Final Exam

16-281: General Robotics

2026

This exam is being administered in person. You should write all of your work and solutions by hand. We will collect your answers when the test is finished.

- You will have 2 hours to complete this exam
- There are 6 sections on 16 pages. Make sure you have all of the pages. There may be blank pages throughout the sections.
- When making drawings, please be precise. Rounded edges should look rounded, sharp edges should look sharp, sizes should be close to scale. Neatness counts.
- Show your work. Partial credit may apply. Where applicable, algebraically justify your work and show intermediate steps to ensure full credit.
- It should be very clear what your final answer is. Circling your final answer is recommended.
- You may need to make certain assumptions to answer a problem. State them (e.g. what is optimal?).
- You are allowed one handwritten two-sided reference sheet for the exam. No cell phones, laptops, neighbors, etc. allowed.
- Good luck and you can do it.

Contents

1 Vision [15 points]

The Carnegie Optics team takes expensive photos for high school yearbooks. They take photos of students in front of green screens and are asking 16-281 students for help to process their images. The photo room has a window so the lighting can vary day to day.

1. The team wants to create a program that thresholds out the green screen. What color representation would be the best for this? In one sentence, how would you tune this threshold for varying light conditions? [3 points]

   You would use **HSV**. Despite the fact it is a "green screen," there are many colors with a high "G" component that are not green (white) and you can more precisely select your preferred hues. Additionally, you would threshold on **value** to control for brightness.

2. Now, the team wants to explore an alternate approach that involves highlighting the edges of each student's profile, but they aren't sure how to proceed. You remember from class that you can convolve images with certain filters to achieve this result. Write out the matrix of the kernel/filter that you would use and explain why you would use it. [4 points] There are many possible answers and explanations. Here are a couple sets of correct matrices.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -8 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \text{(both)}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \text{(both)}$$

Answer the following two questions with the two approaches you suggested above in mind.

3. List two benefits and two drawbacks of each implementation approach. Which is optimal? [6 points]

   Color thresholding

   Benefits: Fast and computationally cheap, Simple to implement, etc

   Drawbacks: Not great if background matches person, Sensitive to shadows or bad lighting, etc

   Edge detection:

   Benefits: Colors on person versus background doesn't matter as much , Fairly invariant to lighting changes, etc

   Drawbacks: More computationally expensive, Could detect additional internal edges, like the end of the shirt, etc

   Optimal with respect to what? Could be different answers, just make sure it makes sense.

4. If the student were wearing a green shirt, would the color-based method from problem 1 still work reliably? In one sentence, explain why or why not. [2 points]

   No, HSV would not work reliably for thresholding because the algorithm would struggle to separate the green from the background to the green from the shirt, which would most likely lead to the student's torso appearing invisible.

2 Control [15 points]

Engineers at a startup are building a self-balancing robot. For their robot, the control system measures the error

$$e(t) = R - \theta(t)$$

and computes the control input (a torque)

$$u(t) = K_p e(t) + K_i \int e(t)dt + K_d \frac{de(t)}{dt} + u_{\text{ff}}(t)$$

where K_p, K_i, and K_d are the proportional, integral, and derivative gains, and $u_{\text{ff}}(t)$ is the feed forward term computed from the desired motion. The block diagram in Figure 1 describes how the control input to the plant is calculated.

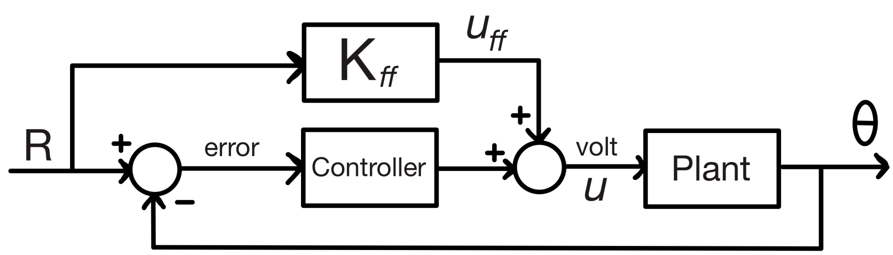


Figure 1: Block Diagram for Balance Bot

1. At one instant in time, suppose: $R = 10, \theta = 8, \int e(t)dt = 1, \frac{de}{dt} = 0$, and the PID gains are $K_p = 2, K_i = 1, K_d = 0.5$. The desired voltage input to the plant at this instant is $u = 10$. What feed forward gain K_{ff} would achieve this? [5 points]

$$u(t) = K_p e(t) + K_i \int (e) + K_d \frac{de}{dt} + u_{ff}(t)$$
$$10 = 4 + 1 + u_{ff}(t)$$
$$u_{ff}(t) = 5$$

2. In practice, the robot dynamics are not perfectly known and are modeled imperfectly which in turn affects the performance of the feedforward term. Explain how the PID feedback term compensates for these errors. [3 points]

   The PID feedback term acts as a corrective loop that constantly monitors the error between the robot's desired trajectory and its actual measured state. While an imperfect feedforward term might miss the mark, the PID controller senses this deviation and applies restorative forces to pull the robot back on track, effectively "filling in the gaps" left by the inaccurate model.

