15-859N
9/25/13

Graph Sparsifiers

Goal Input: $G=(V,E,w)$ where $|E| >> |V|$

Output: $H=(V,E',w')$ where $|E| \approx |V|$.

s.t. H similiar G.

Question: What is similiar?

Ans: Cut preserving. $\forall S \subseteq V$

$$\left|\{w_{ij} \mid i \in S \;\&\; j \notin S\}\right|_1 \approx \left|\{w'_{ij} \mid i \in S \;\&\; j \notin S\}\right|_1$$

Spectral Preserving

$$(1-\varepsilon) G \preccurlyeq H \preccurlyeq (1+\varepsilon) G$$

HW Show Spectral ⇒ Cut Preserving!

Random Sampling

$G = (V, E, W)$

edge	e_1	$\cdots$	e_m
prob	p_1	$\cdots$	p_m

$\sum p_i = 1$

Sample: 1) Pick an edge e_i using probs p_i

2) Return $(1/p_i)e_i$

K-Sample: 1) Pick Sample$_1$, ..., Sample$_k$ (replacement)

2) $H = \{\text{Sample}_1 + \cdots + \text{Sample}_k\}$

Return H/k

Question: What $p_1, \ldots, p_m$, and k work?

Sample: Can be viewed as matrix.

K-Sample: Sum of matrix.

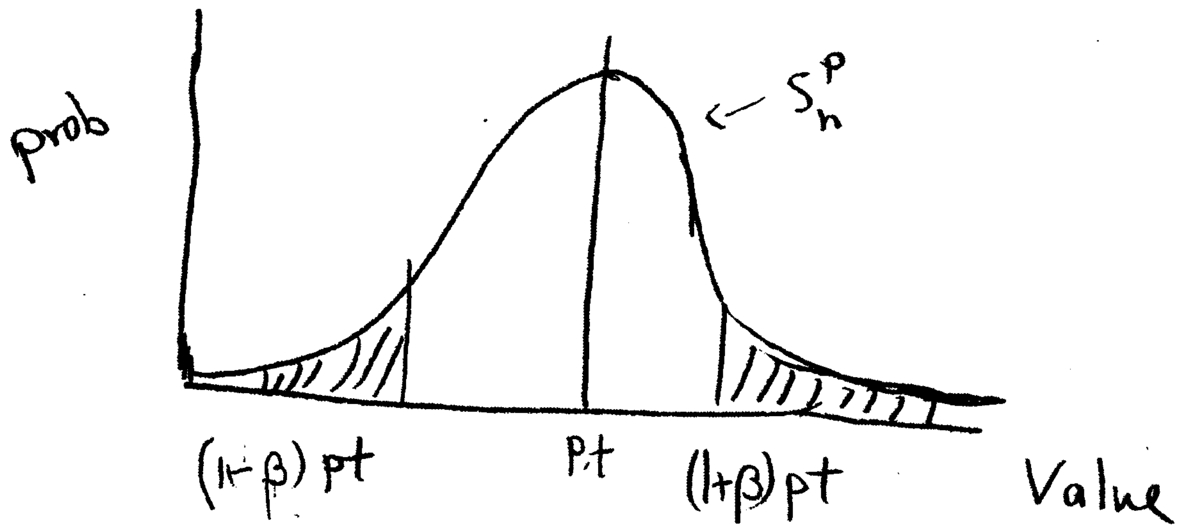
Chernoff Bounds

Let $X_1, \cdots X_t$ be independent 0/1 random variables

Assume $Prob(X_i = 1) = p$

The binomial random variable is

$$S_n^p = X_1 + \cdots + X_t$$



$$\text{Expect}(S_t^p) = \sum E(X_1) = p.t$$

<u>Thm</u> $\text{Prob}(S_t^p < (1-\beta)pt) < e^{-\beta^2 pt/2} \quad \forall 0 \le \beta < 1$

<u>Thm</u> $\text{Prob}(S_t^p > (1+\beta)pt) < e^{-\beta^2 pt/3} \quad \forall 0 \le \beta < 1$

Ahlswede-Winter Thm

Note $\min_{\mu} (A \preceq \mu B)$ is $\tau(A/B)$

(AW Thm) Let $Y_1, \dots, Y_k$ be iid random spsd $n \times n$ matrices st 1) $E(Y_i) = Z$ & 2) $Y_i \preceq \mu Z$ then

$$\Pr\left[(1-\varepsilon)Z \preceq \sum_{i=1}^{k} \frac{Y_i}{k} \preceq (1+\varepsilon)Z\right] \geq 1 - 2n e^{-c\varepsilon^2 k/\mu}$$

