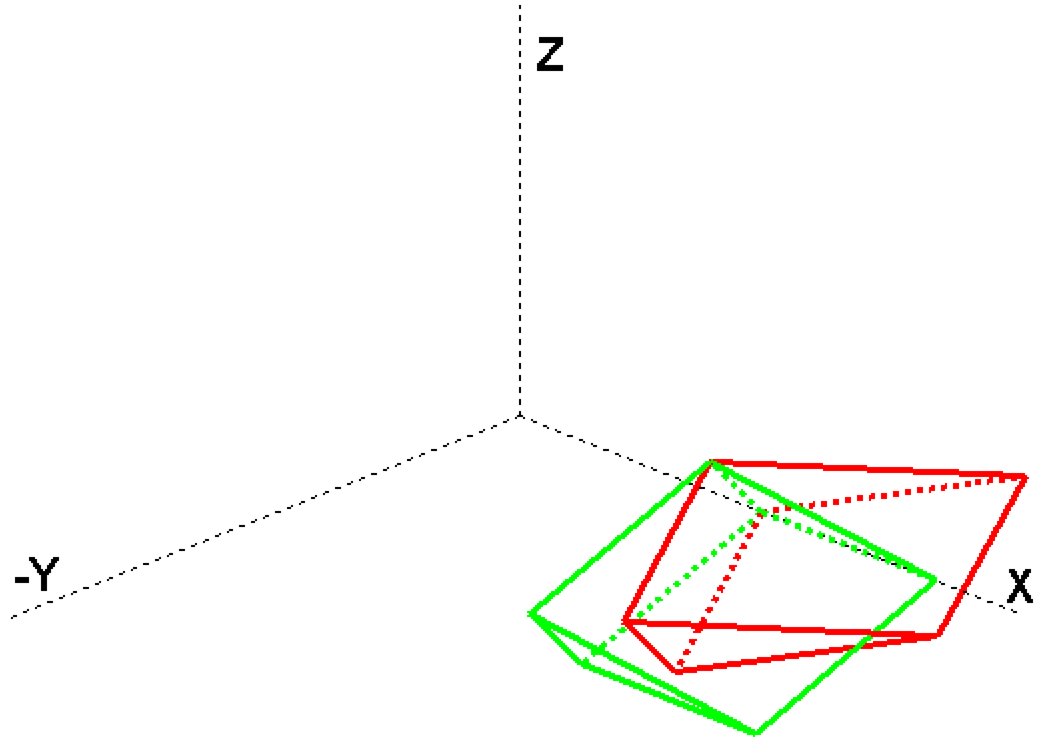
$$\text{Rotation about X - axis by } \theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation about X



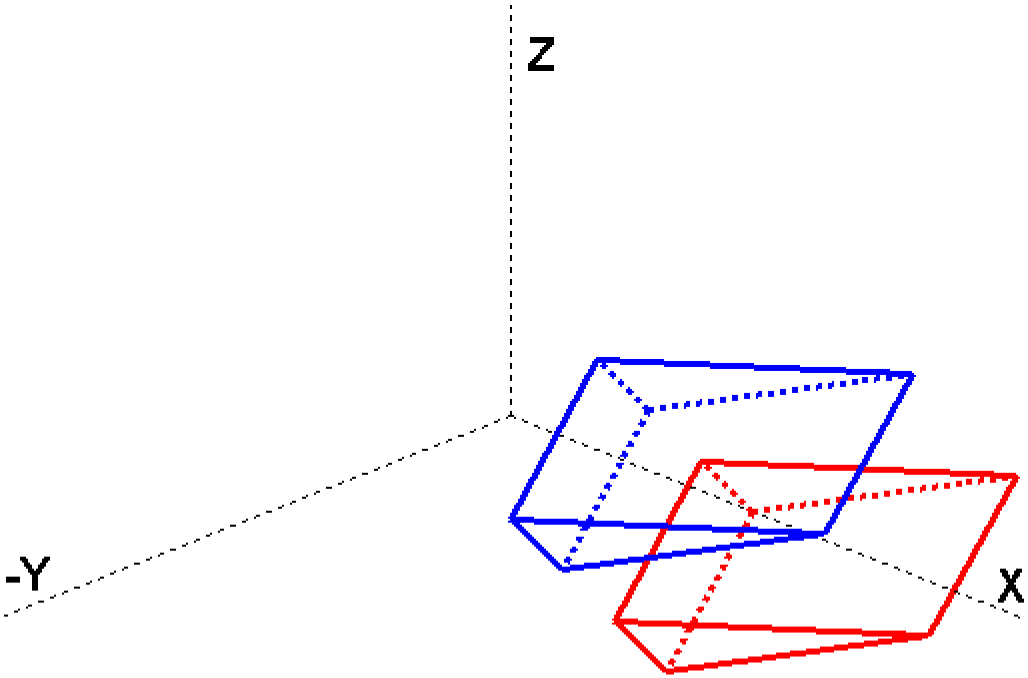
$$\text{Translation along X - axis} = \begin{pmatrix} 1 & 0 & 0 & p^x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Rotation about Z



