15-859N

11/15/13

Random Walks

Probability Space

Sample Space: All walks on a graph G.

Events: All walks starting with say

$X_0, X_1, \cdots X_k$

Prob: $\Pr(X_0) \cdot \Pr(X_0, X_1) \cdots P(X_{k-1}, X_k)$

Random Varibles:

Cover Time: $C = \begin{cases} \text{first time all of } V \text{ is visited} \\ \text{undef} \quad \text{o.w} \end{cases}$

Irreducible:

Recurrent:

Steady state $\pi_1 \cdots \pi_n$ (stationary)

Reversible: $\pi_i P_{ij} = \pi_j P_{ji}$

Random ST by Random Walks

Alg WalkTree Input: $G=(V,E)$ connected undirected (unit weight)

1) Pick $s \in V$ arbitrarily
2) Do random walk from s
3) Collect edge $(i,j) \in E$ if first visit to j
4) Return tree

Thm The WalkTree is a random ST.

Preliminaries

Random walk on strongly connected directed graphs.

ST $\equiv$ convergent rooted trees

$G=(V,P) \quad \sum_j P_{ij} = 1 \quad P_{ij} = W_{ij}$

$w(T) = \prod_{e \in T} w(e)$

$\mathcal{T}_i(G) \equiv ST(G)$ rooted at i

$\mathcal{T}(G) \equiv$ All rooted ST.

Markov Chain Tree Thm Let $\pi_i \equiv$ stationary prob of being at i

$$\pi_i = \frac{\sum_{T \in \mathcal{T}_i(G)} w(T)}{\sum_{T \in \mathcal{T}(G)} w(T)}$$

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pf $W = X_0, X_1, \dots,$ a (random) walk.

Def $B_t \equiv$ (backwards tree at time t)

$I = \{X_0, \dots, X_t\}$

$\ell(i,t) = \underset{0 \le j \le t}{\operatorname{argmax}} \{X_j = X_i\}$

$E(B_t) = \{(X_{\ell(i,t)}, X_{\ell(i,t)+1})\}$ $i \in I - X_t$

root is X_t

t	0	1	2	3	4	5	6	7	8	9
X	X_2	X_7	X_1	X_8	2	8	1	8	2	8

B_5

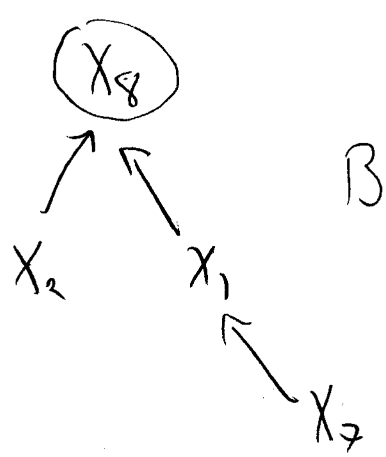
$\ell(0,5) = \emptyset$

$\ell(1,5) = 2 \qquad X_1 \to X_8$

$\ell(2,5) = 4 \qquad X_2 \to X_8$

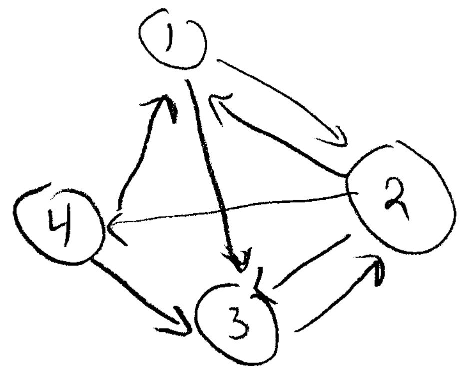
$\ell(7,5) = 1 \qquad X_7 \to X_1$

$\ell(8,5) = 5 \qquad X_8 \cancel{\to} X_1$



B_5

eg

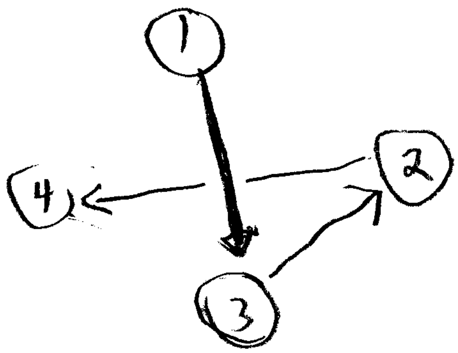


Time 0 1 2 3 4 5

$W = (1, 2, 1, 3, 2, 4, 3 \cdots)$

B_5

root 4



Note: $\{W\} \supseteq V$ then conver ST.

