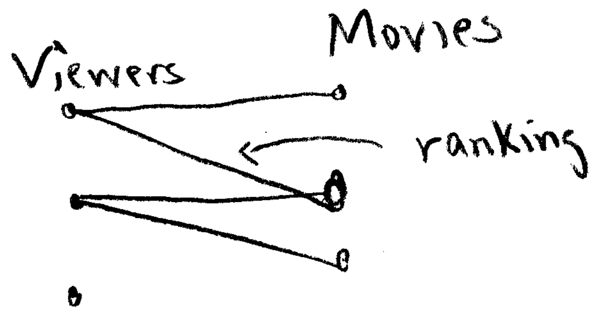
15-859N
9/9/13

Resistive Model of a Graph & Random Walks

<u>Modivation</u>: Making a recommendation (NETFLIX)



<u>Question</u>: Should we recommend M to V?

Score(V, M)

<u>Idea 1</u> $\text{Score}(V,M) = \dfrac{1}{\text{graph dist from } V \text{ to } M}$

$$W_{ij} = \frac{1}{\text{rank}_{ij}}$$

$$\text{Score}(V,M) = \frac{1}{\min\limits_{VPM} W(P)}$$

<u>Idea 2</u> $W(P) = \min\limits_{e \in P} (\text{rank}(e))$

$$\text{Score}(V,M) \equiv \max_{VPm} W(P)$$

Problem For 1) and 2) extra paths do not improve score

Idea 3 $\text{Score}(V, M) \equiv$ Max flow from V to M.

Problem Shorter paths do not improve score

Idea 4 View edges as conductors

$\text{Score}(V, M) =$ effective conductance

Idea 5 Consider random walk from V to M

$$\text{Score}(V, M) = \text{hit}(V, M) + \text{hit}(M, V)$$

$\text{hit}(V, M) \equiv$ Expect length random walk from V, M

We show 4) & 5) are related.

Hypothesis:

effective Conductance &

Commute time $\equiv hit(v,m) + hit(m,v)$

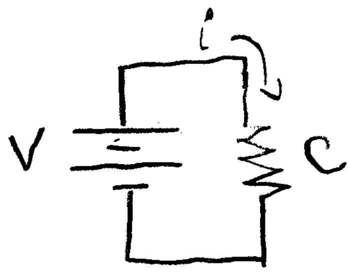
are better scores.

To do:

1) Give formal definitions.
2) Develop basic theory.
3) Give efficient algs.
4) Find apps.

Resistance Theory

Ohms Law:



$C \equiv$ conductance

$R \equiv$ resistance

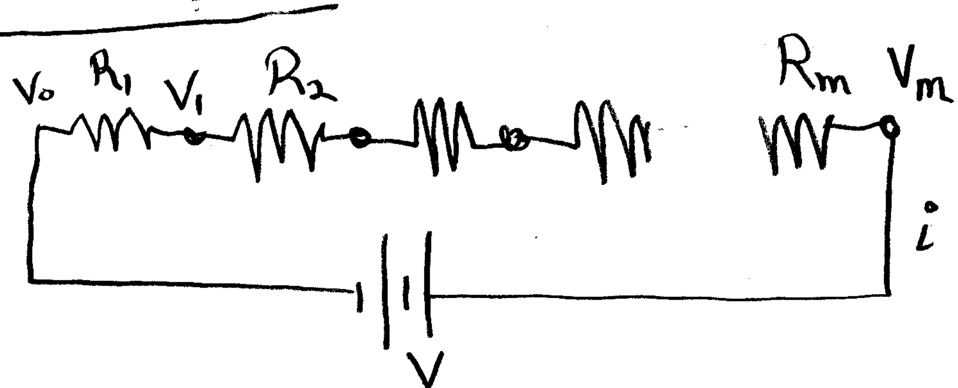
$V \equiv$ voltage

$i \equiv$ current

$$C = 1/R \qquad i = C \cdot V = V/R$$

Facts HW

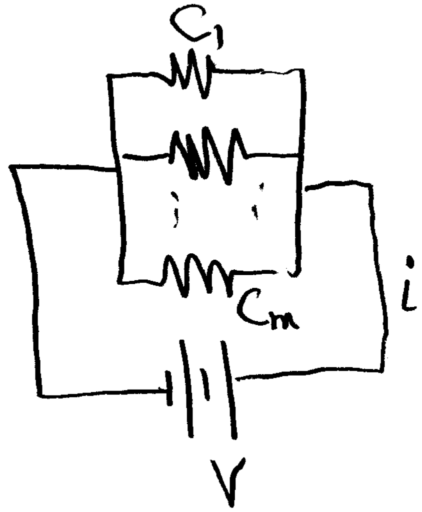
Resistors in series



$$R = R_1 + \cdots + R_m \qquad C = \frac{1}{(1/C_1 + \cdots\; 1/C_m)} = ?$$

i.e. $i = V/R$

Conductors in Parallel

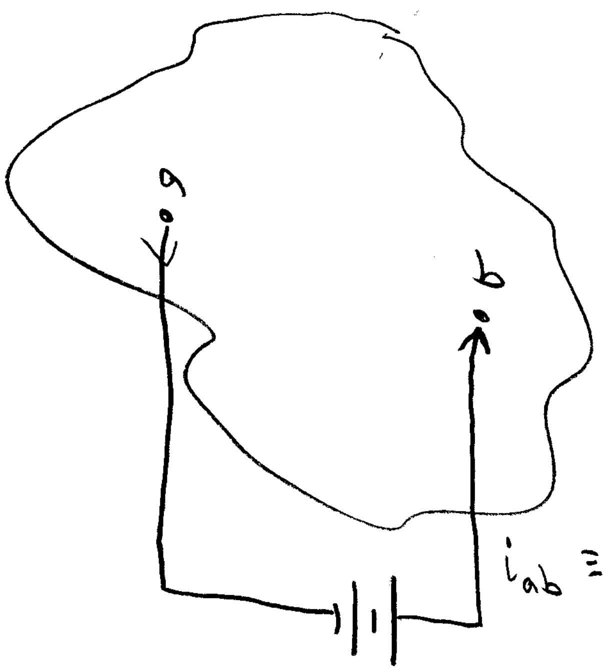


$$C = C_1 + \cdots + C_m$$

ie $i = V \cdot C$

Effective Resistance/Conductance

Let G be a network of resistors



$i_{ab} \equiv$ current

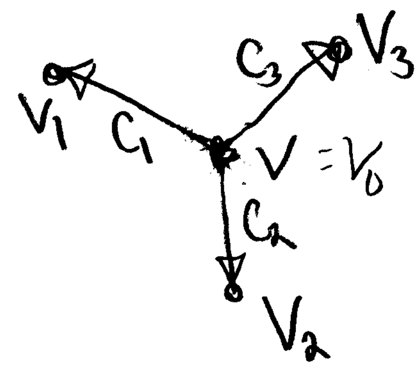
$V_{ab} \equiv$ voltage

Def $R_{ab} = V_{ab}/i_{ab}$ $\quad C_{ab} = 1/R_{ab}$

Computing effective resistance

Case: <u>Kirchhoff's Law</u> ie flowin = flowout

an example



by Ohm's Law

$$i_1 = C_1(V - V_1)$$
$$i_2 = C_2(V - V_2)$$
$$i_3 = C_3(V - V_3)$$

residual current $\equiv i_1 + i_2 + i_3$

by Kirchhoff

$$i_1 + i_2 + i_3 = 0$$

$$C_1(V - V_1) + C_2(V - V_2) + C_3(V - V_3) = 0$$

$$(C_1 + C_2 + C_3)V = C_1V_1 + C_2V_2 + C_3V_3$$

$$C = C_1 + C_2 + C_3$$

$$CV = C_1 V_1 + C_2 V_2 + C_3 V_3$$

$$V = \frac{C_1}{C} V_1 + \frac{C_2}{C} V_2 + \frac{C_3}{C} V_3$$

V is convex combination of $V_1, V_2 V_3$

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residual current $= CV - C_1 V_1 - C_2 V_2 - C_3 V_3$

