Spectral
9/16/13

Matrices considered

A adj $\quad A = A^T \quad D \equiv \deg$

Laplacian $D - A$

Transition $A^T D^{-1} = A D^{-1}$

Symmetric $D^{-1/2} A D^{-1/2}$

Normalized Lap $\quad I - D^{-1/2} A D^{-1/2}$

Goal Bound and compare eigens.

Bounding Eigenvalues

Let A adj matrix & D degree matrix.

Note If $\mu_1 \geq \cdots \geq \mu_n$ are eigenvalues of $A \equiv \lambda(A)$

iff $d-\mu_1 \leq \cdots \leq d-\mu_n$ are eigens of $dI-A$

Let $\mu_1 \geq \cdots \geq \mu_n$ be $\lambda(A)$ $\quad A \equiv adj(G)$

d_{max}, d_{ave} be max, ave degree.

Lemma $\quad d_{ave} \overset{(a)}{\leq} \mu_1 \overset{(b)}{\leq} d_{max}$

pf (a) $\mu_1 = \max_x \frac{x^T A x}{x^T x} \geq \frac{1^T A 1}{1^T 1} = \frac{\sum_{(i,j)} A(i,j)}{n} = \frac{2m}{n} = d_{ave}$

(b) Let $Ax = \mu_1 x$ & pick v s.t. $x_v \geq x_u \ \forall u$

$$\mu_1 = \frac{(Ax)_v}{x_v} = \frac{\sum_u x_u A_{vu}}{x_v} \leq \sum_u A_{vu} \leq d_{max}$$

<u>Lemma</u> If G is connected then

G is d-reg iff $\mu_1 = d = d_{max}$

$\Rightarrow$ $A\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = d\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

$(\Leftarrow)$ $d \cdot x_v = \sum_{(v,u)\in E} x_u \Rightarrow x_u = x_v$

<u>Thm</u> (Sym Perron-Frobenius) Let G sym, connected weighted ≥ 0. $A = \mathrm{adj}(G)$ & $\mu_1 \geq \cdots \geq \mu_n = \lambda(A)$ then

a) $\mu_1 \geq -\mu_n$

b) $\mu_1 > \mu_2$

c) $Ax = \mu_1 x$ then $x > 0$

<u>Lemma</u> Suppose as above & $Ax = \lambda x$ if $\overset{x \neq 0}{x \geq 0}$ then $x > 0$.

pf $\lambda > 0$

Suppose $\exists v$ st $x_v = 0$ pick v s.t. $(v,u) \in E$ & $x_u \neq 0$, $x_v = 0$

$$0 = \lambda x_v = (Ax)_v = \sum_{(v,u)\in E} A_{vu} x_u > 0 \quad \text{contra!}$$

pf (SPF) $Af_i = \mu_i f_i$

$$\mu_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

Def $|x|$ s.t. $|x|_i = |x_i|$

$$|x|^T |x| = \sum |x|_i \, |x|_i = \sum x_i^2 = x^T x$$

$$f_1^T A f_1 = \sum A_{uv} f(u) \cdot f(v) \le \sum A_{uv} |f(u)| \, |f(v)|$$

$$= |f|^T A |f|$$

c) WLOG $f_1 \geq 0 \overset{\text{Lemma}}{\Longrightarrow} f_i > 0$

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d) let $Af_n = \mu_n f_n$

$$|\mu_n| = |f_n^T A f_n| \le |f_n|^T A |f_n| \le \mu_1$$

Case b) Suppose $Af_2 = \mu_2 f_2$

By Spectral Thm pick f_2 s.t. $f_2^T f_1 = 0$

ie f_2 takes on positive & neg values.

$$\mu_2 = \underbrace{\frac{f_2^T A f_2}{f_2^T f_2}}_{A} \overset{*}{\leq} \underbrace{\frac{|f_2|^T A |f_2|}{|f_2|^T |f_2|}}_{B} \leq \mu_1$$

Subcase $\forall v \; f_2(v) \neq 0$ ie $|f_2| > 0$

$\exists (u,v) \in E$ s.t. $f_2(u) \cdot f_2(v) < 0$

Consider Term in A: $A_{uv} f_2(u) f_2(v) < 0$

$0 < A_{uv} |f_2(u)| |f_2(v)|$

thus $*$ is strict.

Subcase $\exists u$ s.t. $f_2(u) = 0$

Suppose $\mu_2 = \mu_1 \Rightarrow Af_2 = \mu_1 f_2$ (eigenvector)

$\therefore |f_2|$ eigenvector but $|f_2|(u) = 0$ contra!