15-857N
10/14/13

Solving Linear Systems

M is $n \times n$ sparse matrix over some field.

$w = \#$ nonzeros elements $= O(n)$ $\quad w = w(M)$

Goal: given M, b solve $Mx = b$

1) Direct Methods
2) Iterative Methods
3) Hybrid Methods

Direct Methods

Suppose L is lower triangular i.e. $L = \begin{bmatrix} * & & 0 \\ & \ddots & \\ * & & * \end{bmatrix}$

Claim $Lx = b$ can be solved in $O(w(L))$ by back substitution

Pf $\quad l_{11} x_1 = b_1 \Rightarrow x_1 = b_1 / l_1$ if $l_1 \neq 0$

solve $\begin{pmatrix} l_{22} & & 0 \\ & \ddots & \\ * & & l_{nn} \end{pmatrix} \begin{pmatrix} x_2 \\ \vdots \\ x_n \end{pmatrix} = b'$

Suppose $M = LU$ L lower tri
U Upper tri

Then to solve $LUX = b$ in $O(w(L) + w(U))$ time.

<u>Alg</u> solve $LY = b$
solve $UX = Y$

Computing an LU decomposition
(Cholesky Decomposition)
Gaussian Elimination - Pivoting

Suppose $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

subtract row 1 from 2 & 3

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Goal: subtract row 2 from row 3

Stuck!

1) Swap rows 2 & 3

2) start over

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>In General</u> (Assume we have pre ordered rows)
($M \leftarrow PM$)

$$\text{Let } L_i = \underbrace{\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ 0 & & * & \ddots & \\ & & \vdots & & \\ & & * & 0 & 1 \end{pmatrix}}_{i}$$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ & & * \;\; 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ \vdots & * & 1 & \\ 0 & * & * & 1 \end{pmatrix}\begin{pmatrix} 1 & & 0 \\ * & \ddots & \\ \vdots & 0 & \\ * & & 1 \end{pmatrix} M = U$$

ie $\quad L_{n-1} \cdots L_2 L_1 M = U$

<u>note</u> $L_{n-1} \cdots L_1 \equiv$ lower tri

The product is a mess

Consider $L^{-1} = L_1^{-1} \cdots L_{n-1}^{-1}$

Claim: if $L_1 = \begin{pmatrix} 1 & & & \\ a_1 & 1 & & \\ \vdots & & \ddots & \\ a_{n-1} & & & 1 \end{pmatrix}$ then $L_1^{-1} = \begin{pmatrix} 1 & & & \\ -a_1 & 1 & & 0 \\ \vdots & & \ddots & \\ -a_{n-1} & & & 1 \end{pmatrix}$

if $L_2 = \begin{pmatrix} 1 & & & \\ 0 & 1 & & 0 \\ \vdots & b_1 & \ddots & \\ \vdots & \vdots & 0 & \\ 0 & b_{n-2} & & 1 \end{pmatrix}$

then $= L_1^{-1} L_2^{-1} = \begin{pmatrix} 1 & & & \\ -a_1 & 1 & & 0 \\ \vdots & -b_1 & \ddots & \\ \vdots & \vdots & & \\ -a_{n-1} & -b_{n-2} & 0 & 1 \end{pmatrix}$

<u>Claim</u> $L^{-1} = nI - L_1 - \cdots - L_{n-1}$

$M = L^{-1} U$

When can we pick any pivot order?

Answer Symmetric positive definite (important case)

ie 1) Symmetric $M^T = M$

2) Positive definite ie $X \neq 0 \Rightarrow X^T A X > 0$

ie eigenvalues of $A > 0$

Ex If M is spd then Diagonal$(M) > 0$

and stays positive under Gaussian elimination.

