Bounding Eigenvalues Cont

9/18/13
Spectral

Thm (Courant-Fischer) $A^{n\times n}$ sym with eigvs $\mu_1 \geq \cdots \geq \mu_n$ Then

$$\mu_k = \max_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S)=k}} \min_{x \in S} \frac{x^T A x}{x^T x} = \min_{\substack{T \subseteq \mathbb{R}^n \\ \dim(T)=n-k+1}} \max \frac{x^T A x}{x^T x}$$

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eg μ_1 using S $\quad Ax = \mu_1 x$

Suppose max is me & min is you

I would pick $S = \langle x \rangle$ you would be forsed to pick αx.

pf For case S. $(\geq)$

pick $S = \langle x_1, \dots x_k \rangle$ Let $x = \sum c_i x_i$

$$\frac{x^T A x}{x^T x} = \frac{\sum \mu_i c_i^2}{\sum c_i^2} \leq \frac{\sum \mu_k c_i^2}{\sum c_i^2} = \mu_k$$

(≤) To show $\forall S$ $\dim(S)=k$

$$\min_{x\in S} \frac{x^TAx}{x^Tx} \le \mu_k$$

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pf Let $T = \langle x_k, \cdots, x_n \rangle$ $\dim(T) = n-k+1$

$\therefore$ $T \cap S \neq 0$

$$\min_{x\in S} \frac{x^TAx}{x^Tx} \le \min_{x\in S\cap T} \frac{x^TAx}{x^Tx}$$

Thus $x = \sum_{i=k}^{n} c_i x_i$

$$\frac{x^TAx}{x^Tx} = \frac{\sum_k^n \mu_i c_i^2}{\sum c_i^2} \le \frac{\sum_k^n \mu_k c_i^2}{\sum c_i^2} = \mu_k$$

$\square$

~~Thus eigens for K_n are $0, \underbrace{n, \dots, n}_{n-1}$. i.e. $\forall x \perp \mathbf{1}$ $L_{K_n} x = nx$.~~

New notation: $A \succcurlyeq 0$ if $\forall x \; x^T A x \geq 0$ A sym.

In general $A \preccurlyeq B$ if $\forall x \; x^T A x \leq x^T B x$

ie $B - A \succcurlyeq 0$

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note $A \preccurlyeq B$ & $B \preccurlyeq C$ then $A \preccurlyeq C$

$A \preccurlyeq B \Rightarrow A + C \preccurlyeq B + C$

Sym A, B, C.

Graphs G & H $\quad G \preccurlyeq H$ if $L_G \preccurlyeq L_H$

Claim $\quad G \preccurlyeq G \cup H$.

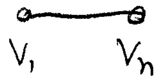
Thm $A \preceq B \Rightarrow \lambda_k(A) \leq \lambda_k(B)$

pf (CF)

$$\lambda_k(B) = \min_{\dim(T)=n-k+1} \max_{x \in T} \frac{x^T A x}{x^T x}$$

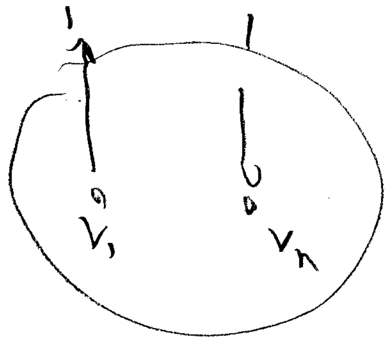
$$\geq \min_{\dim(T)=n-k+1} \max_{x \in T} \frac{x^T B x}{x^T x}$$

$$= \lambda_k(B)$$

G graph & $G_{1n} \equiv$ 

<u>Claim</u> $\alpha = \min\limits_{\alpha} \left(G_{1n} \preceq \alpha G \right)$ iff $ER_{1n} = \alpha$

<u>Recall</u> ER_{1n} test



return $V_n - V_1$

ie solve $LV = \begin{pmatrix} +1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}$ return $V_n - V_1$

$$\delta_{1n} = \begin{pmatrix} +1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \qquad ER_{1n} = \delta_{1n}^T L^+ \delta_{1n}$$

potential flow $C \partial^T V = f$

$$ER_{1n} = f^T R f = (C\partial^T V)^T R C \partial^T V = V^T \partial C R C \partial^T V$$
$$= V^T L V$$

