15-859N
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# Min Energy Flow using Random POCS

Goal: solve $Lv = d$ $\quad L\bar{v} = d$

where $L = \partial_1 C \partial_1^T \quad R = C^{-1}$

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**Let** $f_p = C\partial_1^T v$ st $\partial_1 f = d$

**Def** energy of a flow $f$. $f^T R f$

Thompson: min energy flow $f_p$.

**pf** $f_c$ circulation

$$\partial_1(f_p + f_c) = d \qquad f_p^T R f_c = 0$$

$$(f_p + f_c)^T R (f_p + f_c)$$

$$= f_p^T R f_p + 2\underset{\overset{\shortparallel}{0}}{f_c^T R f_p} + f_c^T R f_c$$

$$= f_p^T R f_p + f_c^T R f_c$$

Idea Lets compute min energy flow $f$ st $\partial_1 f = d$.

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We need lower bd on $f_p^T R f_p$!

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Consider $2v^T d - v^T L v = E(v)$.

$$\nabla E = 2d - 2Lv$$

$$\nabla E(v) = 0 \Rightarrow Lv = d$$

min or max at $\bar{v}$ st $L\bar{v} = d$

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Substitue $\bar{v}$

$$E(\bar{v}) = 2\bar{v}^T d - \bar{v}^T L \bar{v} = 2\bar{v}^T d - \bar{v}^T d = \bar{v}^T d$$

but $\bar{v}^T d = \bar{v}^T L \bar{v} = \bar{v}^T \partial_1 \underbrace{C \partial_1^T \bar{v}} = \bar{v}^T \partial_1 f_p$

$$= (RC\partial_1^T \bar{v})^T f_p$$

$$= f_p^T R f_p$$

Claim $\bar{v}^T d$ is a max

$$2(\bar{v}+v)^T d - (\bar{v}+v)^T L(\bar{v}+v)$$

$$= 2\bar{v}^T d + 2v^T d - \bar{v}^T L \bar{v} \underbrace{- 2v^T L \bar{v}}_{-2v^T d} - v^T L v$$

$$= \bar{v}^T d + 0 - v^T L v$$

$$= v^T d - v^T L v \quad \text{where } v^T L v \geq 0$$

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$$E(v) \leq E(\bar{v}) = E(f_p) \leq E(f_p + f_c)$$

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Idea: Find some f st $\partial_1 f = d$

Keep adding circulation to min energy.

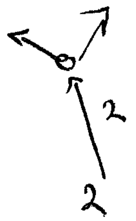
**Pick:** ST T of G

In particular: T LSST

**Claim** $\exists !$ flow f st $\partial_1 f = d$

$f(e) = 0$ for $e \notin T$. say $f_0$

Do a BFS from a leaf of T.



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Tree induced voltage

$$V_T(f) = V(a) = \sum_{e \in P(s,a)} f(e) r_e \quad \text{some fixed } s \in V.$$

**Def** Tree Condition Number

$T \subseteq G$

$e \in G \setminus T$ we define $ER_T(e) = \sum_{e \in R(a,b)} r_e$

KOSZ: $R_e = ER_T(e) + r_e$

Tree Cond. #: $\tau(T) = \sum_{e \in E \setminus T} \frac{R_e}{r_e} =$

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$$STr_T(e) = \frac{ER_T(e)}{r_e}$$

$$STr_T(G) = \sum_{e \in G} STr_T(e)$$

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$$\tau(T) = \underbrace{\sum_{e \in E \setminus T} str_T(e)} + m - n + 1$$

$$STr_T(G) - (n-1) + m - (n-1)$$

$$STr(G) + m - 2n + 2$$

Let $f_e$ be unit flow on cycle $C_e$

Goal: $\min_{\alpha} (f+\alpha f_e)^T R (f+\alpha f_e)$

**Claim** same as proj onto space

$\{f : f^T R f_e = 0\}$ R-orthgonal

ie $\alpha$ st $(f+\alpha f_e)^T R f_e = 0$

$$f^T R f_e + \alpha f_e^T R f_e = 0$$

$$\alpha = -\frac{f^T R f_e}{R_e}$$

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**Def** $\Pi_e(f) = f + \alpha f_e$

$$E(f) - E(\underset{\substack{\uparrow \\ \text{R-orth}}}{\Pi_e(f)}) = \alpha^2 f_e^T R f_e = \frac{(f^T R f_e)^2}{R_e}$$

Note

$$(v+w)^T R (v+w) - w^T R w = v^T R v$$

## KOSZ Alg $L_G x = d, K$

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1) $T \equiv LSST(G)$
2) Compute tree flow $f_0$ st $\partial_1 f_0 = d$
   $f = f_0$
3) for K rounds
    a) Pick $e \in E \setminus T$ proportional $\frac{R_e}{r_e}$
    b) $f = \Pi_e(f)$
4) Return f
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$$p_e = \frac{R_e}{\tau r_e}$$

Def: $\Delta_e(f) = f^T R f_e$

**Lemma** (Tree Gap) $\partial_1 f = d$ $\quad v = v_T(f)$

$$gap(f,v) = \sum_{e \in E \setminus T} \frac{(f^T R f_e)^2}{r_e}$$

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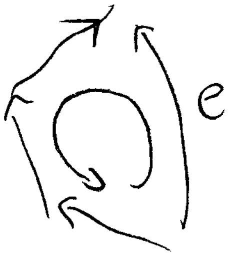
$$gap(f,v) = f^T R f - (2v^T d - v^T L v)$$

(calculation)

$$= (Rf - \partial_1^T v) R^{-1} (Rf - \partial_1^T v)$$

$$= \sum_{e \in E} \frac{1}{r_e} \left( f(e) r_e - \Delta_v(e) \right)^2$$

$e \in T$ then $\Delta_v(e) = f(e) r_e$



$e = (a,b)$

$$\Delta_v(e) = \sum_{e \in P(a,b)} f(e) r_e$$

$$f(e) r_e - \sum_{e' \in P} f(e') r_{e'} = f^T R f_e$$

## Lemma (Expected Progress)

$$E_x\left[ E(f_i) - E(f_{i-1}) \,\middle|\, gap(f_{i-1}, V_{i-1}) \right] = *$$

$$= -\frac{gap(f_{i-1}, V_{i-1})}{\tau}$$

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Pf

$$* = E_x\left[ \overbrace{\sum_{e \in E \setminus T} p_e \left( -\frac{(f_{i-1}^T R f_e)^2}{R_e} \right.}^{**} \,\middle|\, gap(f_{i-1}, V_{i-1}) \right]$$

$$p_e = \frac{1}{\tau} \cdot \frac{R_e}{r_e}$$

$$** \quad \sum_{e \in E \setminus T} -\frac{1}{\tau} \frac{\cancel{R_e}}{r_e} \left( \frac{(f_{i-1}^T R f_e)^2}{\cancel{R_e}} \right.$$

$$\frac{1}{\tau} \sum -\frac{(f_{i-1}^T R f_e)^2}{r_e} = -\frac{1}{\tau}\, gap(f_{i-1}, V_{i-1})$$