Goal: Pick pivot order to minimize$(W(L))$

for spd matrices

Def Graph of $M^{n \times n} \equiv (V, E) = G$

$V = \{V_1, \dots, V_n\}$

$(V_i, V_j) \in E$ if $M_{ij} \neq 0$

G is undirected

Shur complement

Suppose $L_1 M = \begin{pmatrix} a & * & \cdots & * \\ 0 & & & \\ \vdots & & M_1 & \\ 0 & & & \end{pmatrix}$

$$L_1 M L_1^T = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & M_1 & \\ 0 & & & \end{pmatrix}$$

$$(L_1 M L_1^T)^T = (L_1^T)^T M^T L_1^T = L_1 M L_1^T$$

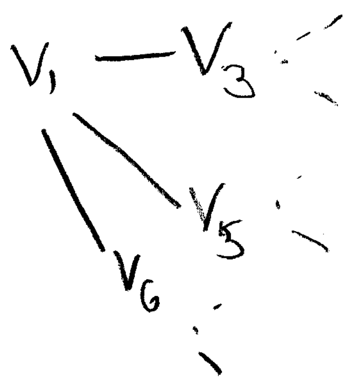
$$\Rightarrow M_1^T = M_1$$

Def $M_1 \equiv$ Shur Complement

Question Graph(M_1)?

$G = \text{Graph}(M)$

eg



$$M = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left(\begin{array}{ccccccc} * & 0 & * & 0 & * & * & 0 \cdots \\ 0 & & & & & & \\ * & & & ? & & & \\ 0 & & & & & & \\ * & & & & & & \\ * & & & & & & \\ 0 & & & & & & \end{array} \right) \end{array}$$

$$L_1 M = \begin{array}{c|c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} \\ \\ 3 \\ \\ 5 \\ 6 \end{array} & \left(\begin{array}{cccccc} * & 0 & * & 0 & * & * \\ 0 & & & & & \\ 0 & & * & & * & * \\ 0 & & & & & \\ 0 & & * & & * & * \\ 0 & & * & & * & * \end{array} \right) \end{array}$$

possibly new edges in M'

$E(3,5)$ new $E(3,1)$ & $E(1,5)$

$E(3,6)$ new $E(3,1)$ & $E(1,6)$

Graph theoretic view of pivoting

pivot(V_i) $\equiv$ 1) Make a clique of Neig(V_i)
2) remove V_i and its edges

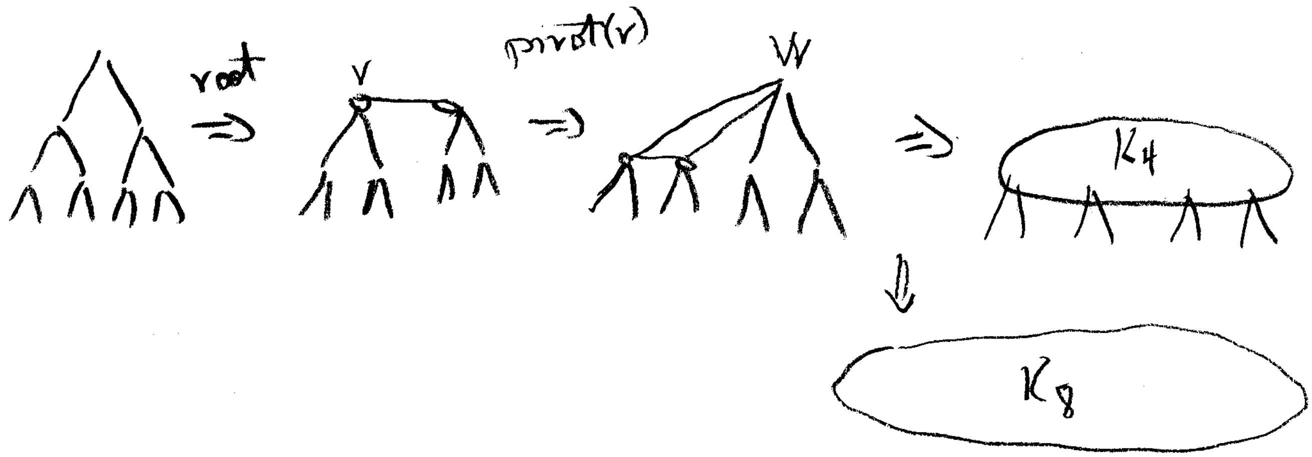
Fill $\equiv$ edges added at step 1

Cost of pivot(V_i) $\equiv$ $O(d_i^2)$ where
$d_i^2 \equiv$ degree of V_i at pivot-time.