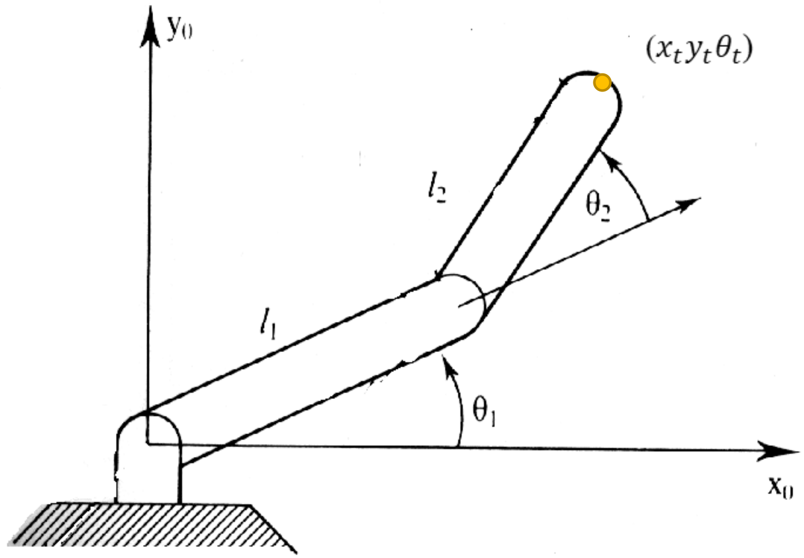
$$\text{Rotation about local Z - axis} = \begin{pmatrix} \cos[\theta] & -\sin[\theta] & 0 & 0 \\ \sin[\theta] & \cos[\theta] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Translation in Z



$$\text{Translation along local Z - axis} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & p^z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example Problem 1



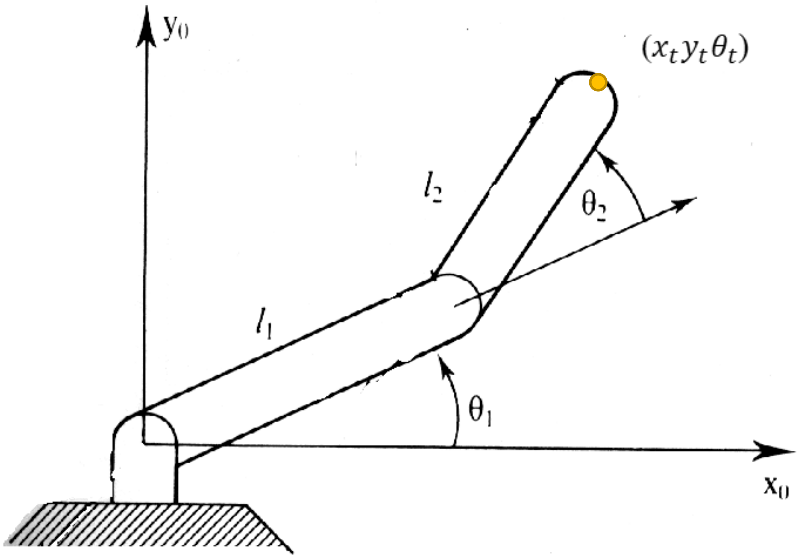
Set up:

- You have an RR robotic arm with base at the origin.
- The first link moves th1 with respect to the x-axis. The second link moves th2 with respect to the first link.

