15-859N
10/16/13

Graph Cuts & Eigenvalues

2-types of cuts
edge-cuts & vertex-cuts

Today edge-cuts

$G=(V,E,w)$ undirected weighted graph

Def $$\text{Cap}(A,B) = \sum_{\substack{(a,b)\in E \\ a\in A,\, b\in B}} w(a,b)$$

Note: $\text{Cap}(A,A) = \text{Cap}(A)$

$\text{Cap}(A) = \sum_{i\in A} d_i$

Cut measures

1) Quotient cuts or Isoperimetric number

$$\delta(G) = \min_{S\subseteq V} \frac{\text{Cap}(S,\bar{S})}{\min\{|S|,|\bar{S}|\}}$$

2) Sparsest Cut

$$\alpha(G) = \min_{S \subseteq V} \frac{Cap(S,\bar{S})}{|S|\,|\bar{S}|}$$

Note $\frac{n}{2}\alpha(G) \leq \varrho(G) \leq n\,\alpha(G)$

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3) Conductance

$$\Phi(G) = \min_{S \subseteq V} \frac{Cap(S,\bar{S})}{\min\{Cap(S), Cap(\bar{S})\}}$$

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4) Sparest Cut

$$\bar{\alpha}(G) = \min_{S \subseteq V} \frac{Cap(S,\bar{S})}{Cap(S) \cdot Cap(\bar{S})}$$

another view

$$Demand(v_i, v_j) \equiv d_i \cdot d_j$$

$$Demand(S,\bar{S}) = \sum_{\substack{s \in S \\ s' \in \bar{S}}} d_s \cdot d_{s'}$$

$$= \left(\sum_{s \in S} d_s\right)\left(\sum_{s' \in \bar{S}} d_{s'}\right) = Cap(S) \cdot Cap(\bar{S})$$

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more generally

$$\bar{\alpha}(G) = \min_{S \subseteq V} \frac{Cap(S, \bar{S})}{Demand(S, \bar{S})}$$

some demand fcn

eg product demand

ie $Demand(v_i, v_j) = P_i \cdot P_j$

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note $\Phi(G)$ is $\approx$ $P_i = \dfrac{d_i}{\sum d_j}$

Main Thms

$L \equiv$ Laplacian of G $\quad \lambda_2 = \lambda_2(L)$

Thm1 $\quad \lambda_2/2 \le q(G) \le \sqrt{2\Delta\lambda_2}$

$\Delta \equiv \max_i \{d_i\}$

$\mathcal{L} \equiv$ Normalized Laplacian of G $\quad \lambda_2 = \lambda_2(\mathcal{L})$

ie $\mathcal{L} = D^{-1/2} L D^{-1/2}$

Thm2 $\quad \lambda_2/2 \le \Phi(G) \le \sqrt{2\lambda_2}$

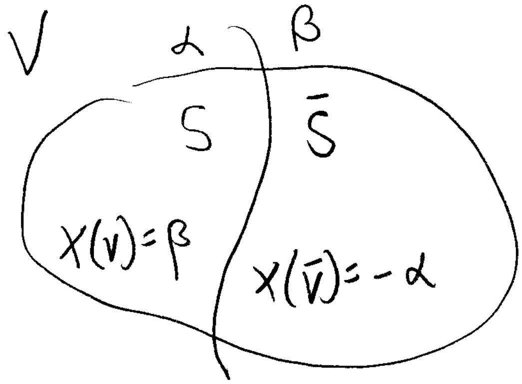
<u>Claim</u> $\lambda_2(L) \le 2 \cdot g(G)$

ie $\exists$ vector x st $x^T \underline{1} = 0$ & $\dfrac{x^T L x}{x^T x} \le 2 \cdot g$

<u>pf</u> Let $S \subseteq V$ be st. $|S| \le n/2$

$$\frac{Cap(S,\bar{S})}{|S|} = g$$

set $\alpha = |S|$ & $\beta = |\bar{S}|$



<u>Def</u> $X_i = \begin{cases} \beta & \text{if } i \in S \\ -\alpha & \text{o.w} \end{cases}$

$$x^T \cdot \underline{1} = \alpha \cdot \beta - \alpha\beta = 0$$

$$\frac{x^T L x}{x^T x} = \frac{Cap(S,\bar{S})(\alpha+\beta)^2}{\alpha\beta^2 + \beta\alpha^2} = \frac{Cap(S,\bar{S})\, n^2}{\alpha \cdot \beta \cdot n} = \left(\frac{Cap(S,\bar{S})}{\alpha}\right)\left(\frac{n}{\beta}\right)$$

$$\le g \cdot 2$$

Pf of upper bd for Thm 2.