Pivot Stratagy

EG $G \equiv$ balanced binary tree

1) Consider pivoting from root to leaves



Fill $= \Omega(n^2)$ we get clique on leaves

$K_{n/2}$

Work the work to pivot $K_{n/2}$ on leaves

$\Omega(n^3)$

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2) Consider pivoting leaves to root

note All pivots are on degree one nodes!

1) no fill

2) $O(n)$ work

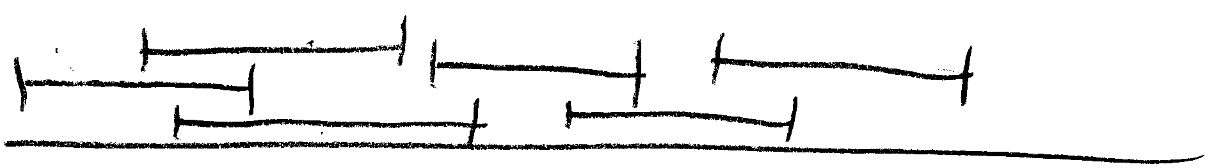
Perfect Elimination

Def A perfect elimination order is one with <u>no</u> fill.

Note Trees have perfect elim orders

Def G is an <u>interval</u> graph if

1) each vertex corresponds to an interval
2) $(v_i, v_j) \in E$ if $I_i \cap I_j \neq \emptyset$

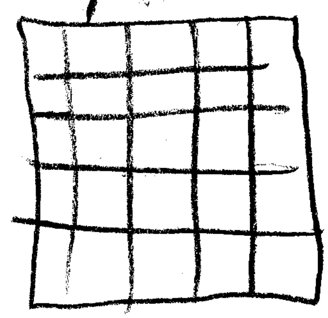


Fact Interval graph has a perfect elim order.

Note If I has left most right end point then $Neig(I)$ is a clique.

Pivot Strategies for Sq Mesh

Input $Ax = b$ where $A^{n \times n} =$



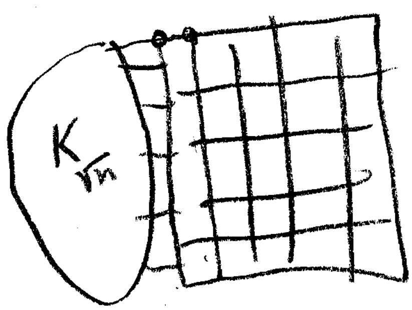
$\sqrt{n}$

Known: $\text{Fill}(\text{Mesh}) = \Theta(n \log n)$

$\text{Work}(\text{Mesh}) = \Theta(n^{3/2})$

Try: One column at a time

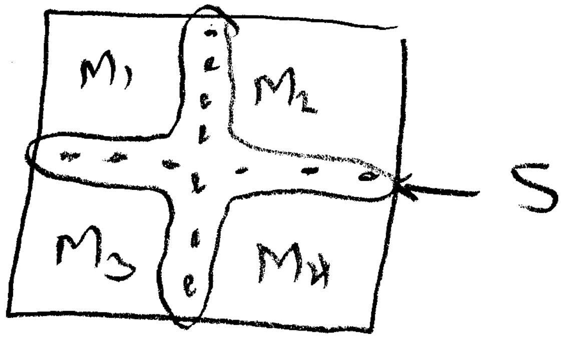
After first column



Each column will be a clique $\Omega(n^{3/2})$ fill

$\Omega(n^2)$ work.

Try Nested Disection



$$\text{Pivot}(M) = (\text{Pivot}(M_1), \text{Pivot}(M_2), \text{Pivot}(M_3), \text{Pivot}(M_4), S)$$

Goal: Bd Fill & work for ND!