What is $V^T L_{G_{1n}} V$? $= (V_n - V_1)^2 = ER_{1n}^2$

Claim $G_{in} \preceq \alpha G \Rightarrow ER_{in} \leq \alpha$

f unit potential flow with potential v

$$ER_{in} = f^T R f = v^T L v \geq \frac{v^T G_{in} v}{\alpha} = \frac{(ER_{in})^2}{\alpha}$$

$$\Rightarrow \alpha \geq ER_{in}$$

(HW) $ER_{in} \leq \alpha \Rightarrow G_{in} \preceq \alpha G$

Upper bounding λ_2

(CF) $\lambda_2 = \min_{v^T 1 = 0} \frac{v^T L v}{v^T v}$

$\forall x$ st $x \perp 1$ $\quad \lambda_2 \le \frac{x^T L x}{x^T x}$

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Lower bding λ_2!

$K_n \equiv$ Complete graph

$$L_{K_n} = \begin{pmatrix} n-1 & & -1 \\ & \ddots & \\ -1 & & n-1 \end{pmatrix}$$

$$\begin{pmatrix} n-1 & -1 & \cdots \\ -1 & n-1 & \\ \vdots & -1 & \\ -1 & \vdots & \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} n \\ -n \\ 0 \\ \vdots \end{pmatrix} = n \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ \vdots \end{pmatrix} \cdots \quad X_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ -1 \end{pmatrix}$$

$X_1 \cdots X_{n-1}$ are $n-1$ indep eigenvector value n.

$\lambda(K_n) = \{0, n, \cdots, n\}$

Let P_n be unit wieght path graph.

$$X = \left(-n/2, -n/2+1, \cdots, +n/2\right)$$

$$X^T L(P_n) X = \sum (X_i - X_{i+1})^2 = n-1$$

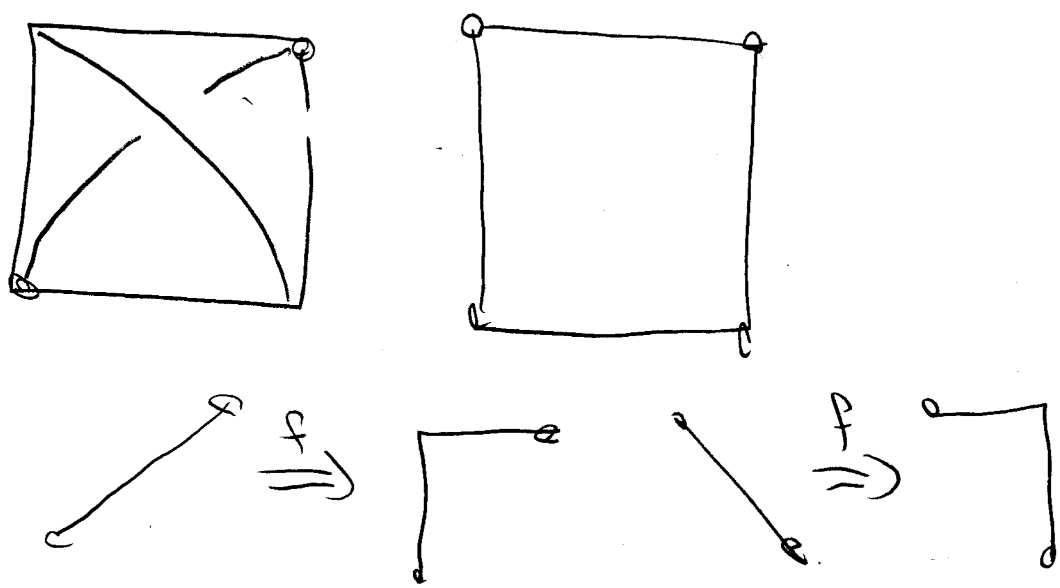
$$X^T X = 2 \sum_{i=1}^{n/2} i^2 \approx 2 \cdot \frac{(n/2)^3}{3} = \frac{n^3}{12}$$

$$\lambda_2 \leq \frac{n-1}{n^3/12} \leq \frac{12}{n^2}$$

Def G (guest), H (host)