After cover time we have a random walk on rooted trees of G

$B_t, B_{t+1}, B_{t+2}, \cdots$

One recurrent class

$\pi(T) \equiv$ stationary prob of T

Note

$$\Pi_i = \lim_{N\to\infty} \frac{1}{N} \sum_{0\le t\le N} P_r(X_t = i)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_t P_r(B_t \text{ rooted at } i)$$

$$= \sum_{T\in T_i} \pi(T)$$

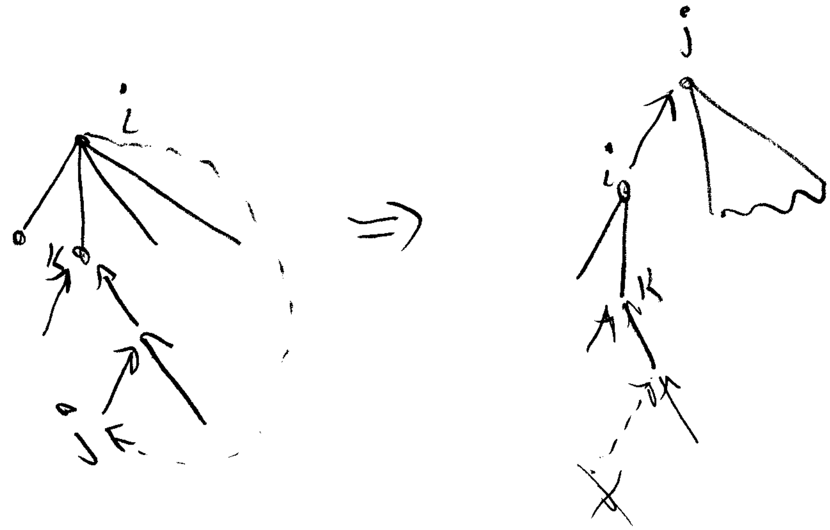
To show $\pi(T) \approx w(T)$

Let $T^{(i)} \equiv$ ST rooted at i

Find precursors of $T^{(i)}$!

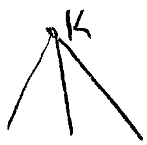
First we consider successor of $T^{(i)}$ say $T^{(j)}$

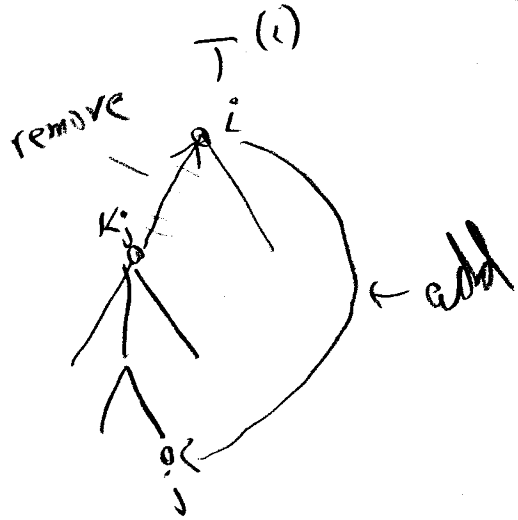
$T^{(i)}$



$$T^{(j)} = T^{(i)} + [i,j] - [j, P(j)]$$

precursor $T^{(k)}$





$$T^{(k)} = T^{(i)} + [i,j] - [K_j, i]$$

$$\nabla(T^{(i)}) = \sum_{(i,j)\in E} \nabla\left(T^{(i)} + [i,j] - [k_j, i]\right) P_{k_j i} \qquad (*)$$

<u>Claim</u> $W(T^{(i)})$ satisfies $(*)$

$$\sum_{(i,j)\in E} W\left(T^{(i)} + [i,j] - [k_j, i]\right) P_{k_j i}$$

$$= \sum_{(i,j)\in E} \frac{W(T^{(i)}) \cdot P_{ij}}{\cancel{P_{k_j i}}} \cancel{P_{k_j i}} = W(T^{(i)})$$

<u>Thus</u> Markov Chain Tree Thm!

Def (Forward Tree) F_t

$$f(i,t) = \underset{0 \le j \le t}{\arg\min} \{ X_j = V_i \} \quad \text{(first visit)}$$

Edges $\{ (X_{f(i,t)}, X_{f(i,t)-1}) \mid i \in I - X_0 \}$

Thm Let c = cover time for G, stationary $\equiv \pi_1 \cdots \pi_n$

$F_c \equiv$ forward tree at time c, G is undirected

$$P_r(B_c = T) = P_r(F_c = T) = \frac{w(T)}{\sum_{T' \in T(G)} w(T')} \quad \left(\text{starting from stationary} \right)$$

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pf Since started from stationary

$$P_r(X_0 = V_0, \ldots, X_k = V_k) = P_r(X_0 = V_k, \ldots, X_k = V_0)$$

Thus $Pr(B_K = T \mid \pi) = Pr(F_K = T \mid \pi)$

pass to limit!

Cor 1 M random walk on undirected $G = (V, E)$ starting at i

C is the cover time starting from π

Tree F_C is uniform random ST rooted at i

Pf $w(T) = d_i / \prod_{j \in V} d_j$ T rooted at i

Cor 2 Hypo same as Cor 1 but we return undirected F_C then get random tree.

Pf 1-1 correspondence with ST.