While testing, the engineers find that the robot is able to balance near upright, but there is a persistent small tilt error when an external disturbance force is applied. One engineer proposes increasing only the derivative gain K_d to eliminate this error.

3. Is increasing K_d alone an effective way to remove steady-state error? Explain why or why not. [2 points]

   No, because increasing the derivative term only addresses the initial overshoot, meaning steady state error might get worse.

4. Which PID term is directly responsible for eliminating steady-state error caused by a constant disturbance, and why? [2 points]

   K_i is responsible because it "accumulates" errors over time and thus is able to address the difference in the real world and the desired state.

5. Another engineer suggests that instead of adding a feedforward term, they could simply increase K_i to reduce steady-state error. Compare how increasing K_i and adding a feedforward term affect steady-state error. Which approach guarantees elimination of steady-state error, and why? [3 points]

   Adding a feedforward term is a good idea. Feedforward is often able to account for instability added by including an integral term.

3 Path Planning [15 points]

1. Label the grid in Fig. 2 with the numbers that would be produced by a wavefront planner using 8-point connectivity. When doing so, label the goal with 0. Draw a path from start to goal that could be returned by this wavefront planner. [4 points]

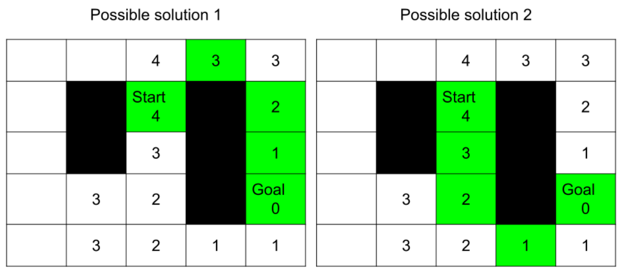


Figure 2: Grid for wavefront question. Black cells are obstacles.

2. Draw the configuration space for the triangular robot in Fig. 3. The robot is only capable of *translational* movement. Also take note of the reference point (bottom left of the triangle)! Treat the environment boundary as an obstacle. [4 points]

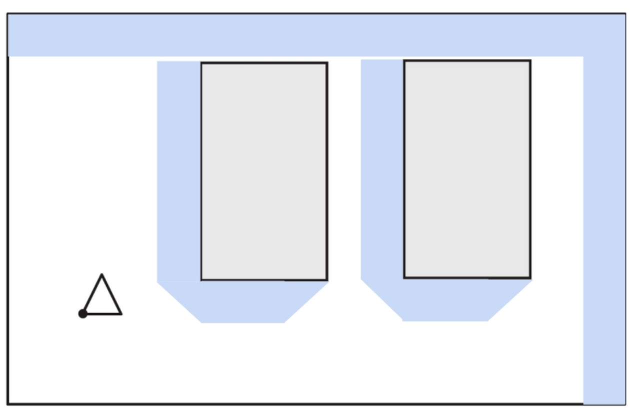


Figure 3: Triangle robot exploring a fixed environment

3. Draw the c-space with the robot centroid as the reference in Fig. 8. Treat the environment boundary as an obstacle. [4 points]

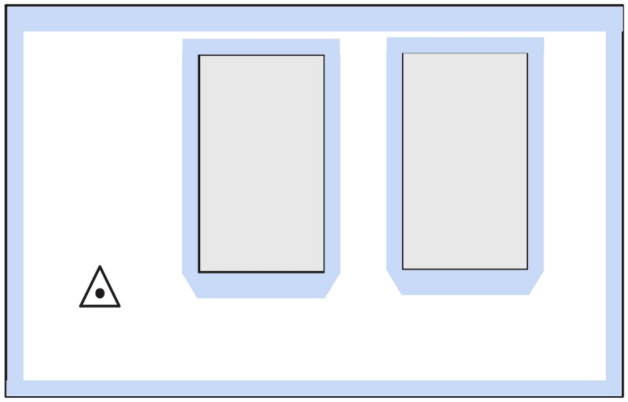


Figure 4: Triangle robot exploring a fixed environment

4. How would the dimensionality of the configuration space change if the triangle was capable of rotational movement as well? [3 points]

   Answer: The configuration space would become 3-dimensional, since it would include planar positions and rotations.

