# $O(m \log n)$ Laplician solver (cont)

Goal: Approx $L_G x = b$
Using Precond Iter methods.

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Last time chain of subgraphs

$G \supseteq H_0 \supseteq H_1 \supseteq \cdots \supseteq H_{\log n}$ (Precond chain)

1) Each with same vertex set
2) $E(H_{i+1}) \subseteq E(E_i)$ weights may be bigger!

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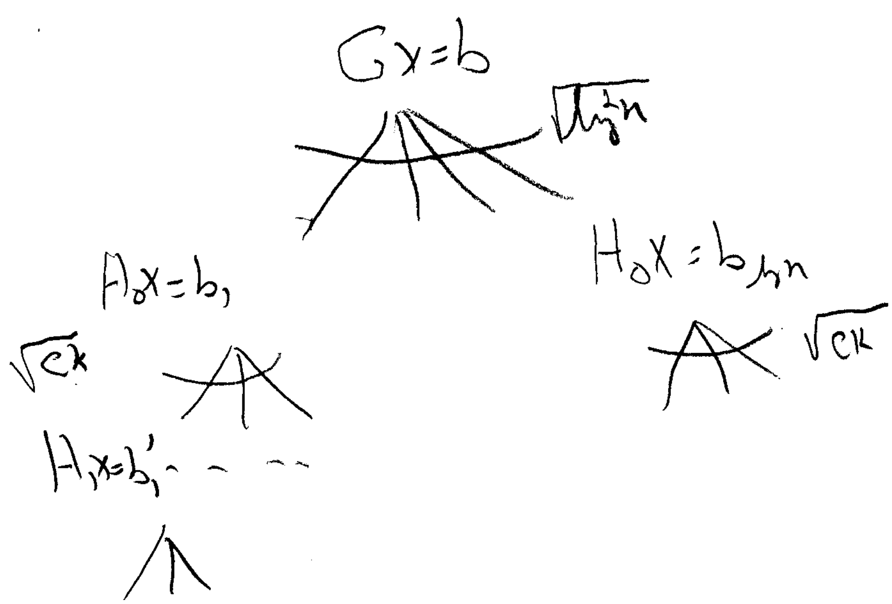
## Condition Numbers

$k(G, H_0) = O(\log^2 n)$ $\quad |E(G)| \approx |E(H_0)|$

$K(H_i, H_{i+1}) = O(k)$ $\quad |E(H_i)| \geq k|E(H_{i+1})|$

# Tree of Recursive calls




Work per call $O(n, m)$ where $n$ = # vert
$m$ = # non tree edges.

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<u>Note</u> $O(\lg n)$ calls to Spine-heavy solver.

## Greedy Elim-1-2

**Goal** 1) Use GE to remove all degree 1 & 2 nodes.
2) But keep spanning tree.

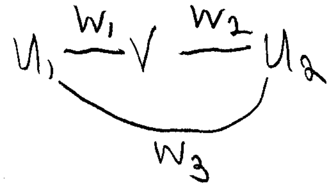
Greedy Elim

Input: $G = (V, E, w)$ & ST $T$ of $G$

Output: $\hat{G} = (\hat{V}, \hat{E}, \hat{w})$ & ST $\hat{T}$ of $\hat{G}$

1) $\hat{G} := G$; $E_{\hat{T}} = E_T$

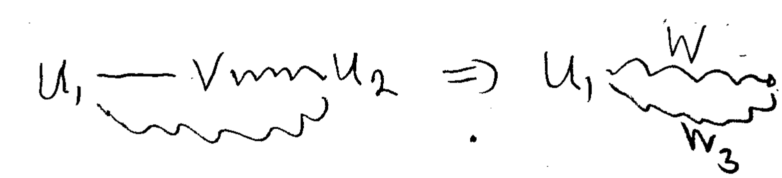
2) while $\exists v$ of degree $\leq 2$ do
   if degree$(v) = 1$ then remove $v$.
   if degree$(v) = 2$




— tree edges
~ non-tree edges

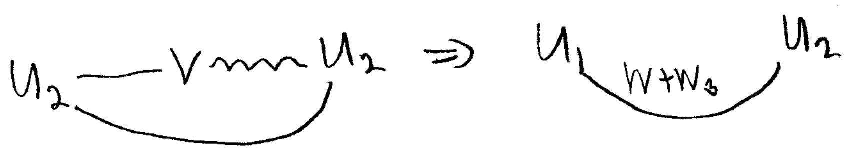
$$W \Leftarrow \frac{w_1 \cdot w_2}{w_1 + w_2}$$

