Personal PageRank & Spilling Paint

$G=(V,E)$ undir (weighted?)

A adj $D \equiv$ deg matrix

Regular random walk

$$p^{(t+1)} = A^T D^{-1} p^{(t)} \qquad W = A^T D^{-1}$$

Lazy

$$p^{(t+1)} = \frac{1}{2}\left(I + A^T D^{-1}\right) p^{(t)}$$

$$\frac{1}{2}\left(I + A^T D^{-1}\right) = \hat{W}$$

Two possible goals.

1) Recommendations.

2) Graph Cut.

2-Views

$0 \le u \in \mathbb{R}^n \quad |u|_1 = 1$ $\qquad \mu = \chi_\mu$

Walks with resets

$$p^{(t+1)} = \alpha\mu + (1-\alpha) W p^{(t)}$$

or " " $\hat{W}$ " "

Thm $\exists!\, P_u \ni \; P_u = \alpha\mu + (1-\alpha) W P_\mu$

$$P_u - (1-\alpha) W P_u = \alpha u$$

$$(I - (1-\alpha)W) P_u = \alpha u$$

but $(I - (1-\alpha)W)$ non-sing $\quad |\lambda((1-\alpha)W)| < 1$

Thus

$$P_u = (I - (1-\alpha)W)^{-1} \alpha\mu$$

$$(I-X)^{-1} = (I + X + X^2 + X^3 + \cdots)$$

$$P_n = \alpha \sum_{t \geq 0} (1-\alpha)^t W^t u$$

View 2

Suppose we have $s^{(t)}$ stuck paint.

$r^{(t)}$ remaining paint.

they evole

$$s^{(t+1)} = s^{(t)} + \alpha r^{(t)}$$
$$r^{(t+1)} = (1-\alpha) W r^{(t)}$$

Where is the stuck paint?

$$s^{\infty} = \alpha \sum_{t \geq 0} r^{(t)} = \alpha \sum (1-\alpha)^t W^t r^{(0)}$$
$$= \alpha \sum (1-\alpha)^t W^t u$$

Thus $P_u = s^{\infty}$

<u>Claim</u> Reg & Lazy Walk diff by a constant.

$$s^{\infty} = \alpha\left(I-(1-\alpha)\hat{W}\right)^{-1}u \qquad \hat{W} = I/2 + W/2$$

$$= \alpha\left(\frac{1+\alpha}{2}I-\left(\frac{1-\alpha}{2}\right)W\right)^{-1}u$$

$$= \frac{2\alpha}{1+\alpha}\left(I-\left(\frac{1-\alpha}{1+\alpha}\right)W\right)^{-1}u$$

$\beta = \frac{2\alpha}{1+\alpha}$ note $1-\beta = 1-\frac{2\alpha}{1+\alpha} = \frac{1-\alpha}{1+\alpha}$

$$= \beta\left(I-\beta W\right)^{-1}u$$

Pushing wet paint

Suppose s dry paint
r wet

$$p_{s,r} = s + \alpha \sum_{t \geq 0} (1-\alpha)^t W^t r = s + \alpha (I - (1-\alpha) W)^{-1} r$$

Consider a partial update at some u.

Update(u)

$$s'(u) = s(u) + \alpha r(u)$$

$$p'(u) = 0$$

$$p'(v) = p(v) + \frac{1-\alpha}{d(u)} p(u) \quad \forall v \in N(u)$$

Lemma $P_{s',r'} = P_{s,r}$

pf see notes

Alg Approx Painting

Init: $s \equiv 0 \quad r = \chi_u$

While $\exists\ r(u) \geq \varepsilon d(u)$

Pick max $r(u)/d(u)$

Update(u)

Computing small conductance sets

$S \subseteq V$

<u>Def</u> $\mathrm{Vol}(S) = d(S) = \sum_{v \in S} d(v)$

$\partial S = \{(x,y) \mid x \in S \ \& \ y \notin S\}$

$|\partial S| = \sum_{e \in \partial S} w(e)$

Conductance $\Phi(S) = \dfrac{|\partial(S)|}{\min\{d(S), d(\bar{S})\}} = \phi(S)$

$\chi_S(u) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{o.w.} \end{cases}$