Question:

- What is the position and orientation of the end effector of the robotic arm?

Geometric Approach

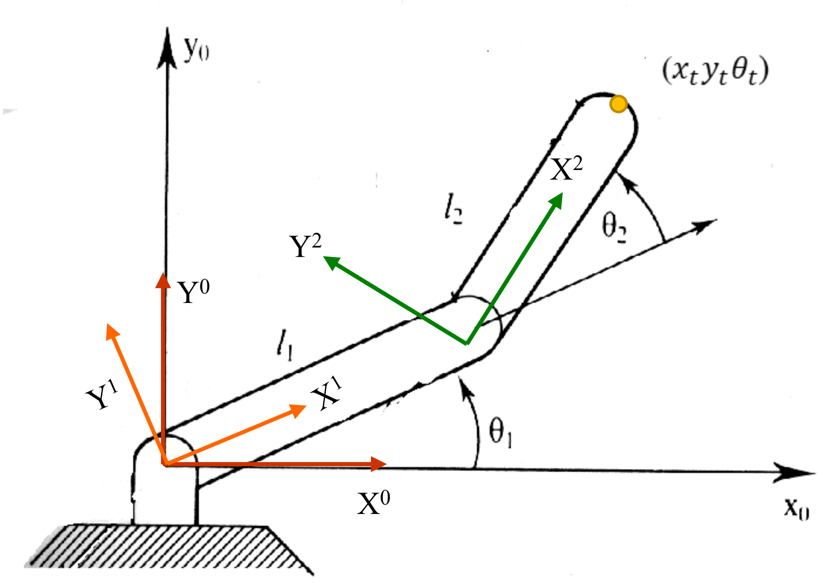


$$\theta t = \theta 1 + \theta 2$$

$$x_t = l1 * \cos(\theta 1) + l2 * \cos(\theta 1 + \theta 2)$$

$$y_t = l1 * \sin(\theta 1) + l2 * \sin(\theta 1 + \theta 2)$$

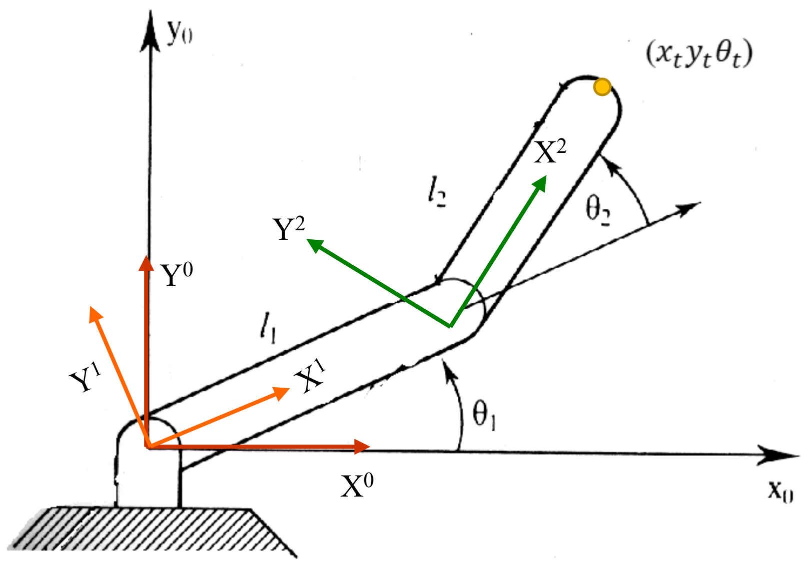
Algebraic Approach



- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix}\cos(\theta_1) & -\sin(\theta_1)\\ \sin(\theta_1) & \cos(\theta_1)\end{bmatrix}$.

$$\bar{V}^{X0Y0} = \begin{bmatrix}\cos(\theta_1) & -\sin(\theta_1)\\ \sin(\theta_1) & \cos(\theta_1)\end{bmatrix}\bar{V}^{X1Y1}$$