Consider spring-mass version $LX = \lambda D X$

Given $X^T \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = 0$

We show $\Phi(G) \leq \sqrt{2 \frac{X^T L X}{X^T D X}}$ (*)

The proof is effective!

ie given X we find a "good" cut.

A "threshold cut"

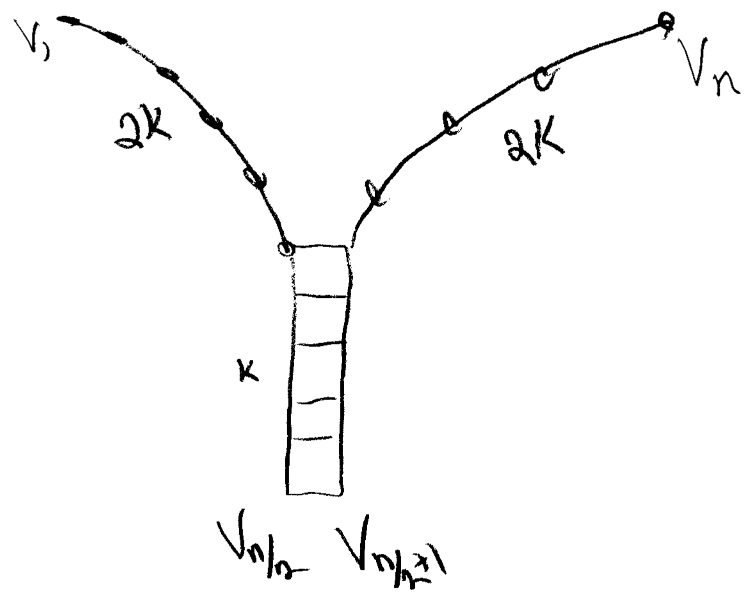
Suppose $f: V \longrightarrow R$

sort V so that $f(v_i) \geq f(v_{i+1})$

Def ith threshold cut $\equiv S_i = \{v_1, \dots, v_i\}$

<u>Note</u> $S_{n/2}$ may not be "good".

Consider Bug-Graph



<u>Claim</u> If $Lf = \lambda_2 f$ then f is odd with respect to its reflection.

The threshold cut for $S_{n/2}$ gives $\frac{K}{3K} \approx 1/3$

$$g \approx \Phi \approx \frac{1}{2K} = \frac{3}{n}$$

Preliminaries $A \equiv$ adj matrix

$$L = D - A \qquad z^T L z = \sum_E w_{ij} (z_i - z_j)^2$$

The sum-Laplacian $P = D + A$

Claim $$zPz = \sum_E w_{ij} (z_i + z_j)^2$$

Fact $P = 2D - L$

Cauchy-Schwartz

$$(z^T L z)(z^T P z) = \left(\sum w_{ij} (z_i - z_j)^2\right)\left(\sum w_{ij} (z_i + z_j)^2\right)$$

$$\left(\sum \left(\sqrt{w_{ij}}\, |z_i - z_j|\right)^2\right)\left(\sum \left(\sqrt{w_{ij}}\, |z_i + z_j|\right)^2\right)$$

$$\geq \left(\sum w_{ij} \,|z_i^2 - z_j^2|\right)^2$$

Here $a = (\cdots, \sqrt{w_{ij}}\, |z_i - z_j|, \cdots)$
$b = (\cdots, \sqrt{w_{ij}}\, |z_i + z_j|, \cdots)$

<u>The proof</u> z st $z^T D \bar{1} = 0$ $z_1 \geq z_2 \geq \cdots \geq z_n$

$$\Phi_i = \frac{Cap(S_i, \bar{S}_i)}{\min\{Cap(S_i), Cap(\bar{S}_i)\}}$$

To show $\exists i$ $\Phi_i \leq \sqrt{2\frac{z^T L z}{z^T D z}}$

We modify z & G only decreasing $\frac{z^T L z}{z^T D z}$ s.t.

