15-859N
10/25/13

# Arnoldi Iteration
# Lanczos Iteration

Goal! Compute some of the eigens of $A$.

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Def $A$ similiar to $A'$ if $A' = B^{-1}AB$

Eigens of $A'$ are just a change of variables of $A$.

if $Ax = \lambda x$ then $A'z = \lambda z$ for $z = B^{-1}x$

Pf $A'z = B^{-1}ABB^{-1}x = B^{-1}Ax = \lambda B^{-1}x = \lambda z$

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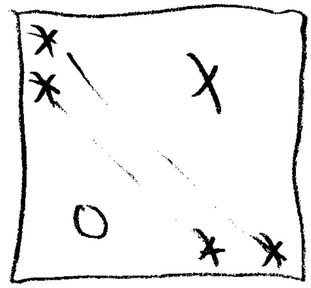
Question If $A$ is rational can we find similiar rational $A'$ close to diagonal?

Can we pick $B$ to be orthonormal?
ie. $B^TB = I$

<u>Def</u> $H$ is Hessenberg form if

$H =$ 

<u>Note</u> If $H$ is sym then it is tridiagonal

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2-methods we will <u>not</u> do

Jacobi & Householder

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<u>Goal:</u> Given $A$ find similiar Hessenberg $H$.

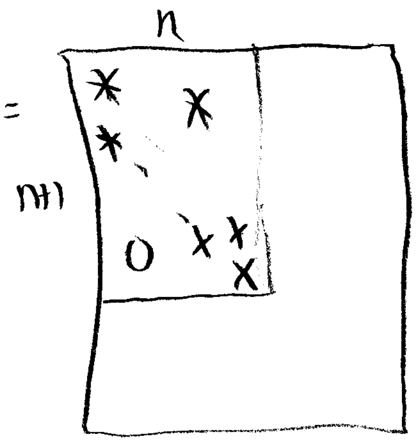
<u>Find</u> $Q$ st $Q^T A Q = H \wedge Q^T Q = I$

ie $AQ = QH$

$A$ is $m \times m$.

We construct $Q$ & $H$ a column at a time.

$$Q_n = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} \qquad \tilde{H}_n =$$




$\tilde{H}_n$ is $(n+1) \times n$

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Suppose we start with $b$

Set $q_1 = b/\|b\|$

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Solve for $q_2$ & $h_{11}$ & $h_{21}$

$$V = Aq_1 = \begin{pmatrix} | & | & \\ q_1 & q_2 & \cdots \\ | & | & \end{pmatrix}\begin{pmatrix} h_{11} \\ h_{21} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = h_{11} q_1 + h_{21} q_2$$

$$q_1^T q_2 = 0 \qquad q_1^T q_1 = 1$$

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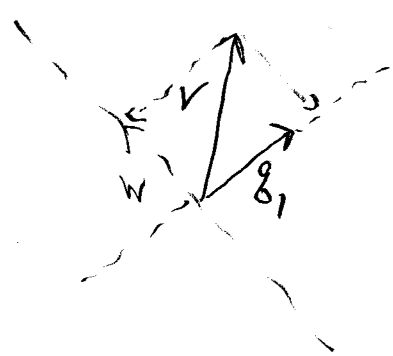
$h_{11} = q_1^T v$

$w = v - h_{11} q_1$

$h_{21} = \|w\|$

$q_2 = w / h_{21}$



<u>Given</u> $Q_n$ & $\tilde{H}_n$ find $Q_{n+1}$ & $\tilde{H}_{n+1}$

ie $$A\begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_{n+1} \\ | & & | \end{pmatrix}\begin{pmatrix} h_{11} & \cdots & h_{1n} \\ h_{21} & & | \\ & \ddots & | \\ 0 & & h_{n+1\,n} \end{pmatrix}$$

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Solve $V = A q_n = h_{1n} q_1 + \cdots + h_{n+1,n} q_{n+1}$

for $i \le n$ $q_i^T q_{n+1} = 0$ & $q_{n+1}^T q_{n+1} = 1$

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<u>Answer</u> 1) project $v$ onto $\langle q_1 \cdots q_n \rangle$

2) Find orthogonal residual.

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<u>Alg</u> Arnoldi Iter

Input $A, b$

$q_1 = b/\|b\|$

for $n = 1, 2, \cdots$

$V = A q_n$

⋮

for $j = 1$ to $n$

$h_{jn} = q_j^T V$

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$W = V - \sum_j h_{jn} q_j$

$h_{n+1,n} = \|W\|$ $(= 0)$?

$q_{n+1} = w / h_{n+1,n}$

<u>Claim</u> $\mathcal{K}_n = \langle b, Ab, \dots A^{n-1}b \rangle = \langle q_1, \dots, q_n \rangle$

$\|q_i\| = 1$ & $q_i^T q_j = 0$ for $i \neq j$

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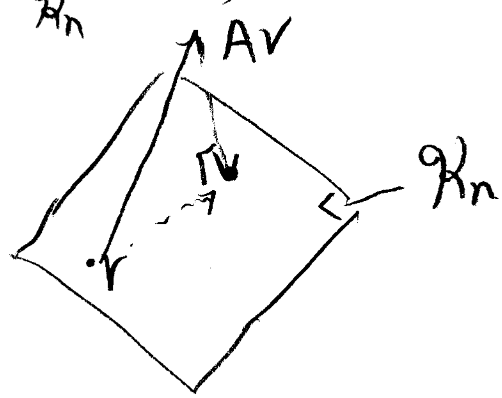
Consider $A$ as acting on $\mathcal{K}_n$

ie $v \in \mathcal{K}_n$ $\quad \bar{A}_n(v) = \mathrm{Proj}_{\mathcal{K}_n}(Av)$

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a) Working in standard basis

<u>Note</u> $\mathrm{Proj}_{\mathcal{K}_n}(v) = Q_n Q_n^T v$

$$\bar{A}_n = Q_n Q_n^T A$$

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b) Consider $\underset{\text{local}}{\wedge}$ basis $\begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$

If $w \in \mathbb{R}^n$ in local basis then $v = Q_n w$ is in standard.