View of Fill for Planar Graphs

Input: 1) Triangulated planar graph G
2) Pivot strategy $v_1 \cdots v_n$

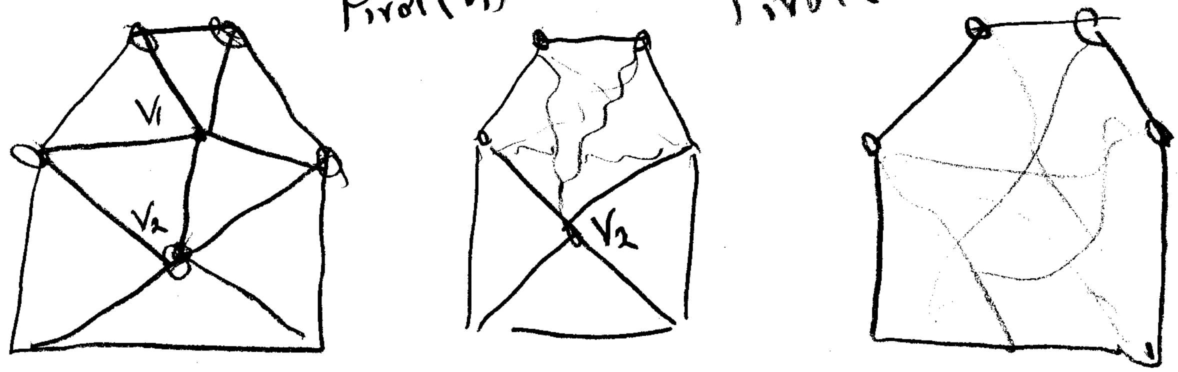
Two Views: 1) $G_i \equiv G \setminus V_1 \cdots V_i$

2) $F_i \equiv G$ after pivot $V_1, \cdots, V_i$

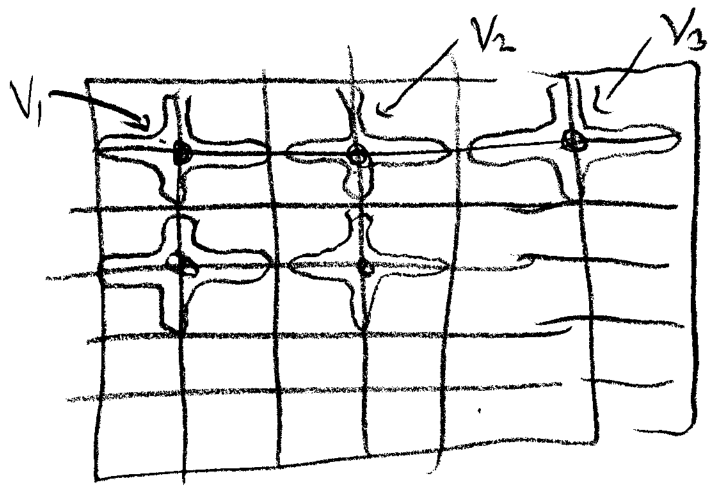
View of elimination – Pivot (V_i)

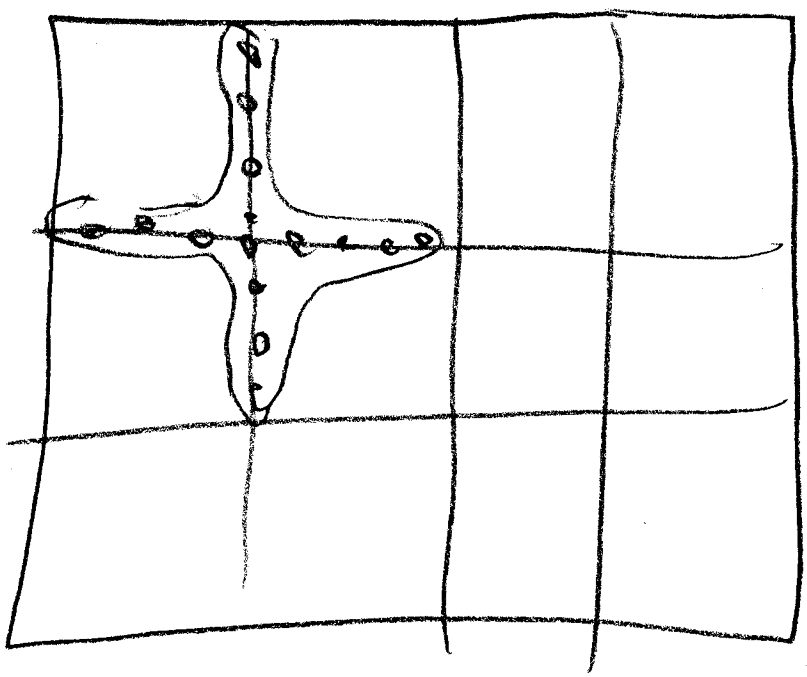
Given G_{i-1} 1) remove V_i from G_{i-1} forming new faces
2) Form a clique out of vertices on a face.