Let P be our stationary dist with resets

$$q_v(u) = \frac{P_v(u)}{d(u)}$$

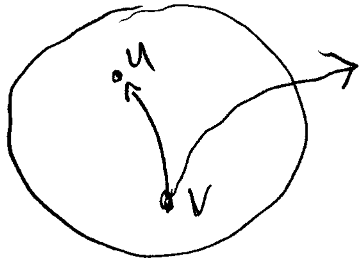
sort $q_v(u)$ giving $q_v(1) \geq q_v(2) \geq \cdots \geq q_v(n)$

<u>Def</u> $S_k = \{1, \dots, k\}$ $\quad S_k \subseteq V$

<u>Lemma</u> $\displaystyle\sum_{ij \in E \;\; i \leq k < j} q(i) - q(j) \leq \alpha \qquad S = S_k$

first $\displaystyle \chi_S^T W P = \sum_{(u,v) \in E,\, u \in S,\, v \in V} W_{uv} P(v)$

$$= \sum_{uv \in E,\, u \in S\, v \in S} q(v) + \sum_{uv \in E,\, u \in S\, v \notin S} q(v)$$



$$\sum_{v\in S} p(v) - \sum_{uv\in E,\, u\notin S,\, v\in S} \delta(v) \qquad + \sum \delta(v)$$

$$\left.\chi_S^T p - \sum_{uv\in E,\, u\in S,\, v\notin S} \delta(u) - \delta(v) \right\} \text{LHS}$$

$$p = \alpha \chi_u - (1-\alpha) W p$$

$$Wp = (p - \alpha\chi_u)(1-\alpha)^{-1} \geq (p - \alpha\chi_u)$$

$$\chi_S^T W p \geq \chi_S^T p - \alpha \qquad (u \in S)$$

$$\chi_S^T p - \sum \quad \geq \chi_S^T p - \alpha$$

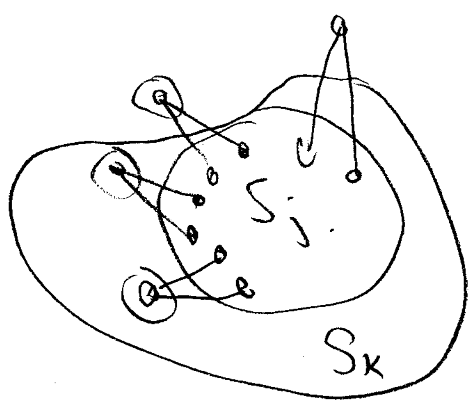
$$\alpha \geq \sum \delta(i) - \delta(j)$$

<u>Lemma</u> $\phi(S_j) \geq 2\theta$ Then $\exists k > j$ s.t

1) $d(S_k) \geq (1+\theta)\, d(S_j)$

2) $g(j) - g(k) \leq \dfrac{\alpha}{\theta d(S_j)}$

<u>Pf</u> Set $k = \underset{k}{\operatorname{argmin}} \left\{ |\partial S_j \cap E(S_k)| \geq \dfrac{|\partial S_j|}{2} \right\}$



$|\partial(S_j)| \geq 2\theta\, d(S_j)$

ok

$$\alpha \geq \sum_{ab \in E;\, a \leq j;\, b \geq k} g(a) - g(b) \geq \theta d(S_j)(g(j) - g(k))$$

This gives 2)

<u>Lemma</u> $\phi(S_j) \geq 2\theta \;\; \forall j$ st $d(S_j) \leq 2m/3$

$$h = \arg\min_h \{ d(S_h) \geq 2m/3 \}$$

then $\forall i \leq h$ $\quad g(h) \geq g(i) - \frac{2\alpha}{\theta^2 d(S_i)}$

Let $K_1, \cdots K_t$ be seq of K from last lemma

$$g(i) - g(K_1) \leq \frac{\alpha}{\theta d(S_i)}$$

$$g(K_1) - g(K_2) \leq \frac{\alpha}{\theta d(S_{K_1})} \leq \frac{\alpha}{\theta(1+\theta) d(S_i)}$$