<u>Def</u> $\beta = \arg\max_i \{ Cap(\bar{S}_i) \geq Cap(S_i) \}$

1) $z_\beta = 0$ (shifting z)

2) $\forall (i,j) \in E$ $z_i \cdot z_j < 0$ (pinch (i,j) at β)

1) Shifting z: $y = z + \alpha \vec{1}$

$$\frac{yLy}{y^T D y} = \frac{(z+\alpha 1)^T L (z+\alpha \bar{1})}{(z+\alpha 1) D (z+\alpha \bar{1})} = \frac{z^T L z}{z^T D z + 2\alpha z^T D \bar{1} + \alpha^2 \sum d_i} \leq \frac{z^T L z}{z^T D z}$$

note $z^T D \bar{1} = 0$

pinching an edge at zero

$z_i \cdot z_j < 0$

<u>Claim</u> $\dfrac{y^T L' y}{y D' y} \le \dfrac{z^T L z}{z^T D z}$

<u>Note</u> $y^T D' y = z^T D z$

<u>To Show</u> $y^T L' y \le z^T L z$

Term $(z_i - z_j)^2$ replaced with $(z_i - 0)^2 + (0 - z_j)^2$

but $(z_i - z_j)^2 = z_i^2 - 2 z_i z_j + z_i^2 > z_i^2 + z_j^2$

For sym-Laplacians

$$\frac{y^T P' y}{y^T D' y} = \frac{y^T(2D'-L')y}{y^T D' y} \le \frac{y^T(2D')y}{y^T D' y} = 2$$

$$2\,\frac{z^T L z}{z^T D z}\cdot (y^T D y)^2 \ge (y^T P' y)(y^T L' y) \ge \left(\sum_E w_{ij}\,|y_i^2 - y_j^2|\right)^2$$

$$\sqrt{2\,\frac{z^T L z}{z^T D z}}\,(y^T D y) \ge \overset{\text{(Left)}}{\sum_{\substack{E\\ i<j\le\beta}} w_{ij}\,(y_i^2 - y_j^2)} + \overset{\text{(Right)}}{\sum_{\substack{E\\ j>i\ge\beta}} w_{ij}\,(y_j^2 - y_i^2)} \qquad (*)$$

Lets assume $y_i > y_{i+1}$

WLOG Assume G is weighted path graph

$$V_i \overset{w_{ij}}{\frown} V_j \Rightarrow V_i \overset{w_{ij}}{\frown} V_k \overset{w_{ij}}{\frown} V_j$$

new edge weights w_i from V_i to V_{i+1}

$$\Phi_i = \frac{w_i}{Cap(V_i)} \qquad \Phi^*_{Left} = \min_{i \le \beta} \Phi_i \qquad Cap(V_0) = 0$$

$$\sum_{i=1}^{\beta-1} w_i (y_i^2 - y_{i+1}^2) = \sum_{i=1}^{\beta-1} \Phi_i \, cap(V_i)(y_i^2 - y_{i+1}^2)$$

$$\geq \Phi^*_{Left} \sum_{i=1}^{\beta-1} cap(V_i)(y_i^2 - y_{i+1}^2)$$

$$= \Phi^*_{Left} \sum_{i=1}^{\beta-1} (cap(V_i) - cap(V_{i-1})) y_i^2$$

$$= \Phi^*_{Left} \sum_{i=1}^{\beta-1} d_i y_i^2 = \Phi_{Left} \, y_{left} D y_{left}$$

$$\sqrt{2\,\frac{z^T L z}{z^T D z}}\,(y^T D y) \geq \Phi^*_{left}\, y_{left}^T D y_{left} + \Phi^*_{right}\, y_{right}^T D y_{right}$$

$$\geq \Phi^* (y_l^T D y_l + y_r^T D y_r)$$

$$= \Phi^* y^T D y$$

Thus

$$\sqrt{2\,\frac{z^T L z}{z^T D z}} \geq \Phi^*$$