Path Embedding: $f: E_G \rightarrow$ Paths in H

eg $G = K_4$ $H = C_4$



$$\text{Conjestion} \equiv \max_{e \in H} \left| \{ P \mid e \in \text{Path} \} \right|$$

$$\text{Dilation} \equiv \max_{P} \{ |P| \}$$

In example: Conj = 3 & Dil = 2

Mediant of Fractions

Fractions $\left(\frac{a_1}{b_1}, \cdots, \frac{a_n}{b_n}\right)$

$$\text{mediant} \equiv \frac{\sum a_i}{\sum b_i}$$

Claim: $\dfrac{\sum a_i}{\sum b_i} \le \max\limits_i \dfrac{a_i}{b_i} \qquad a_i > 0 \quad b_i > 0$

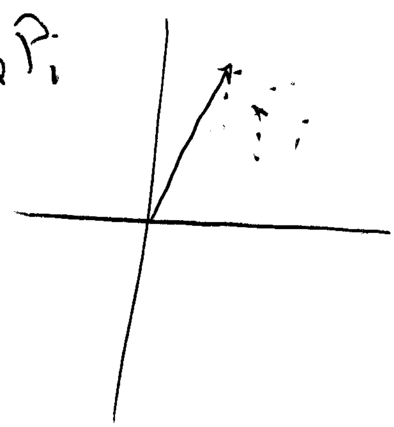
picture proof

Consider points in $\mathbb{R}^2$ $\overset{P_1}{(b_1, a_1)}, \cdots\cdots, \overset{P_n}{(b_n, a_n)}$

Let $\bar{P}$ average or center of mass.

$\text{Slope}(P_i) = \dfrac{a_i}{b_i} \qquad \text{Slope}(\bar{P}) = \dfrac{\sum a_i}{\sum b_i}$

$\text{Slope}(\bar{P}) \le \max \text{Slope } P_i$



Laplacian's as sums of Laplician's

$$E_{ij} = \begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 1 & & -1 & \\ 0 & & 0 & & 0 \\ & -1 & & 1 & \\ 0 & & 0 & & 0 \end{pmatrix} \end{array}$$

$i<j$

Def A is diagonal dominate if $A_{ii} \geq \sum_{j \neq i} |A_{ij}|$

Claim $L = L(G)$ then $L = \sum_{(i,j) \in E} w_{ij} E_{ij}$

Claim A is SDD then $A = \sum_{\substack{i \neq j \\ A_{ij} < 0}} -A_{ij} E_{ij} + \sum_{\substack{A_{ij} > 0 \\ i \neq j}} A_{ij} P_{ij} + D'$

$$P_{ij} = \begin{array}{cc} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$ & D' is non-neg diagonal.

<u>Thm</u> $f: G \to H$ with conjestion c & dilation d

then $c \cdot d\, L_H \succeq L_G$

<u>pf</u> $L(G) = G \quad L(H) = H$

$E_1 \cdots E_m$ edge subgraphs of G

$P_1 \cdots P_m$ the path of H $\quad \sum P_i \preceq c \cdot H$

$$\frac{x^T G x}{x^T H x} \le \frac{x^T \sum E_i x}{(1/c) x^T \sum P_i x} = \frac{c \sum x^T E_i x}{\sum x^T P_i x} \le c \cdot \operatorname{Max} \frac{x^T E_i x}{x^T P_i x}$$

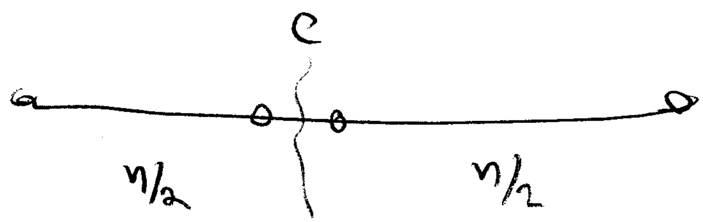
$$\le c \cdot d$$

ie $\quad c \cdot d\, x^T H x \ge x^T G x$

Back to λ_2 for P_n

$f: K_n \xrightarrow{\text{path}} P_n$

Congestion



$$c = (n/2)^2$$

dilation $\equiv n-1$

By Thm: $\lambda_2(K_n) \le c \cdot d \cdot \lambda_2(P_n)$

$$\frac{n}{c \cdot d} \le \lambda_2(P_n)$$

$$\frac{n}{(n/2)^2 (n-1)} = \frac{4}{n(n-1)} \le \lambda_2(P_n)$$