Case 1




(2 samples)

Case 2



Case 3 $u_1 - v - u_2 \Rightarrow u_1 \overset{w+w_3}{\text{——}} u_2$

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Claim $\hat{T}$ is ST of $\hat{G}$ & $|\text{nontree}(\hat{G})| \leq |\text{nontree}(G)|$

if $K = |\text{nontree}(G)| = K$ then $|\hat{V}| \leq 2K$ & $|\hat{E}| \leq 3K$

pf

Let $n_1$ = # leaves in $\hat{T}$

$n_2$ = # degree 2 node in $\hat{T}$

$n_3$ = # " " $\geq 3$ " ".

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note $n_3 < n_1$ thus $n = n_1 + n_2 + n_3 \leq 2n_1 + n_2 \leq 2K$

$\hat{G}$ has $2K-1$ tree edges & $K$ nontree edges

$m \leq 3K$

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Claim $\text{Stretch}_T(e) \geq \text{stretch}_{\hat{T}}(e)$

# Gaussian Elim Details

Reducing solve $Hx = b$ to $\bar{H}z = \bar{b}$

$$H = L\begin{pmatrix} I & 0 \\ 0 & \bar{H} \end{pmatrix} L^T$$

solve $L\begin{pmatrix} I & 0 \\ 0 & \bar{H} \end{pmatrix} L^T = b$

1) solve $Ly = b$

2) solve $\begin{pmatrix} I & 0 \\ 0 & \bar{H} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

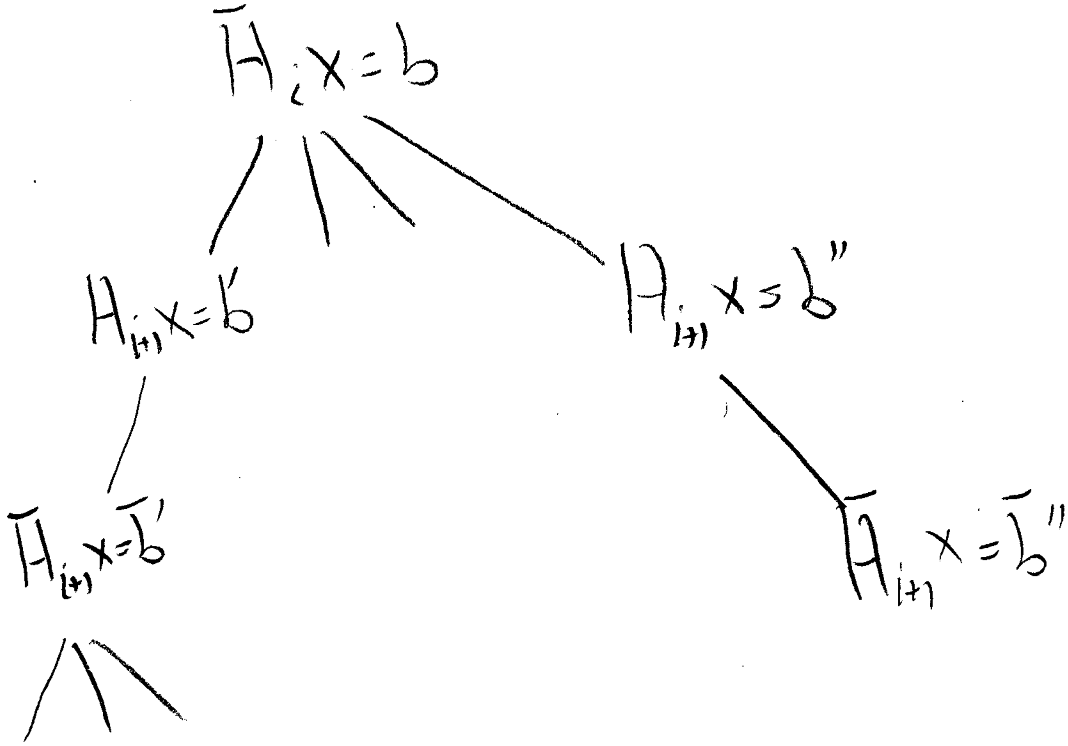
   a) $z_1 = y_1$

   b) solve $\bar{H}z_2 = y_2$

3) solve $L^T x = z$

# Tree of Recursive calls

Top level unchanged

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<u>Claim</u> Solving spine-heavy Laplacians in $O(m+n)$ time.

<u>Note</u> for sufficiently large $k$ work per level is geometrically decreasing.