Pivot(V_1) Pivot$(V_2$

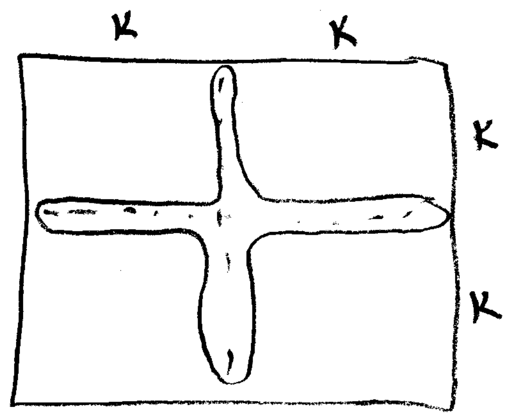


Analysis of pivoting on crosses for sq Mesh





Removing Crosses of Size $2K \times 2K$



View pivot of a cross in 2 steps

1) Pivot horizontal part
2) Pivot out vertical part.

Computing the Fill for some K

$$\# \text{crosses} = \left(\frac{\sqrt{n}}{2K}\right)^2 = \frac{n}{4K^2}$$

1) Initially the faces are size $4K$.
The first pivot will form a new face from two faces
This fill $= (3K)^2$ and $\frac{n}{2K^2}$ faces
The remaining pivots have no fill

$$Fill_H = \left(\frac{n}{2K^2}\right)(3K)^2 = \frac{9}{2}n$$

2) Initially faces have size 6K.

Only the first pivot per vertical part has fill.

$$\text{Fill}_V = \left(\frac{n}{4K^2}\right)(4K)^2 = 4n$$

$$\therefore \text{ Total fill } \sum_{i=1}^{\log\sqrt{n}} \frac{17}{2} n \approx \frac{17}{2} n \log n = O(n \log n)$$

The Work for ND

1) (# crosses) (# pivots) $\left(\frac{\text{face size}}{2}\right)^3$

$$\left(\frac{n}{4k^2}\right)(2k)\left(\frac{(7k)^2}{2}\right) \leq \frac{49}{4} nk = cnk$$

2) $$\left(\frac{n}{4k^2}\right)(2k)\left(\frac{(10k)^2}{2}\right) = cnk$$

Total Work $$\left(\sum_{i=1}^{\log \sqrt{n}} cn2^i\right) = O(n^{3/2})$$

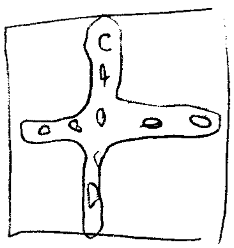
$k = 2^i$

Block Gaussian Elimination

$$\begin{pmatrix} I & 0 \\ -BA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B^T \\ B & C \end{pmatrix} = \begin{pmatrix} A & B^T \\ 0 & C-BA^{-1}B^T \end{pmatrix}$$

Shur complement $D = C - BA^{-1}B^T$

pivot-out crosses



$A^{4k \times 4k}$

B has only $8k$ nonzero rows

Alg	work	time
1) compute A^{-1}	$O(k^\alpha)$	$O(\lg k)$
2) BA^{-1}	$O(k^\alpha)$	$O(\lg k)$
3) $(BA^{-1})B^T$	$O(k^\alpha)$	$O(\lg k)$
	$O(k^\alpha)$	$O(\lg k)$ per cross

Work per level

$$\left(\frac{n}{4k^2}\right) c k^{\alpha} = c k^{\alpha-2} n$$

$k = 2^i$

Total Work $$cn \sum_{i=1}^{\lg\sqrt{n}} (2^i)^{\alpha-2} = cn(\sqrt{n})^{\alpha-2} = c n^{\alpha/2}$$