$\bar{A}_n$ in local basis is:

$$\underbrace{Q_n^T Q_n}_{I} Q_n^T A Q_n (w)$$

$$\bar{A}_n \approx Q_n^T A Q_n \quad \text{(in local basis)}$$

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What is $Q_n^T A Q_n$?

$$Q_n^T A Q_n = Q_n^T Q_{n+1} \tilde{H}_n = H_n$$

where
$$H_n = \begin{array}{c} \\ n \end{array} \begin{array}{c} n \\ \left[\begin{array}{c|c} H_n & \\ \hline & \end{array}\right] \end{array} \leftarrow H$$

Note
$$Q_n^T Q_{n+1} = \begin{array}{c} \\ n \end{array} \begin{array}{c} n+1 \\ \left[\begin{array}{ccc|c} 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{array}\right] \end{array}$$

<u>Thus</u> $\bar{A}_n$ is just $H_n$

<u>Def</u> $\lambda(H_n)$ are called Arnoldi eigenvalues or Ritz values.

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<u>Note</u> A sym $\Rightarrow H_n$ is tridiagonal.

Alg has a 3-term recurrance

alg is called <u>Lanczos</u>

Still need eigen alg for sym tridiagon syms.

# Arnoldi & Polynomial Approx

Note $x \in \mathcal{K}_n$ $x = c_0 b + c_1 Ab \cdots c_{n-1} A^{n-1} b$

$\exists$ poly $g(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1}$

st $x = g(A)b$

Let $\mathbb{P}^n = \{$monic poly of degree $n\}$

ie $c_n = 1$

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Arnoldi/Lanczos Approx Prob

Find $p^n \in \mathbb{P}^n$ min $\|p^n(A)b\|$

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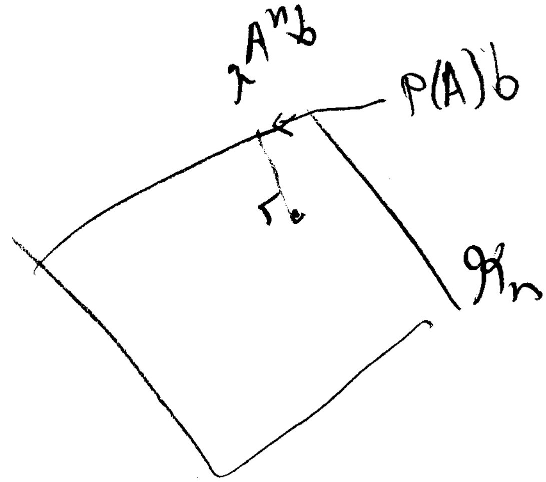
Thm $\mathcal{K}_n$ not of full rank then $p^n$ is unique

and $p^n = \text{char}(H_n)$

Pf $p \in P^n \Rightarrow p(A)b = A^n b - Q_n y$ for some $y \in \mathbb{R}^n$

If $\|p^n(A)b\|$ min $\Rightarrow p^n(A)b$ is orthogonal complement of the proj $A^n b$ onto $\mathcal{K}_n$




ie $Q_n^T p^n(A) b = 0$ (1)

<u>Claim</u> $p^n(H_n) = 0$

We know $A = QHQ^T$

Thus $Q_n^T p^n(QHQ^T) b = 0$

but $Q_n^T Q\, p^n(H) Q^T b$

<u>note</u> $Q^T b = \begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

The first columns of $P^n(H)$ ~~is~~ only depend on first $\underline{n}$ columns of $H$

$$\bar{H}_n = \left(\begin{array}{ccc|c} | & & | & 0 \\ h_1 & \cdots & h_n & \\ | & & | & 0 \end{array}\right) = \left(\begin{array}{c|c} H_n & 0 \\ \hline 0 \quad * & 0 \end{array}\right)$$

$h_{n+1\,n}$

$$Q_n^T Q\, P^n(\bar{H}_n) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \qquad (*)$$

$$Q_n^T Q = \begin{array}{cc} & m \\ n & \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \end{array}$$

~~(2) iff $P^n(H_n) = 0$~~

~~Cayley-Hamilton $\Rightarrow$ $P^n = \mathrm{char}(H_n)$~~

Thus $Q_n^T Q\, P^n(\bar{H}_n)\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$ = top $n$ elements in first column of $P^n(\bar{H}_n)$

<u>Note</u> $h_{n+1\,n}$ only effects the $n+1$ first element in last column of $P^n(H)$

$$\therefore\ Q_n^T Q\, P^n(H)\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} = 0 \text{ iff } \qquad P^n(H_n)\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} = 0$$

iff first column $(P^n(H_n)) = 0$

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Let $g = \text{char}(H_n) \Rightarrow g$ is monic

$g(H_n) = 0$ by Cayley-Hamilton Thm

Thus 1st-col $(g(H_n)) = 0$

So $g$ works but we know it is unique.

QED