15-859N
11/4/13
part 1

Maximum Flow (undirected)

1) Unit capacities

2) $G=(V,E)$, $s,t\in V$ $|V|=n$ & $|E|=m$

$$B^m_\infty = \{ y \in \mathbb{R}^m \mid \forall i \; -1 \le y_i \le 1 \}$$

3) $$e_{ts} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \end{pmatrix} \begin{matrix} \\ \\ \\ s \\ \\ \\ t \\ \\ \\ \end{matrix}$$

<u>The Max flow prob</u> $\max_{F\geq 0} \exists_{f\in\mathbb{R}^m} \left(\partial_1 f = F e_{ts} \;\&\; f \in B^m_\infty \right)$

We will do binary search for F.

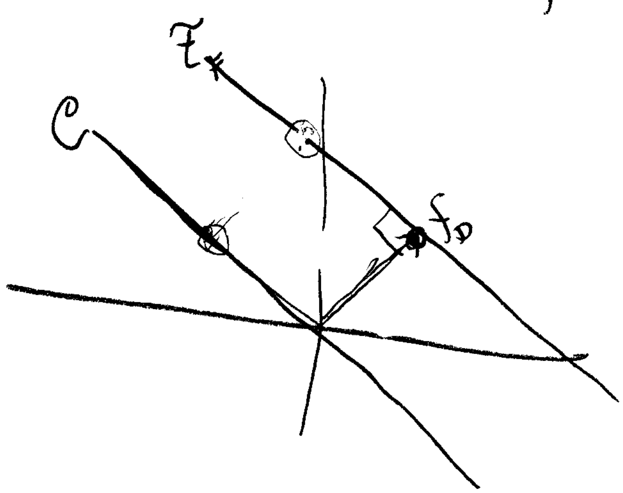
<u>Def</u> $\mathcal{C} = \{ f : \partial_1 f = 0 \}$

$\tilde{\tau}_F = \{ f : \partial f = F e_{ts} \}$

<u>Thompson</u> Min energy flow is a potential flow.

$\partial_1 f_p = F e_{ts}$ st $\exists v \;\; f_p = \partial_1^T v$

Since circulation $\perp$ potential flow (unit wieghts)



Projecting onto $\tilde{\tau}_F \equiv \mathrm{Proj}_{\tilde{\tau}_F}(f)$

Write $f = f_c + f_p'$ return $f_c + f_p = \bar{f}$

to show 1) $\bar{f} \in \tilde{Z}_F$ clear

2) $f - \bar{f} \perp \bar{f} - f_p$

2) $f_c + f_p' - f_c + f_p = f' - f_p \in$ Potential

$\bar{f} - f_p = f_c - f_p + f_p = f_c \in$ Cycle

Finding potential part:

$\partial_1^T v = f\,?$ $\quad \partial_1 \partial_1^T v = \partial_1 f$

$V = (\partial_1 \partial_1^T)^+ \partial_1 f$

$f_p = \underbrace{\partial_1^T (\partial_1 \partial_1^T)^+ \partial_1}_{\Pi} f$

$f_c = (I - \Pi) f$

Thus $\bar{f}$ L_2 closest point to $\tilde{Z}_F$

What is closest point to B_∞^m?

More general K convex set

$$\mathrm{Proj}_K(x) = \underset{y \in K}{\mathrm{argmin}} \; \|x - y\|_2$$

<u>Lemma</u> (Proj onto convex set)

(i) Proj_K is 1-Lipschitz

ie $\|\mathrm{Proj}_K(x) - \mathrm{Proj}_K(y)\|_2 \leq \|x - y\|_2$

ii) $f(x) = \frac{1}{2}\|x - \mathrm{Proj}_K(x)\|_2^2$

a) convex

b) Gradient $\nabla f(x) = x - \mathrm{Proj}_K(x)$

1-Lipschitz

<u>Note</u> (hw)

$\mathrm{Proj}_\infty^n(f) \equiv$ remove overflow.

Natural alg (POCS)

<u>Input</u> G, s, t, F

Init: $f_p = F \Pi e_{ts}$; $f = f_p$

Repeat

$\bar{f} = \text{removeoverflow}(f)$

$f_c = (I - \Pi)\bar{f}$

$f = f_c + f_p$