4 Graph Search [15 points]

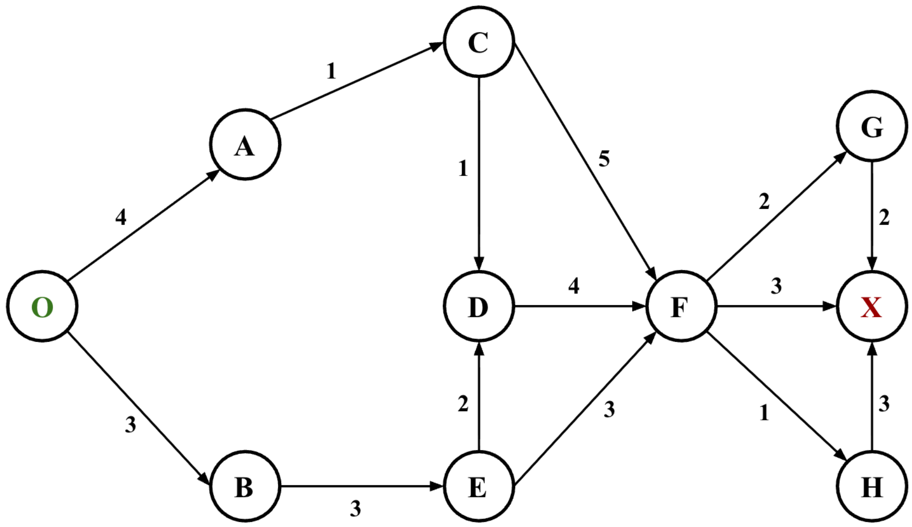


Figure 5: Graph to Search

1. Perform Dijkstra's search on the graph shown above. O is the starting node and X is the goal node. Break ties alphabetically. List the nodes that are EXPANDED and added to the closed list in the order in which they are expanded. [6 points]:

   OBACDEFHGX

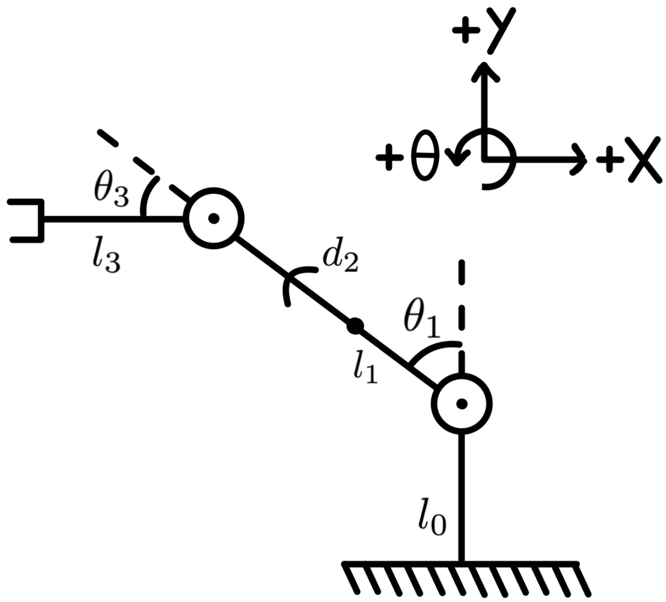
2. What is the path found by Dijkstra's search, and how long is it? [6 points]:

   OBEFX. Length: 12

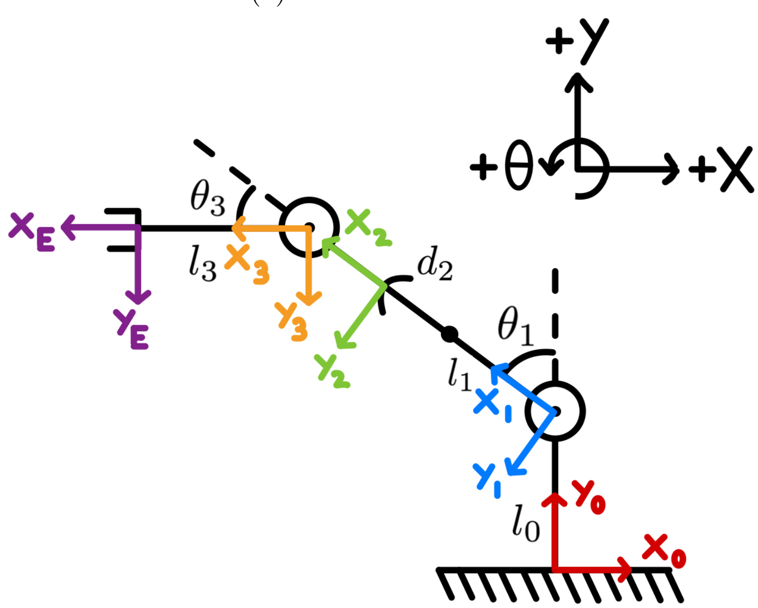
3. If this were to be switched to an A* problem, give an example of an admissible versus inadmissible heuristic value for node E. [3 points]:

   Admissible: 1-6 Inadmissible: 7-∞

5 Forward Kinematics [20 points]



(a) Planar RPR Arm



(b) RPR Arm With Drawn Coordinate Frames

Figure 6: Design for a Planar RPR Arm

Use Figure 6 as a means to answer the following questions in this section. Note that when $d_2 = 0$, the distance between the two revolute joints is l_1.

1. If the robot is commanded to the joint configuration $\theta_1 = 90°$, $d_2 = 1$ cm, and $\theta_3 = -90°$, what are the exact (x, y, θ) coordinates of the end-effector in the base frame? Solve geometrically. [5 points]

   We are given the joint configuration $\theta_1 = 90°$, $d_2 = 1$ cm, and $\theta_3 = -90°$.

   - The base frame 0 has its origin at $(0, 0)$.
   - Joint 1 is located at $(0, l_0)$ along the $+Y_0$ axis.
   - The angle θ_1 is measured counter-clockwise from the vertical dashed line (which is parallel to $+Y_0$). Setting $\theta_1 = 90°$ points link l_1 strictly leftwards along the $-X_0$ direction.
   - The distance to Joint 3 is $l_1 + d_2 = l_1 + 1$ cm.
   - Therefore, Joint 3 is positioned at $(0, l_0) + (-(l_1 + 1), 0) = (-(l_1 + 1), l_0)$.
   - θ_3 is measured from the extended axis of link l_1. A rotation of $\theta_3 = -90°$ (clockwise) from the leftward $-X_0$ direction points link l_3 directly upwards along the $+Y_0$ direction.
   - The end-effector E is at a distance l_3 along link l_3.
   - Thus, the exact coordinates are $(-(l_1 + 1), l_0) + (0, l_3) = (-(l_1 + 1), l_0 + l_3)$.
   - The end-effector orientation θ_E points upwards, which corresponds to $90°$ (or $\pi/2$ radians) relative to the base $+X_0$ axis.

   Answer: $x = -(l_1 + 1)$, $y = l_0 + l_3$, $\theta = 90°$.

For the following questions, feel free to plug in $\theta_1 = 90°, d_2 = 1$ cm, $\theta_3 = -90°$ up front to simplify the math.

2. Find the homogeneous matrix H_1^0 that maps frame 1 to frame 0 [2 points]

   Frame 1 is translated by l_0 along Y_0. Since θ_1 is measured from the vertical $+Y_0$ axis, the angle of X_1 relative to X_0 is $90° + \theta_1$.

$$H_1^0 = \begin{bmatrix} \cos(\theta_1 + 90°) & -\sin(\theta_1 + 90°) & 0 \\ \sin(\theta_1 + 90°) & \cos(\theta_1 + 90°) & l_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta_1 & -\cos\theta_1 & 0 \\ \cos\theta_1 & -\sin\theta_1 & l_0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Find the homogeneous matrix H_2^1 that maps frame 2 to frame 1 [2 points]

   Frame 2 is translated along X_1 by distance $l_1 + d_2$ without any rotation relative to Frame 1. (although the picture shows it as in the middle of d_2 so not sure)

$$H_2^1 = \begin{bmatrix} 1 & 0 & l_1 + d_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Find the homogeneous matrix H_3^2 that maps frame 3 to frame 2 [2 points]

   Frame 3 shares the same origin as Frame 2 but is rotated by θ_3.

$$H_3^2 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. Find the homogeneous matrix H_E^3 that maps frame E to frame 3 [2 points]

   Frame E is translated along X_3 by distance l_3 with no rotation.

$$H_E^3 = \begin{bmatrix} 1 & 0 & l_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Compute the full forward-kinematics transform H_E^0 in terms of H_1^0, H_2^1 H_3^2, H_E^3, and any other necessary variables. Write out the full matrix. [3 points]

