# Faster Random Spanning Trees

## Two Types

1) Determinant Based methods using Matrix Tree Thm

$O(n^{2.38})$

2) Random Walk Based methods

a) Aldous-Broder: $O(\text{cover-time})$

HW cover-time $= O(m \cdot n) = O(m \cdot \text{diameter})$

b) Wilson:

Random Tree(r)

1) Tree ← r

2) While $\exists v \notin$ Tree

Walk ← randomWalk(v, Tree)

Tree ← Tree + Erase Cycles(Walk)

Thm 1) RandomTree generate a random ST
2) Expect time = mean hitting time = $\tau$

$$\tau = \sum_{v,w} \pi(v)\,\pi(w)\,H(v,w)$$

no proof

Today (Kelner, Madry, Propp)
Random Walk using Short Cutting

Thm Short Cutting $O(m\sqrt{n})$ work
$O(m\sqrt{n})$ expected walk.

We start with $O(m^{3/2})$ bd.

# Outline of Alg.

1) Find decomp into subgraphs $G_1 \cdots G_k$
   a) $\text{dia}(G_i) = \sqrt{n}$
   b) edge between subgraphs $O(\sqrt{n})$

2) Using Laplician solver
   Compute transition prob between nodes in
   $\partial G_i$.

3) Run Short Cutting some node $r$.
   a) Start by running simple Random Walk.
   b) once a $G_i$ is "covered" use transition prob
   from 2) to shortcut $G_i$

## The $(\phi,\gamma)$-decomp

$(D_1, \dots, D_k, S, C)$ partition

1) $S \subseteq V(G)$
2) $D_i$ are vertex disjoint induced subgraphs
3) $\bigcup V(D_i) = V(G) \setminus S$
4) $C = E(G) \setminus \bigcup E(D_i)$

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Def 1) $U(D_i) \subseteq V(D_i)$ incident to $C$.

Def 2) $C(D_i) \subseteq C$ incident to $D_i$

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**Def** $(\phi,\gamma)$-decomp

1) $|C| \leq \phi |E(G)|$
2) $\gamma(D_i) \equiv \text{dia } D_i \leq \gamma$
3) $|C(D_i)| \leq |E(D_i)|$

for us: $\phi = 1/\sqrt{m}$ $\quad \gamma = \tilde{O}(m^{1/2})$

1) $|C| \leq \sqrt{m}$
2) $\gamma(D_i) = \tilde{O}(m^{1/2})$

The Walk $X$

$X \equiv$ Random walk starting from stationary!

Random variable: $\tau \equiv$ cover time

Thus $E(\tau) = O(mn)$

Random Var: $Z \equiv$ # traversals of $C$ before $\tau$

Thus $E(Z) = O((\phi m) \cdot n)$

Ran Var: $Z_i \equiv$ # traversals of edges $E(D_i)$ before $\tau$

Thus $\tau = \sum Z_i + Z$

Ran Var: $\tau_i \equiv$ time when $D_i$ "covered"

Ran Var: $Z_i^* \equiv$ # traversals of $E(D_i)$ before $\tau_i$

<u>Lemma</u> $E(Z_i^*) = \tilde{O}(|E(D_i)|\, \gamma(D_i))$

# Transition Probs

Fix $D_i$

**Def** $P_v(e) \equiv$ prob of $X$ leaving $D_i$ on edge $e$ after entering via $v \in \partial(D_i)$

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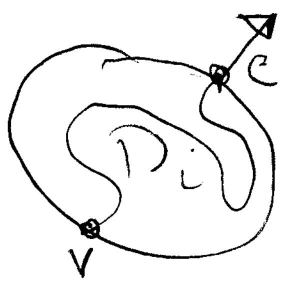


**Lemma** $\exists\ \tilde{O}(\phi m^2 \log 1/\delta)$ alg to compute $\delta$-approx of all $P_v(e)$.

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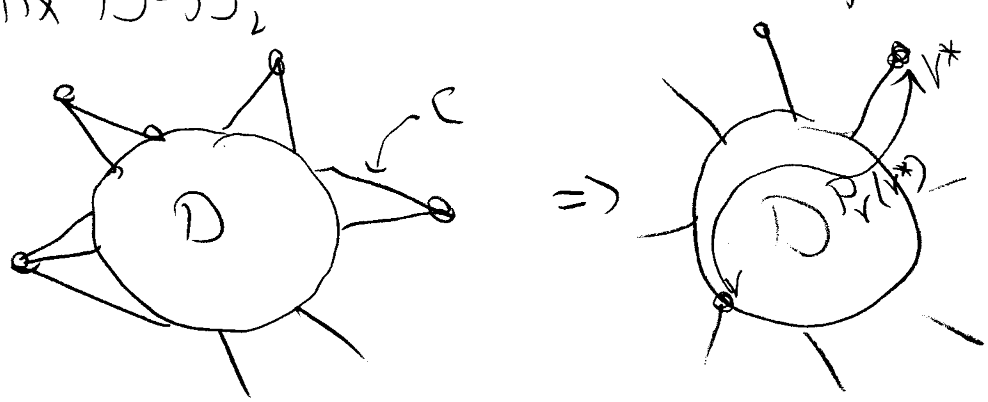
fix $D = D_i$

split vertices



**Recall** Prob visiting $v^*$ starting at $v$ before visiting $\mathcal{U}$

Alg

1) set volt $(v^*) = 1$; volt$(\mathcal{U}) = 0$

2) Compute volt$(v)$ = Prob.

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solve $L \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ ? \\ \vdots \\ ? \end{pmatrix} = \begin{pmatrix} ? \\ \vdots \\ ? \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

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**Note** Need only solve per edge $e \in C$.

**Cost to solve** $\tilde{O}(|E(D_i)| + |C(D_i)|) = \tilde{O}(|E(D_i)|)$

\# solves $2|C|$

Total cost $\equiv \tilde{O}\left(|C| \sum_i |E(D_i)| \lg 1/\delta\right) = \tilde{O}(\phi m^2 \lg 1/\delta)$

## Length of Random Short cut walk.

$$\sum_i E(z_i^*) + 3E(z) = O(m\gamma + \phi n)$$