Algebraic Approach

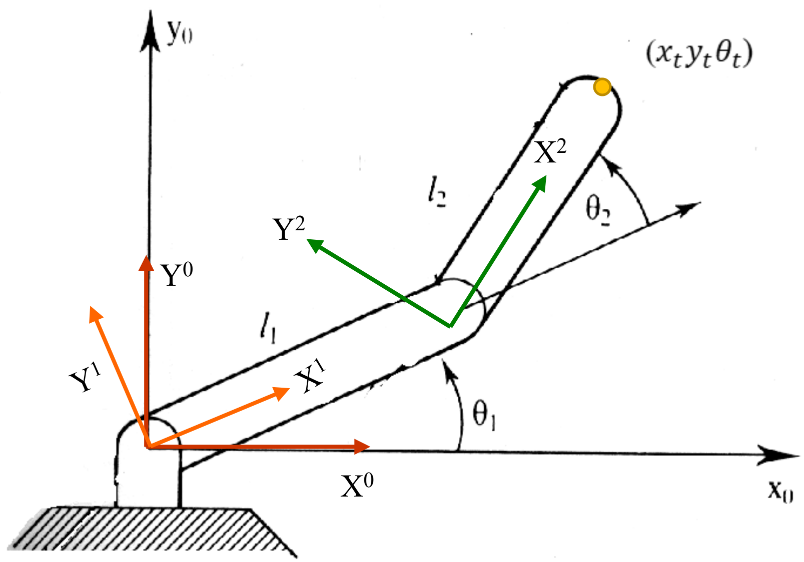


- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix}\cos(\theta_1) & -\sin(\theta_1)\\ \sin(\theta_1) & \cos(\theta_1)\end{bmatrix}$.

- In the X1Y1 frame, the X2X2 frame is at position $\begin{bmatrix} l_1 \\ 0 \end{bmatrix}$ and orientation $\begin{bmatrix}\cos(\theta_2) & -\sin(\theta_2)\\ \sin(\theta_2) & \cos(\theta_2)\end{bmatrix}$.

$$\bar{V}^{X0Y0} = \begin{bmatrix}\cos(\theta_2) & -\sin(\theta_2)\\ \sin(\theta_2) & \cos(\theta_2)\end{bmatrix}\left(\begin{bmatrix} l_1 \\ 0 \end{bmatrix} + \begin{bmatrix}\cos(\theta_2) & -\sin(\theta_2)\\ \sin(\theta_2) & \cos(\theta_2)\end{bmatrix}\bar{V}^{X2Y2}\right)$$

Algebraic Approach



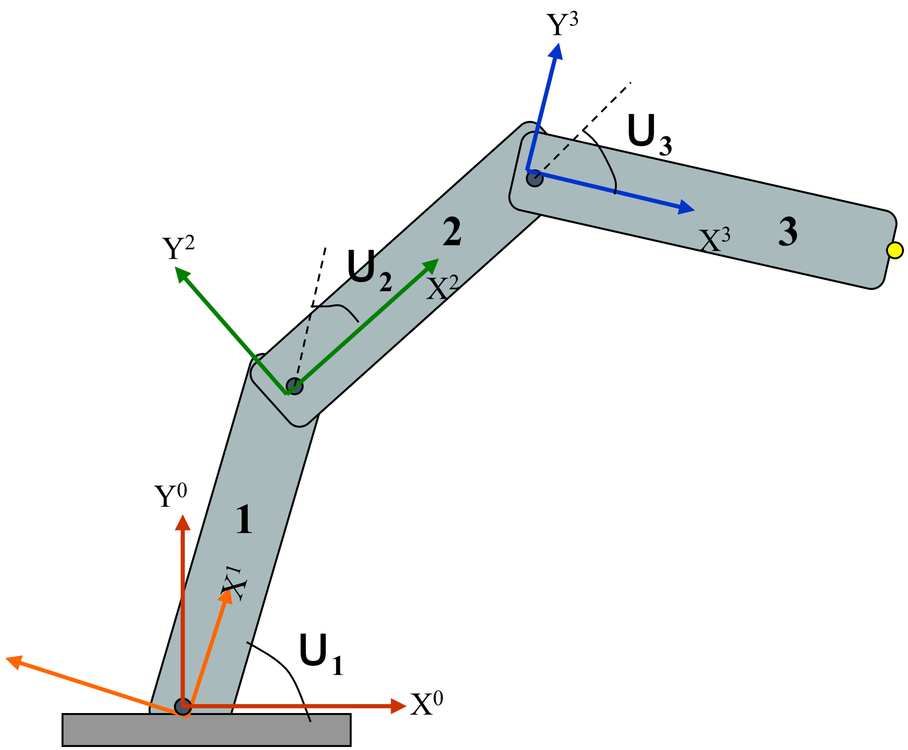
- In the X0Y0 frame, the X1X1 frame is at orientation $\begin{bmatrix}\cos(\theta_1) - \sin(\theta_1) \\ \sin(\theta_1) \ \cos(\theta_1)\end{bmatrix}$.