$\vdots$

$$g(i) - g(h) \leq \frac{\alpha}{\theta d(S_i)} \left(1 + \frac{1}{1+\theta} + \frac{1}{(1+\theta)^2} + \cdots \right)$$

$$\leq \frac{\alpha}{\theta d(S_i)} \frac{1+\theta}{\theta} \qquad \theta \leq 1$$

$$\leq \frac{2\alpha}{\theta^2 d(S_i)}$$

$S \subseteq V$

$$\pi_S(u) = \begin{cases} \frac{d(u)}{d(S)} & \text{if } u \in S \\ 0 & \text{o.w.} \end{cases}$$

$$p_{\pi_S} = \alpha \sum (1-\alpha)^t W^t \pi_S$$

<u>Claim</u> $\chi_{\bar{S}}^T W^t \pi_S \leq t\,\phi(S)$

case $t=1$

$$\sum_{v \notin S} \sum_{uv \in \partial S} \frac{1}{d(u)} \cdot \frac{d(u)}{d(S)} = \frac{1}{d(S)} \sum_{v \notin S} \sum_{uv \in \partial S} 1 = \frac{|\partial S|}{d(S)} = \phi(S)$$

to check $t > 1$

Lemma $\chi_{\bar{S}}^{T} P_{\pi_S} \leq \phi(S) \frac{1-\alpha}{\alpha}$

$$\text{RHS} = \alpha \sum_{t \geq 0} (1-\alpha)^t \chi_{\bar{S}}^{T} \pi_S$$

$$\leq \alpha \sum_{t \geq 0} (1-\alpha)^t \, t \, \phi(S)$$

$$= \alpha \phi(S) \sum_{t \geq 0} (1-\alpha)^t t \qquad (*)$$

$\beta = 1-\alpha$

$$\frac{1}{(1-\beta)} = \sum_{t \geq 0} \beta^t \Rightarrow + \frac{1}{(1-\beta)^2} = \sum_{t \geq 1} \beta^{t-1} \cdot t$$

$$\frac{\beta}{(1-\beta)^2} = \sum_{t \geq 1} \beta^t \cdot t = \sum_{t \geq 0} \beta^t t$$

$1-\beta = \alpha$

$$(*) = \alpha \phi(S) \left(\frac{1-\alpha}{\alpha^2} \right) = \phi(S) \left(\frac{1-\alpha}{\alpha} \right)$$

Set α st $\phi(S)\left(\frac{1-\alpha}{\alpha}\right) < 1/3$

<u>Lemma</u> $\exists i$ st $d(S_i) \le d(S)$ and

$$g(i) \ge \frac{2/3}{d(S_i)\, H_{2m}}$$

$\gamma = H_{2m}$

<u>pf.</u> ?

Thm

Suppose the following for some S

1) $d(S) \leq 2m/9\gamma$

2) start walk with resets for $u \in S$

prob $= \pi_S$

3) Set α st $\phi(S)\left(\frac{1-\alpha}{\alpha}\right) < 1/3 \approx \phi(S) < \frac{\alpha}{3}$

then $\exists j$ st $\phi(S_j) = O\left(\sqrt{\phi(S)\log m}\right)$ $(*)$

pf Suppose false

set $\sqrt{6\alpha\gamma} \approx \sqrt{2\phi(S)\ln n} = \Theta$

$\forall j \quad \phi(S_j) \geq 2\Theta$

$$g(h) \geq g(i) - \frac{2\alpha}{\Theta^2 d(S_i)} = g(i) - \frac{2\alpha}{6\alpha\gamma\, d(S_i)} \geq$$

$$\frac{2/3}{d(S_i)\gamma} - \frac{1/3}{\gamma d(S_i)} = \frac{1/3}{\gamma d(S_i)}$$

$\forall i \leq h \quad g(i) \geq g(h) \geq \dfrac{1}{3\gamma d(S_i)}$

$$\sum_{i=1}^{n} d(i)\, g(i) = 1$$

$$\sum_{i=1}^{h} d(i)\, g(i) \geq g(h) \sum_{i=1}^{h} d(i) = d(S_h)\, g(h)$$

$$\geq \left(\tfrac{2}{3}m\right) \big/ 3\gamma d(S_i) = \frac{2m}{9\gamma d(S_i)} \geq \frac{2m}{9\gamma d(S)} = (*)$$

$d(S) < \dfrac{2m}{9\gamma} \Rightarrow (*) > 1$ contra!