The general case:

$G = (V, E, C)$ $\quad C: E \to R^+$

$V = \{V_1, \dots, V_n\}$

$$d(V_i) = \sum_{(i,j) \in E} C_{ij}$$

Def: $A_{ij} = \begin{cases} C_{ij} & \text{if } (i,j) \in E \\ 0 & \text{o.w} \end{cases}$

$$\text{Laplacian}(G) = L(G) = L$$

$$L_{ij} = \begin{cases} d(v_i) & \text{if } i=j \\ -c_{ij} & \text{if } (i,j) \in E \\ 0 & \text{o.w} \end{cases}$$

ie $L = D - A$ where $D = \begin{pmatrix} d(v_1) & & 0 \\ & \ddots & \\ 0 & & d(v_n) \end{pmatrix}$

Let V be a voltage setting of nodes

<u>Note</u> $(LV)_i$ = residual current at V_i

Inverse: We inject currents and get voltages.

The net injected must be zero!

Goal: R_{in}

method 1 solve $L\begin{pmatrix} 0 \\ V_2 \\ \vdots \\ V_{n-1} \\ 1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \\ -i \end{pmatrix}$ (*) $i = V/R$, $V = 1$, $R = 1/i$

return $1/i$

(*) is called a boundary valued prob.

In our case V_1 & V_n are the bdary

$(V_1, \ldots, V_n)$ is called harmonic

because $V_i \in$ interior $\Rightarrow$

V_i is convex combination of neighbors

<u>Maximum Principle</u> If f is harmonic then min & max are on bdary

<u>pf</u> $v \in$ interior then $\exists$ neig v_i & v_j st

$$v_i \leq v \leq v_j$$

<u>Uniqueness Principle</u> If f & g are harmonic with same bdary values then $f = g$

<u>pf</u> $f-g$ is harmonic with zero on bdary

$$\Rightarrow f-g \equiv 0 \Rightarrow f = g$$

Method 2 solve $LV = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ return $R_{1n} = V_1 - V_n$

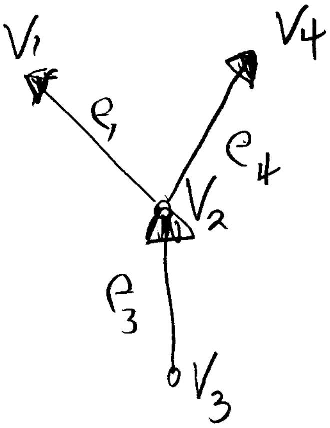
Does V exist?

Another way to view the Laplician

Boundary Operator (Vertex - Edge Matrix)

$$\partial^{n \times m} \quad \begin{array}{c|ccc|} & e_1 & \cdots & e_m \\ \hline v_1 & & & \\ \vdots & & & \\ v_n & & & \\ \hline \end{array}$$

Orient each edge.



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	e_1	e_2	e_3
v_1	1	0	0
v_2	-1	-1	1
v_3	0	0	-1
v_4	0	1	0

Let $c_1 \cdots c_m \equiv$ conductance of $e_1 \cdots e_m$

$$C = \begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_m \end{pmatrix}$$

<u>Note</u> $\partial^T V \equiv$ voltage drop across each edge

$C \partial^T V \equiv$ current flow " "

$\partial C \partial^T V \equiv$ residual current at each vertex

<u>Thus</u> $L = \partial C \partial^T$

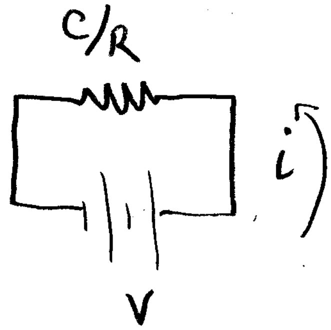
Assume G is a connected graph

$$X^T \partial C \partial^T X = (\partial^T X)^T C \partial^T X = \sum c_{ij} (X_i - X_j)^2 \geq 0$$

$$X^T L X = 0 \text{ iff } \forall e_{ij} \; (X_i - X_j)^2 = 0 = (X_i - X_j)$$

$$\operatorname{Ker}(L) = \left\langle \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right\rangle \quad \operatorname{rank}(L) = n-1$$