- In the X1Y1 frame, the X2X2 frame is at position $\begin{bmatrix} l_1 \\ 0 \end{bmatrix}$ and orientation $\begin{bmatrix}\cos(\theta_2) - \sin(\theta_2) \\ \sin(\theta_2) \ \cos(\theta_2)\end{bmatrix}$.

- In the X2Y2 frame, the end effector is at position $\begin{bmatrix} l_2 \\ 0 \end{bmatrix}$.

$$\bar{V}^{X0Y0} = \begin{bmatrix}\cos(\theta_2) - \sin(\theta_2) \\ \sin(\theta_2) \ \cos(\theta_2)\end{bmatrix} \left(\begin{bmatrix} l_1 \\ 0 \end{bmatrix} + \begin{bmatrix}\cos(\theta_2) - \sin(\theta_2) \\ \sin(\theta_2) \ \cos(\theta_2)\end{bmatrix} \left(\begin{bmatrix} l_2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix}\right)\right)$$

Example Problem 2



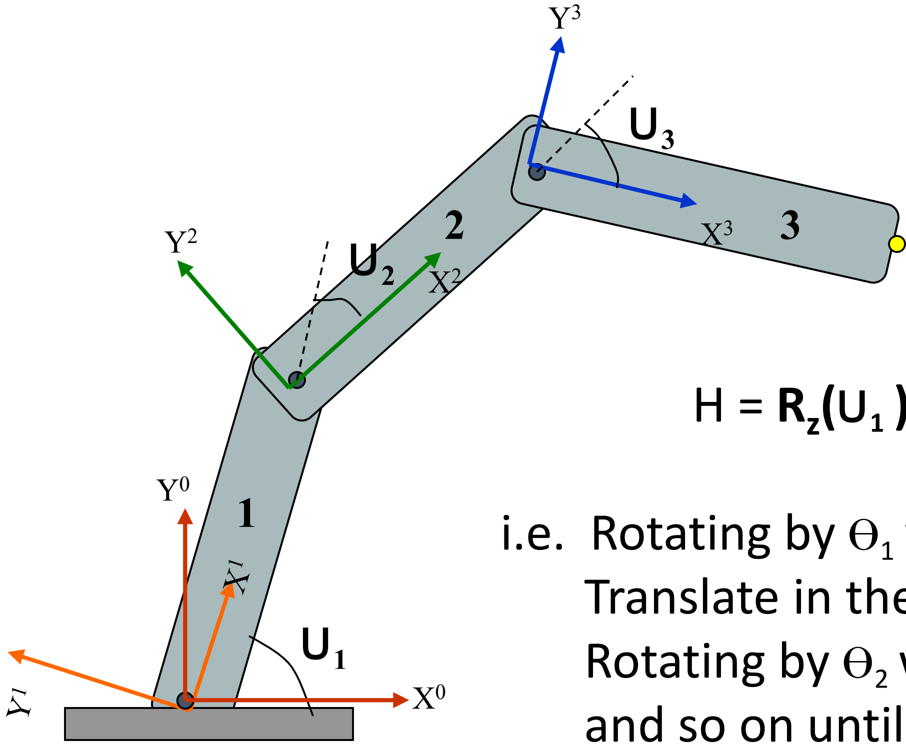
Set up:

- You are have a three-link arm with base at the origin.
- Each link has lengths l_1, l_2, l_3, respectively. Each joint has angles θ_1, θ_2, θ_3, respectively.

Question:

- What is the Homogeneous matrix to get the position of the yellow dot in the X^0Y^0 frame.

