15-859N
11/15/13

Counting & Finding Spanning Trees

$G \equiv$ directed graph

$H \subseteq G$ is <u>divergent ST</u> with root r if

1) r can reach all of $V(G)$ in H.
2) $\text{indegree}(v) = 1$ for $v \neq r$.
3) $\text{indegree}(r) = 0$

<u>Def</u> D is in-degree matrix of G if

$$D(i,j) = \begin{cases} d_{in}(i) & \text{if } i = j \\ -K = -\#\text{edge from } i \text{ to } j & i \neq j \end{cases}$$

<u>Lemma</u> $G=(V,E)$ is a divergent Tree iff

1) $D(i,i) = \begin{cases} 0 & \text{if } i=r \\ 1 & \text{o.w} \end{cases}$

2) det of minor of $D_{r,r}$ (removing rth col & row) $=1$

($\Rightarrow$) 1) clear

2) After permuting row & col

D is upper $\triangle$ & $r=1$

$\det(D_{1,1})=1$

($\Leftarrow$) suppose false.

$\therefore$ G must contain a disconnected component C.
Each node of C has indegree 1 and zero col sums.
Thus $\det(D(C))=0$.
$\Rightarrow \det(D(G))=0$ contra!

Thm # divergent ST rooted at $r \equiv \det(D_{r,r})$

pf WLOG assume $r = v_1$

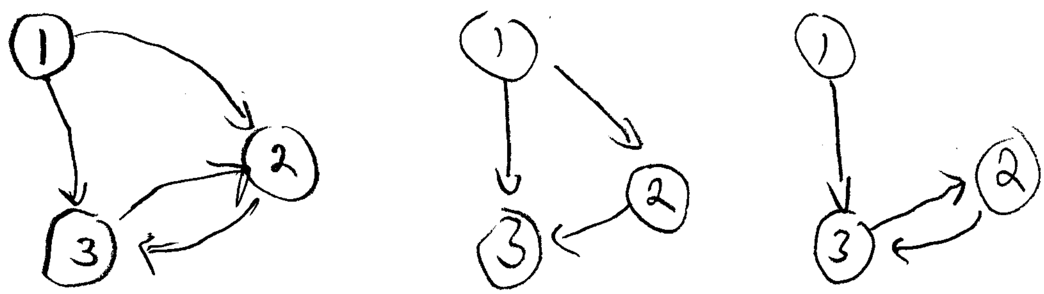
Note: Col sums of D are zero.

Start by replacing first col with $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$\det(D) \doteq \det(D_{1,1})$.

Expand col 2 into subgraphs with fix edge into V_2.

eg



G_1 G_2

$$D = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$|D| = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

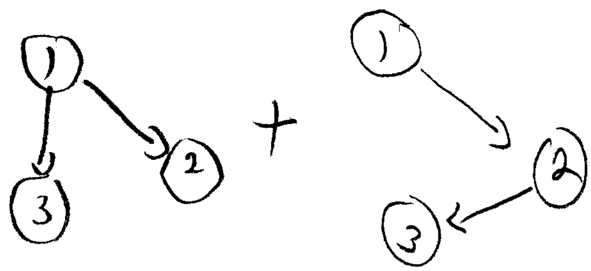
G_1 G_2

For matrix expand next col.

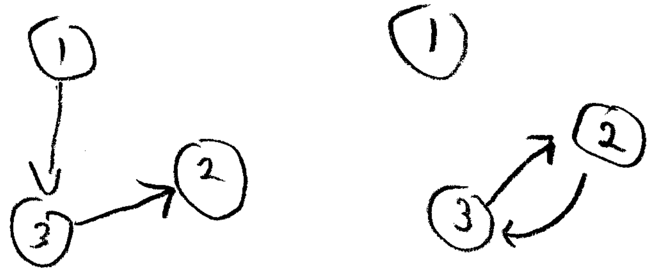
Repeat!

We get exact one matrix per indeg 1 subgraphs.

eg $G_1 \overset{\text{expand}}{\Longrightarrow}$



$G_2 \overset{\text{expand}}{\Longrightarrow}$



Each term is $\begin{cases} 1 & \text{if } \exists \text{ divergent tree} \\ 0 & \text{o.w.} \end{cases}$

$\square$

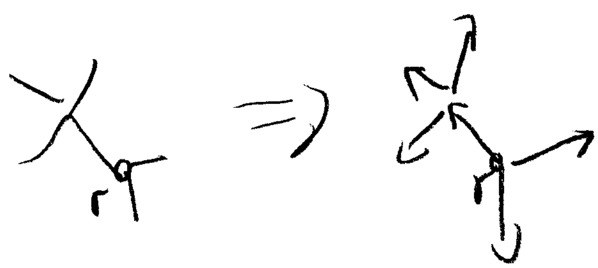
The undirected Case

$G = (V, E)$ undirected

Let $G' = (V, E')$ where each edge $\{i, j\}$ is viewed as (i,j) & (j,i)

Claim $\# ST(G) = \# ST_r(G')$

$f: ST(G) \to ST_r(G')$



1-1 & onto

Note $D(G') \equiv$ Laplacian of G