$\boxed{c \equiv 1/4 \text{ works!}}$

Lets do some examples

1) $Z = I$ & $E_i = \begin{pmatrix} 0 & & \\ & 1 & 0 \\ 0 & & 0 \end{pmatrix}$, with the 1 in position i

$Y \equiv$ pick random $i \in [1, \dots, n]$ return $n E_i$

$$E(Y) = \sum_{i=1}^{n} p(i) n E_i = \sum E_i = I = Z$$

$$\tau(n E_i / I) = n \tau(E_i / I) = n$$

Goal $2n e^{-c\varepsilon^2 K/n} < \delta$

pick $K = c' n \log n$

$2n e^{-c\varepsilon^2 K/n} = 2n\left(e^{\log n}\right)^{-c''} \approx n^{c''-1}$

Ex 2) $Z = L_G = L$ $\quad H_{ij} = \begin{array}{c} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ \begin{matrix} i & j \end{matrix} \end{array} = H_e$

$$L = \sum_{e \in E} H_e \quad \text{(unweighted case)}$$

$Y \equiv$ pick random $e \in E$ return mH_e

$$E(Y) = \sum_{i=1}^{m} p(i)\, m H_{e_i} = \sum H_{e_i} = L$$

$\sigma(mH_e/L) = m\,\sigma(H_e/L) = m\, ER_e$ in G. $\quad (*)$

If e is an isthmus then $ER_e = 1$

Thus $\mu = m$ and AW will require $O(m \log m)$ samples.

Picking edges uniformly is not good!

Lets try picking proportional to ER_e!

Question: What is $\sum_{e\in E} ER_e$ (G connected)

<u>Claim</u> $\sum_{e\in E} ER_e = n-1$ (Trees done!)

Foster: $\sum_{e\in E} w(e)\, ER_e = n-1$

(We will prove later.)

$Y \equiv$ pick an edge e with prob $= \dfrac{w(e)\, ER_e}{n-1} = P_e$

return $\dfrac{w(e)}{P_e} H_e = \dfrac{(n-1)\, w(e)}{w(e)\, ER_e} H_e = \dfrac{(n-1)}{ER_e} H_e$

$$E(Y) = \sum_{e\in E} P_e \left(\frac{w(e)}{P_e}\right) H_e = \sum_{e\in E} w_e H_e = L$$

$$\nabla\left(\frac{n-1}{ER_e} H_e/L\right) = \frac{(n-1)}{ER_e}\nabla(H_e/L) = \frac{n-1}{ER_e} ER_e = n-1$$

$$\nabla(Y/L) \leq n$$

<u>Thus</u> AW gives concentration at

$K = c' n \log n / \varepsilon^2$

<u>Note</u> $L_K = \sum_{i=1}^{K} Y_i / k$ is a Laplacian $\leq K$ edges.

<u>Thm</u> (SS) If H is obtained from G by sampling proportional to $w_e ER_e$ $O(n \log n)$ times then WHP $(1-\varepsilon)G \preceq H \preceq (1+\varepsilon)G$.

Questions

1) $\exists$ linear #edge spectral sparsifiers?
   (yes)
2) Can we find $O(n\log n)$ spectral sparse quickly?
   (yes)
3) Proof of AW?
   (to do!)
4) Do we need exact ER's?
   (no)
5) Proof of Foster Thm?
   (to do!)

What if we use upper estimates of ER_e !

Given $P_e \geq ER_e \;\; \forall e \in E$

set $t = \sum w_e P_e \geq n-1$

$P_e' = \frac{w_e P_e}{t}$ (probabilities)

$Y \equiv$ Pick e with prob P_e'; return $\frac{w_e}{P_e'} H_e = \frac{t}{P_e} H_e$

Clear $E(Y) \equiv L$

$$\nabla(Y_i/L) = \max_e \nabla\left(\frac{t}{P_e} H_e / L\right) =$$

$$= \max_e \left(\frac{t}{P_e} ER_e\right) \leq t \quad (P_e \geq ER_e)$$

AW $\Rightarrow$ concentration for $K = O(t \log t)$

Thus $P_e \geq ER_e$ work but may get more edges!