Current & Energy/Power Dissipation



Newton Energy $\equiv$ Force $\cdot$ speed

$\equiv$ Volt $\cdot$ current

$$\equiv V \cdot i$$

$$\equiv C V^2$$

$$\equiv i^2 R$$

$i = CV$

$iR = V$

Network
$$E = \frac{1}{2} \sum_{x,y} i_{xy} (V_x - V_y)$$

$$V^T L V = V^T \partial C \partial^T V = (\partial^T V)^T C (\partial^T V)$$

$$= \sum_{\substack{\text{Oriented} \\ (x,y) \in E}} C_{xy} (V_x - V_y)^2 = E$$

Two Types of Flow

Def: A flow $f: E \to \mathbb{R}$ (oriented edges)

Def A potential flows $= \{ c\partial^T v \mid v \in \mathbb{R}^n \} = P_G$

Def A circulations(flow) $= \{ f \in \mathbb{R}^n \mid \partial f = 0 \} = C_G$

Assume G is connected & T spanning tree

Claim C_G is a subspace & $\dim(C_G) = m - n + 1$

$m = \#$ edges.

1) Subspace easy

2) $E \setminus T \equiv$ nontree edges of G

   $|E \setminus T| = m - n + 1$

3) Claim any flow on $E \setminus T$ can be extended to C_G on G. (HW)

4) $f, g \in C_G$ and $f \setminus T = g \setminus T$ then $f = g$.

<u>Claim</u> $f_c \in C_G$ & $g_p \in P_G$ then $f_c^T R g_p = 0$

when $R = \begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_m \end{pmatrix}$

<u>pf</u> $\exists V$ st $g_p = C \partial^T V$,

$$f_c^T R g_p = f_c^T R C \partial^T V = f_c^T \partial V = (\partial f_c)^T V = 0^T V = 0$$

note: $RC = I$

<u>Thus</u> C_G, P_G spans $\mathbb{R}^m$ (all flows)

ie $\forall f \in \mathbb{R}^m \ \exists! \, f_c \ \& \ f_p$ st $f = f_c + f_p$

Def $f_a = \sum_{a \neq b} f_{ab}$

Def f is a unit flow from a to b if:

1) f is a flow

2) $-f_a = f_b = 1$

3) $f_x = 0$ for $x \neq a, b$.

Thomson's Principle

1) f is unit Potential flow from a to b
2) g is any flow from a to b.

then $f^T R f \le g^T R g$

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pf

We Know that $g = f + f_c$ $\quad f_c \equiv$ circulation

$$g^T R g = (f+f_c)^T R (f+f_c) = f^T R f + 2 \underbrace{f_c R f}_{0} + f_c^T R f_c$$

$$= f^T R f + f_c^T R f \ge f^T R f$$

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note

Def or Thm Effective resistane from a to b

$$ER_{ab} = f_p^T R f_p$$

$f_p \equiv$ unit potential flow from a to b.

Rayleigh's Monotonicity Law

$\bar{R} \geq R$ then $\overline{ER}_{ab} \geq ER_{ab}$

pf $f \equiv$ unit potential flow in G_R

$g \equiv$ " in $G_{\bar{R}}$

$$\overline{ER}_{ab} = g^T \bar{R} g = \sum_{e \in G} g_e^2 \bar{R}_e$$

$$\geq \sum_{e \in G} g_e^2 R_e$$

$$\geq \sum_{e \in G} f_e^2 R_e \quad \text{(Thomson)}$$

$$= f_e^T R f_e = ER_{ab}$$

HW) Show that R_{ab} is a metric space

ie 1) $R_{ab} \geq 0$

2) $R_{ab} = 0$ iff $a = b$

3) $R_{ab} = R_{ba}$

4) $R_{ac} \leq R_{ab} + R_{bc}$