Algebraic Approach



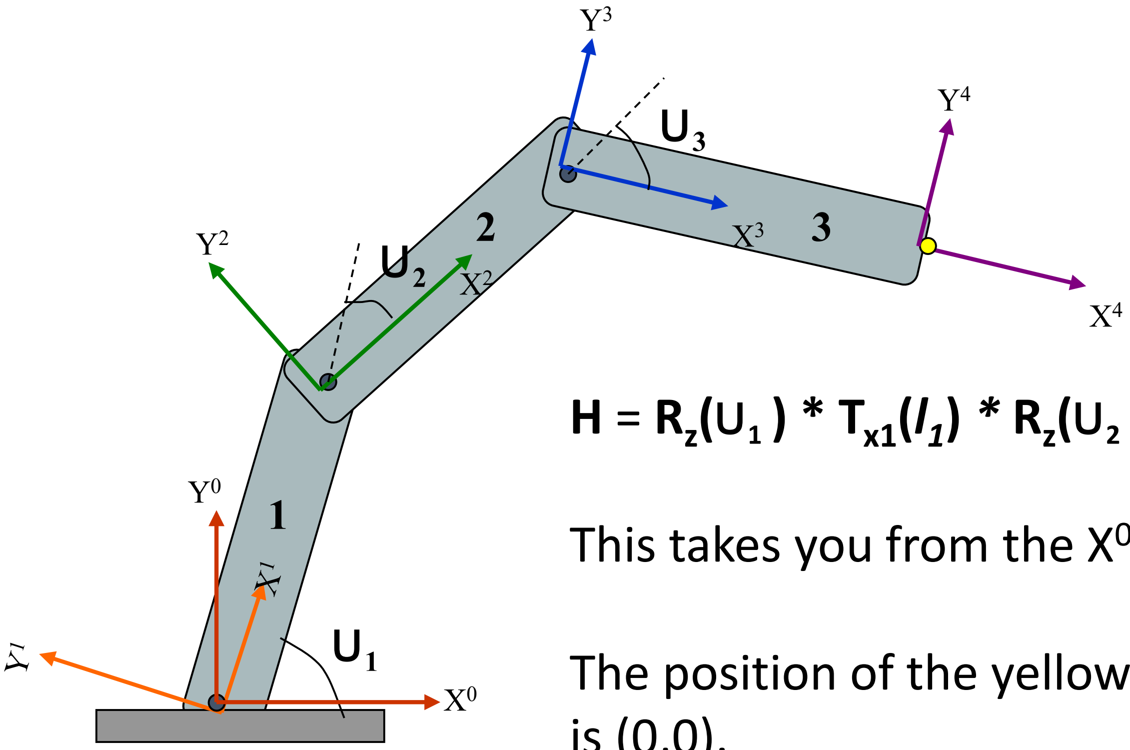
$$H = \mathbf{R_z}(U_1) \mathbf{\ast\ T_{x1}}(\boldsymbol{l_1}) \mathbf{\ast\ R_z}(U_2) \mathbf{\ast\ T_{x2}}(\boldsymbol{l_2}) \mathbf{\ast\ R_z}(U_3)$$

i.e. Rotating by Θ_1 will put you in the X^1Y^1 frame.
Translate in the along the X^1 axis by l_1.
Rotating by Θ_2 will put you in the X^2Y^2 frame.
and so on until you are in the X^3Y^3 frame.

The position of the yellow dot relative to the X^3Y^3 frame is $(l_1, 0)$. Multiplying H by that position vector will give you the coordinates of the yellow point relative the the X^0Y^0 frame.

Slight variation on the last solution:

Make the yellow dot the origin of a new coordinate X^4Y^4 frame



H = $R_z(U_1) * T_{x1}(l_1) * R_z(U_2) * T_{x2}(l_2) * R_z(U_3) * T_{x3}(l_3)$

This takes you from the X^0Y^0 frame to the X^4Y^4 frame.

The position of the yellow dot relative to the X^4Y^4 frame is (0,0).

$$\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = H \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Notice that multiplying by the (0,0,0,1) vector will equal the last column of the H matrix.

Next Class: Inverse Kinematics

Forward Kinematics (angles to position)

What you are given:
The length of each link
The angle of each joint

What you can find:
The position of any point
(i.e. it's (x, y, z) coordinates)

Inverse Kinematics (position to angles)

What you are given:
The length of each link
The position of some point on the robot

What you can find:
The angles of each joint needed to obtain that position