15-859N
11/13/13

Randomized Perfect Bipart Matching

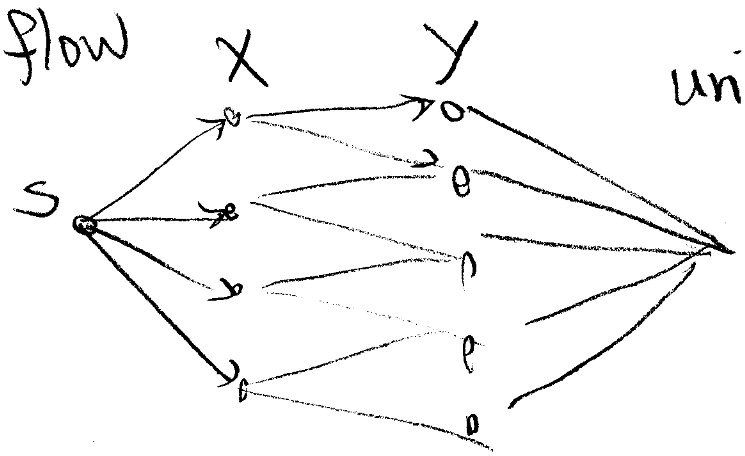
LRS: If G is an st-flow graph (undirected) in $O\left(\frac{1}{\varepsilon}\sqrt{\frac{m}{F}}\, m \log^3 n\right)$ can find flow $(1-\varepsilon)F$ where F is max flow.

Prob: The flow f is not integral even if cap are!

Simple example: G is regular bipartite graph

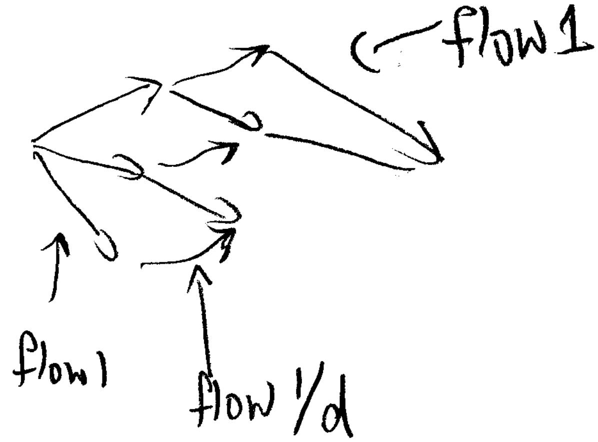
Goal: perfect matching.

Set up flow



unit cap

degree $= d$

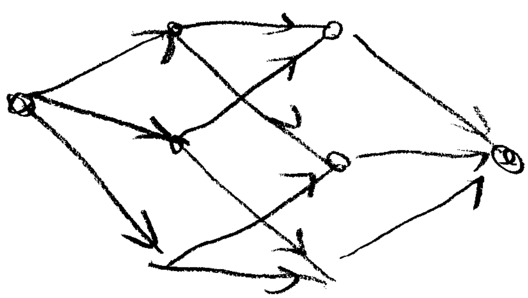


∴ G has perfect matching.

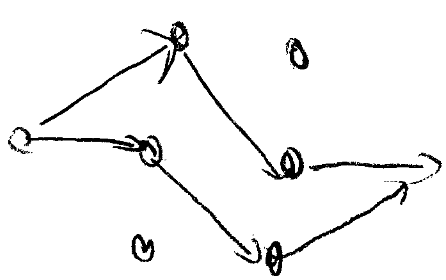
But the flow did not give us the matching.

Simplest Alg Ford-Fulkerson

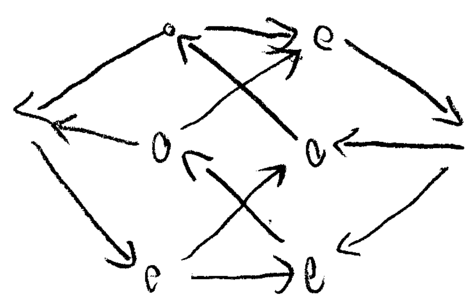
Review



suppose 2 paths



residual



1) Generate augmenting path.

2) update residual!

Thm 1 If we use random walk to find aug paths then $O(n \log n)$ total expect time to find matching.

Let $G=(X \cup Y, E)$ reg bipart graph

M a matching of size k.

$G_M \equiv$ residual graph.

Thm 2 Expect # of back edge on random walk from s to t is $\frac{n}{n-k} - 1$.

Thm 2 $\Rightarrow$ Thm 1

If $b(s)$ backedges then

$$|\text{path}| = 3 + 2(b(s)) = 3 + 2\left(\frac{n}{n-k} - 1\right) = 1 + \frac{2n}{n-k}$$

$$Ex(\text{Total}) = \sum_{k=0}^{n-1} 1 + \frac{2n}{n-k} = n + 2n \sum_{i=1}^{n} 1/i$$

$O(n \lg n)$ □

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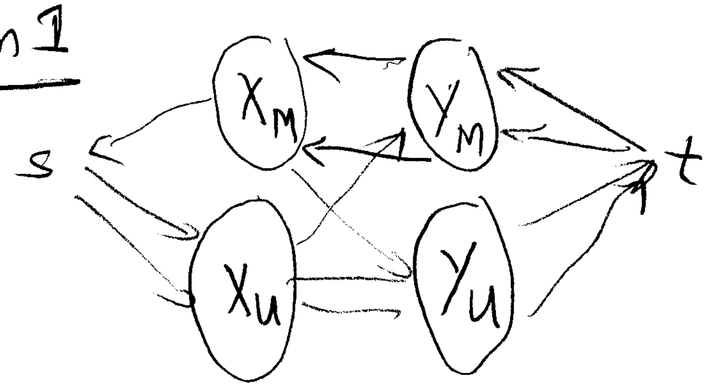
pf Thm 1



Def $b(v) \equiv$ Ex # of backedge on Ran Walk starting at v.

$y \in Y_U$ then $b(y) = 0$

$$b(s) = \frac{1}{n-k} \sum_{x \in X_u} b(x)$$

$$b(y) = \begin{cases} 0 & y \in Y_u \\ 1 + b(M(y)) & y \in Y_M \end{cases}$$

$x \in X_u$

$$b(x) = 1/d \sum_{(x,y) \in E} b(y) \Rightarrow d\, b(x) = \sum_{(x,y) \in E} b(y) \quad (*)$$

$x \in X_M$

$$b(x) = \frac{1}{d-1} \sum_{(x,y) \in E \setminus M} b(y) \qquad (d-1)\, b(x) = \sum_{(x,y) \in E \setminus M} b(y)$$

for $(x,y) \in M$ $\quad b(y) = 1 + b(x)$

$$d\, b(x) = \sum_{(x,y) \in E} b(y) - 1 \qquad (**)$$

$$*+** \quad d\sum_{x\in X} b(x) = -k + \underbrace{\sum_{x\in X}\sum_{(x,y)\in E} b(y)}_{d\sum_{y\in Y} b(y)}$$

but $$\sum_{y\in Y} b(y) = k + \sum_{x\in X_M} b(x)$$

$$d\sum_{x\in X} b(x) = -k + dk + d\sum_{x\in X_M} b(x)$$

$$d\sum_{x\in X_U} b(x) = (d-1)k$$

$$d(s) = \frac{1}{n-k}\sum_{x\in X_U} b(x) = \frac{(d-1)k}{d(n-k)} \le \frac{k}{n-k} = \frac{n}{n-k} - 1$$

$\square$