   The full transformation is $H_E^0 = H_1^0 H_2^1 H_3^2 H_E^3$. Multiplying the matrices yields:

$$H_E^0 = \begin{bmatrix} -\sin(\theta_1+\theta_3) & -\cos(\theta_1+\theta_3) & -l_3\sin(\theta_1+\theta_3) - (l_1+d_2)\sin\theta_1 \\ \cos(\theta_1+\theta_3) & -\sin(\theta_1+\theta_3) & l_3\cos(\theta_1+\theta_3) + (l_1+d_2)\cos\theta_1 + l_0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Based on your solution from H_E^0, what are the exact (x, y, θ) coordinates? Use the same numerical parameters from problem 1 when solving. Is the solution the same as in problem 1? [4 points]

   Substituting $\theta_1 = 90°$, $d_2 = 1$, and $\theta_3 = -90°$ into H_E^0:

- $\theta_1 + \theta_3 = 0°$
- $\sin(0°) = 0$, $\cos(0°) = 1$
- $\sin(90°) = 1$, $\cos(90°) = 0$

$$H_E^0 = \begin{bmatrix} 0 & -1 & -l_3(0) - (l_1+1)(1) \\ 1 & 0 & l_3(1) + (l_1+1)(0) + l_0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -(l_1+1) \\ 1 & 0 & l_0 + l_3 \\ 0 & 0 & 1 \end{bmatrix}$$

Extracting the coordinates from the transformation matrix: $x = -(l_1 + 1)$, $y = l_0 + l_3$, and orientation $\theta = 90°$ (since the X_E axis maps to the unit vector $[0, 1]^T$). This exactly matches the geometric solution found in Problem 1.

6 Inverse Kinematics [20 points]

A planar robotic arm consists of three degrees of freedom: a prismatic joint followed by two revolute joints (PRR configuration). The configuration of the robot is given by:

$$q = (d, \theta_1, \theta_2)$$

The extension of the prismatic joint is constrained as so: $d \in \left[l_1 + l_2,\ 3(l_1 + l_2)\right]$
It is important to note that $l_1 > l_2$.

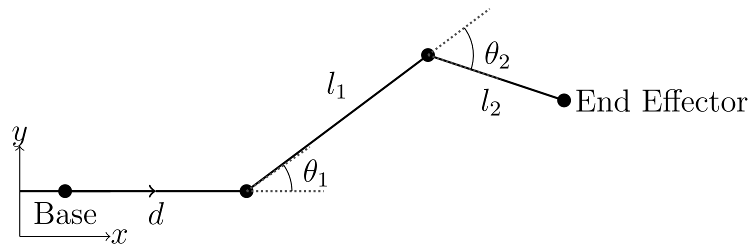


Figure 7: PRR planar manipulator

You are tasked with analyzing the inverse kinematics of this system.

1. Draw the workspace of the arm, i.e. the set of points where the end-effector can reach. Identify how many solutions exist in each region. Put dimensions in your drawing (in terms of the given variables). [10 points]

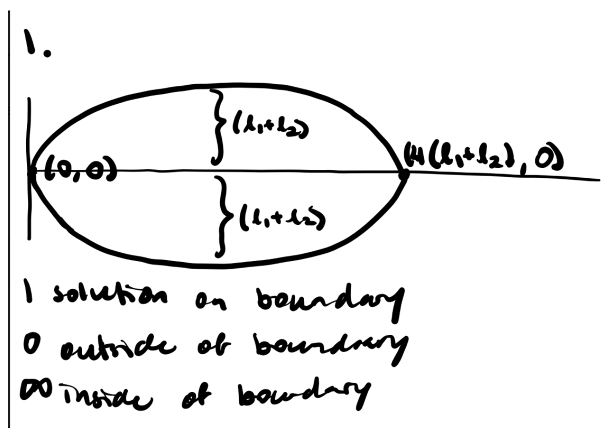


Figure 8: Sketched answer to 6.1

2. Assuming d is fixed, derive expressions for θ_1 and θ_2 using geometric inverse kinematics. You can assume that for the given d, the point will be achievable. [6 points]

   $\theta_2 = \pm \arccos(\frac{(x-d)^2+y^2-l_1^2+-l_2^2}{2l_1l_2})$

   $\theta_1 = \arctan 2(y, x-d) - \arcsin(\frac{l_2 sin(\theta_2)}{\sqrt{(x-d)^2+y^2}})$

3. For any fixed d, explain why multiple solutions may exist, and describe how you would select a preferred solution in practice. [4 points]

   Multiple solutions exist because there is the elbow up and elbow down configuration for each point. Preferred solution is variable, could be because of obstacle avoidance or something else.

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This is the end of the test.