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Fiedler's Thm on Nodal Domains

G connected weighted graph Laplacian L_G

eigenvalues $0 = \lambda_1 < \lambda_2 \cdots \leq \lambda_n$

vectors $v_1, \cdots, v_n$

<u>Note</u> $v_i : V \to \mathbb{R}$

<u>let</u> $f : V \to \mathbb{R}$

<u>Def</u> (Nodal Domains of f)

Let $f^+ = \{ v \in V \mid f(v) > 0 \}$

$f^- = \{ v \in V \mid f(v) < 0 \}$

Induced graph on $W \subseteq V$ is

$G(W) = (W, E_W)$ $E_W = \{ (v,u) \in E \mid v, u \in W \}$

<u>Nodal Domain</u> is a connected component of either $G(f^+)$ or $G(f^-)$

Def (Weak Nodal Domains of f)

$$W_f^+ = \{v \in V \mid f(v) \geq 0\}$$

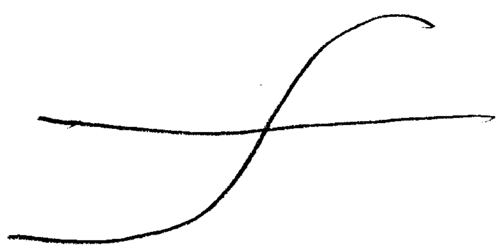
WND $\equiv$ connected components of W_f^+.

We could also use $W_f^- = \{v \in V \mid f(v) \leq 0\}$

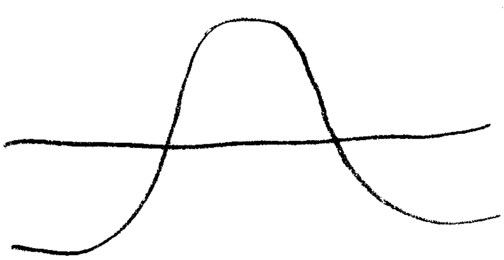
EG. $G = P_n$

$v_1 \equiv (1, \cdots, 1)$ one ND.

v_2

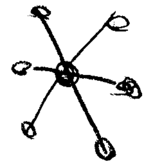


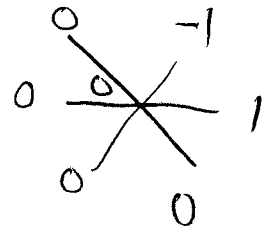
$\Rightarrow$ 2-NDs.

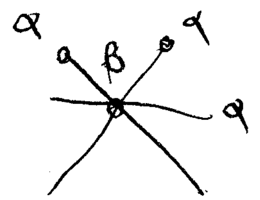


v_3

3-NDs.

$G = S_n$  $n-1$ leaves

$\lambda(S_n) = (0, 1, \dots, 1, \lambda_n)$ 

$\lambda_n =$  $(n-1)\alpha = -\beta$

for $\lambda_2, \dots, \lambda_{n-1}$ one WND

but 2 to $n-1$ ND.

<u>Thm</u> (Fiedler 75) G weighted connected and

$\lambda(L_G) \equiv 0 = \lambda_1 < \dots \le \lambda_n$ with

vectors $v_1 \dots v_n$ then

for $k \geq 2$ v_k has $\le k-1$ weak nodal domains

Preliminaries

Perron-Frobenius for generlized Laplacian

M is generalized Laplacian if

1) $M_{ij} \leq 0$ for $i \neq j$
2) The nonzero off diagonals form a connected undirected graph
3) M_{ii} ?

Cor If M is GL and λ_1 smallest eigenvalue. with vector V_1 then λ_1 has multplicity one and V_1 strictly positive or negative.

pf pick σ s.t $A = \sigma I - M \geq 0$

Let α_1 max eigenvalue of A with strictly positive V_1 by PF Thm.

then $\lambda_1 = \sigma - \alpha_1$ with eigenvector v_1.

<u>Thm</u> (Eigenvalue Interlacing)

A sym $n \times n$ matrix & B is A with same row & column deleted, then

$$\alpha_1 \le \beta_1 \le \alpha_2 \cdots \alpha_{n-1} \le \beta_{n-1} \le \alpha_n$$

where $\alpha_1 \le \cdots \le \alpha_n$ & $\beta_1 \le \cdots \le \beta_{n-1}$ eigens of A & B.

