15-859N
Spectral
9/23/13

Differential Equations on Graphs.

$G = (V, E, C)$ undirected $C : E \to R^+$
thermal conductors

$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ = vector of temperatures

$\partial_1^T u \equiv$ temper drop across each edge.

$C\partial_1^T u \equiv$ thermal flow

$-\partial C \partial_1^T u \equiv$ change in temp at each vertex.

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i.e.

$\frac{\partial u}{dt} = -Lu$ (differential Eq)

Goal given $u^{(0)}$ find $u(t)$

note $\frac{\partial u}{\partial t} \equiv \begin{pmatrix} \frac{du_1}{dt} \\ \vdots \\ \frac{du_n}{dt} \end{pmatrix}$

Matrix Exponential

recall $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Def $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$ (A sym)

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$\frac{de^{At}}{dt} = A + \frac{2A^2t}{2!} + \frac{3A^3t^2}{3!} + \cdots$$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \cdots\right)$$

$$Ae^{At}$$

Claim $u(t) = e^{-Lt}u^{(0)}$ works

$u(0) = u^{(0)}$

$$\frac{du(t)}{dt} = \frac{d(e^{-Lt})}{dt}u^{(0)} = (-L)e^{-Lt}u^{(0)} = -Lu(t)$$

Sym $L = S \Lambda S^T$

$$e^{-Lt} = S e^{-\Lambda t} S^T = S \begin{pmatrix} e^{-\lambda_1 t} & & \\ & \ddots & \\ & & e^{-\lambda_n t} \end{pmatrix} S^T$$

$$S = \begin{pmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{pmatrix}$$ (orthonormal col of eigenvectors)

$$u(0) = c_1 x_1 + \cdots + c_n x_n$$

$$u(t) = c_1 e^{-\lambda_1 t} x_1 + \cdots c_n e^{-\lambda_n t} x_n$$

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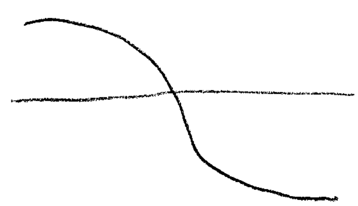
eg $G = P_n$

$x_1 = \begin{pmatrix} \vdots \end{pmatrix}$ no decay

$x_2 =$



$\lambda_2 \approx \frac{1}{n^2}$ $\quad e^{-t/n^2}$

halflife = 1/e-life

$T = n^2$

Springs and Graph Laplacian's

Input: Graph of Springs (Mattress)

$G = (V, E, K)$

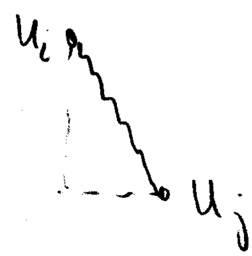
$K_{ij} \equiv$ spring constant for edge E_{ij}

Consider only vertical displacements

$u_i \equiv$ displacement of V_i

Goal: Find solutions to Newton: $f = ma$

Find forces for displacement $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$.



Set force on u_i by spring E_{ij}

$-(u_i - u_j) K_{ij}$

ie linear spring model.

$L = L(G)$

Force vector $-Lu$

Let $m_i \equiv$ mass of V_i $\quad M = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix}$

$m_i > 0$

View u as a fcn of time $u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$

Acceleration: For $u_i(t) \equiv \dfrac{d^2 u_i}{dt^2}$

$$\frac{du}{dt} = \begin{pmatrix} \frac{du_1}{dt_1} \\ \vdots \\ \frac{du_n}{dt_n} \end{pmatrix} \quad \therefore \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \frac{d^2 u}{dt^2}$$

Newton: $-Lu = M\left(\dfrac{d^2 u}{dt^2}\right)$

two ways to solve:

1) Solve for u given initial conditions
say $u(0)$ & $u'(0)$

2) Find steady state solutions

Do 2) using Guess and check.

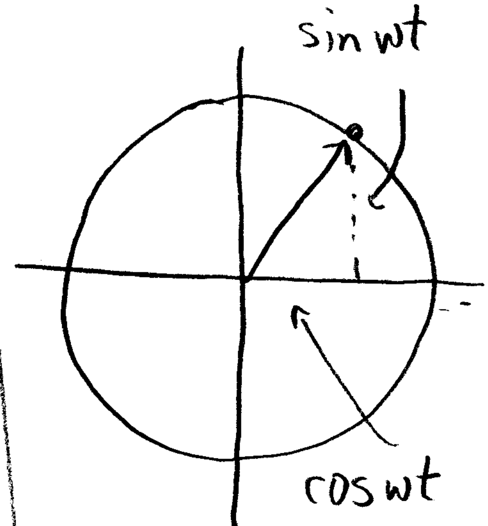
Recall: $e^{iwt} \equiv \cos wt + i \sin wt$

$$\frac{de^{iwt}}{dt} = iw\, e^{iwt} \overset{\text{or}}{\equiv} -w \sin wt + iw \cos wt$$

$$= iw(\cos wt + i \sin wt)$$

$$\frac{d^2 e^{iwt}}{dt^2} = \frac{d(iw\, e^{iwt})}{dt} = i^2 w^2 e^{iwt}$$

$$= -w^2 e^{iwt}$$



$$\frac{d \cos x}{dx} = -\sin x$$

$$\frac{d \sin x}{dx} = \cos x$$

Guess: $u = e^{i\omega t} x$ some vector x.

$$\frac{d^2 u}{dt} = -\omega^2 e^{i\omega t} x$$

Check:

$$M(-\omega^2 e^{\pm i\omega t} x) \stackrel{?}{=} -L(e^{\pm i\omega t} x)$$

iff $\omega^2 M x = L x$ where $\omega = \pm\sqrt{\lambda}$

iff $L x = \lambda M x$ ie (λ, x) is a generalized eigen-pair.

Claim: Eigenvalues of $Lx = \lambda M x$ are real nonneg.

Change of variables $y = M^{1/2} x$ or $x = M^{-1/2} y$

$$L M^{-1/2} y = \lambda M M^{-1/2} y$$

$$\underbrace{M^{-1/2} L M^{-1/2}}_{L'} y = \lambda \cancel{M^{-1/2} M M^{-1/2}} y$$

L' is positive semidefinite

$\therefore$ We have found a space of solutions of dim $= 2n$.

$(\lambda_1, X_1), \cdots (\lambda_n, X_n)$ eigen pairs

$w_1 = \sqrt{\lambda_1} \cdots \quad w_n = \sqrt{\lambda_n}$

$$u(t) = \alpha_1 e^{iw_1 t} X_1 + \cdots \quad + \alpha_n e^{iw_n t} X_n$$

$$\beta_1 e^{-iw_1 t} X + \quad + \beta_n e^{-iw_n t} X_n$$

Let find real solution!

<u>Claim</u> $u(t) = (a_1 \cos w_1 t + b_1 \sin w_1 t) X_1 +$

$$\cdots + (a_n \cos w_n t + b_n \sin w_n t) X_n$$

also work.

$$u(0) = a_1 X_1 + \cdots + a_n X_n$$

$$u'(0) = b_1 w_1 X + \cdots + b_n w_n X_n$$

Normalized Laplacians & Spring Mass Systems

Consider Mass of node = degree!

Eigen pairs $LX = \lambda DX$

$Y = D^{1/2}X \quad D^{-1/2} L D^{-1/2} Y = \lambda Y$

Sym version but

$$D^{-1/2} L D^{-1/2} = D^{-1/2}(D-A)D^{-1/2} = I - D^{-1/2} A D^{-1/2}$$

(the normalized Laplician)