<u>pf</u> (Courant-Fischer)

<u>Cor</u> $\alpha_i \le \beta_i$

<u>Cor</u> If B is a principle submatrix of A

then $\alpha_i \le \beta_i$

(proof of Fiedler 75)

<u>Note</u> $W_k \neq \emptyset$ $\quad v_k \neq 0$ & $v_k \perp \mathbb{1}$.

Suppose $G(W_k)$ has t connected components.

We reorder vertices by component they belong.

$$L_G = \begin{pmatrix} B_1 & 0 & & 0 & C_1 \\ 0 & B_2 & & 0 & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & & 0 & B_t & C_t \\ C_1^T & \cdots & & C_t^T & D \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_t \\ y \end{pmatrix}$$

x_i, B_i are vertices in ith ND.

vertices in $D = \{v \in V \mid v_k(v) < 0\}$ = vertices of y.

$x_i \geq 0$ & $y < 0$

$$\begin{pmatrix} B_1 & & & C_1 \\ & \ddots & & \vdots \\ & & B_t & C_t \\ C_1^T & & C_t^T & D \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_t \\ Y \end{pmatrix} = \lambda_K \begin{pmatrix} X_1 \\ \vdots \\ X_t \\ Y \end{pmatrix}$$

<u>note</u> $B_i X_i + C_i Y = \lambda_K X_i$

<u>where</u> $C_i \leq 0 \wedge Y < 0 \Rightarrow C_i Y \geq 0$ & $C_i Y \neq 0$

$X_i \geq 0$ & $X_i \neq 0$ & $B_i X_i \neq \lambda_K X_i$

$B_i X_i \leq \lambda_K X_i \Rightarrow X_i^T B_i X_i \leq \lambda_K X_i^T X_i$

<u>Thus</u> $\lambda_1(B_i) \leq \lambda_K$

$$\underset{X \neq 0}{\arg\min} \frac{X^T B X}{X^T X} = V_1(B_i)$$

<u>Case 1</u> X_i is not strictly positive $\Rightarrow$

$\Rightarrow X_i$ is not eigenvector of smallest value

$\Rightarrow \lambda_1(B) < \lambda_K$

Case 2 X_i strictly positive

then $X_i^T C_i Y > 0$

$$X_i^T B_i X_i < X_i^T B_i X_i + X_i^T C_i Y = \lambda_k X_i^T X_i$$

Thus $\lambda_1(B_i) < \lambda_k$ ie $B_i w_i = \mu_i w_i$

$\mu_i < \lambda_k$

Consider Principle Submatrix

$$\begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_t \end{bmatrix} \begin{pmatrix} 0 \\ \vdots \\ \mu_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mu_i \begin{pmatrix} 0 \\ \vdots \\ \mu_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We have t eigenvalues $\mu_1 \cdots \mu_t < \lambda_k$

WLOG $\mu_1 \leq \cdots \leq \mu_t$

$\lambda_t \leq \mu_t \Rightarrow t \leq k-1$.

Let $f: V \to R$

<u>Def</u> $\text{support}(f) = \{v \mid f(v) \neq 0\}$

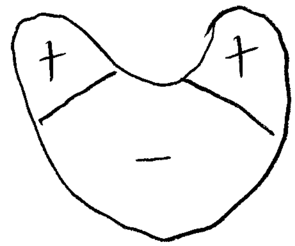
$\text{supp}_+(f) = \{v \mid f(v) > 0\}$

$\text{supp}_-(f) = \{v \mid f(v) < 0\}$

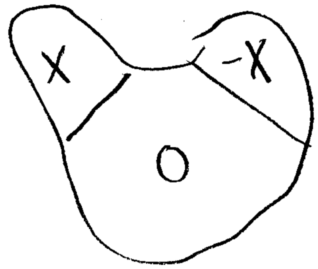
<u>Def</u> If $Ax = \lambda x$ then x has <u>minimal support</u> if $\forall y$ s.t. $Ay = \lambda y \not\Rightarrow \text{support}(y) \subsetneq \text{support}(x)$

<u>Main</u> Q gen-Lap of G, G connected, $Qx = \lambda_2 x$, and x has minimal support then $G(\text{supp}_+(x))$, $G(\text{supp}_-(x))$ are connected.

<u>picture proof</u> Suppose x



Consider $\bar{x} \equiv$



adjust $\bar{x}$ s.t. $\bar{x} \perp \mathbb{1}$

<u>Claim</u> $\dfrac{\bar{x}^T Q \bar{x}}{\bar{x}^T \bar{x}} < \dfrac{x^T Q x}{x^T x}$ or $\text{support}(\bar{x}) \subsetneq \text{support}(x)$