Divide without Division!

$a \in \mathbb{R}$ Goal: approx $1/a$

$\bar{a} = 1 - a \Leftrightarrow a = 1 - \bar{a}$

$$\frac{1}{a} = \frac{1}{1-\bar{a}} = 1 + \bar{a} + \bar{a}^2 + \bar{a}^3 + \cdots \qquad |\bar{a}| < 1$$

ie $0 < a < 2$

$$u^{(k)} = 1 + \bar{a} + \cdots + \bar{a}^{k-1}$$

Recurrance $u^{(k+1)} = 1 + \bar{a}\, u^{(k)}$

Suppose $\approx b/a \approx v^{(k)}$

$$v^{(k)} = b\, u^{(k)} \iff u^{(k)} = v^{(k)}/b$$

$$v^{(k+1)} = b\left(1 + \frac{\bar{a}}{b} v^{(k)}\right) = b + \bar{a}\, v^{(k)}$$

$$= b + (1-a)\, v^{(k)} = v^{(k)} + (b - a v^{(k)})$$

$0 < a < 2$

Error Analysis

$$\varepsilon^{(k)} = \frac{b}{a} - x^{(k)} \qquad \varepsilon^{(0)} = b/a$$

$$\varepsilon^{(k)} = b(1+\bar{a}+\bar{a}^2+\cdots) - b(1+\bar{a}\cdots+\bar{a}^{k-1})$$

$$= b\bar{a}^k(1+\bar{a}+\bar{a}^2+\cdots) = b\bar{a}^k/a = \bar{a}^k\left(\frac{b}{a}\right)$$

$$= \bar{a}^k e^{(0)}$$ converges iff $|\bar{a}|<1$

The Basic Iterative Method

Goal: Solve $Au = b$ $(*)$

Let $u^{(0)}$ our initial guess eg $u^{(0)} = 0$

Richardson's Method

view 1 $$u^{(m+1)} = u^{(m)} + \underbrace{(b - Au^{(m)})}_{\text{residual error}}$$

view 2 $$u^{(m+1)} = (I - A)\,u^{(m)} + b$$

$$u^{(m+1)} = G\,u^{(m)} + b \qquad G = (I - A)$$

Good: 1) $O(w(A))$ work per step. (parallel)

2) A fixed point is a solution

Bad: 1) May not converge.

2) If it does it may do so very slowly

The error for Richardson

Suppose $A\bar{u}=b$ & $G=(I-A)$

Def the error $\varepsilon^{(m)}=u^{(m)}-\bar{u}$

Claim $\varepsilon^{(m)}=G^{m}\varepsilon^{(0)}$

to show $\varepsilon^{(m+1)}=G\varepsilon^{(m)}$

Pf

$$
\begin{aligned}
G\varepsilon^{(m)} &= G(u^{(m)}-\bar{u}) = Gu^{(m)}-(I-A)\bar{u}\\
&= Gu^{(m)}-\bar{u}+A\bar{u}\\
&= Gu^{(m)}-\bar{u}+b\\
&= (Gu^{(m)}+b)-\bar{u}\\
&= u^{(m+1)}-\bar{u} = \varepsilon^{(m+1)}
\end{aligned}
$$

To converge $\lim_{m \to \infty} G^m \varepsilon^{(0)} = 0$

If it is to converge $\forall \varepsilon^{(0)}$ then $\lim_{m \to \infty} G^m = 0$

If A is sym then G is sym

$$G = \sum \lambda_i (v_i v_i^T) \quad \lambda_i \in \lambda(G) \quad v_i \text{ eigen vector}$$

Iff $\forall_i \ |\lambda_i| < 1$ Spectral radius

In general This is not true!

A simple modification to get convergence.

The Extrapolated Method

$A = A^T \quad A \equiv SPD$

Goal Get Richardson to converge

Note $Ax = b$ iff $\gamma A x = \gamma b \quad \gamma \neq 0$

New recurrence. $u^{(k+1)} = u^{(k)} = \gamma(b - Ax)$

or $u^{(k+1)} = (I - \gamma A) u^{(k)} + \gamma b$

We need $\sigma(G_\gamma) \equiv \max_i \{ |\lambda_i(G_\gamma)| \} < 1$ where $G_\gamma = 1 - \gamma A$

Let $M = \lambda_{max}(A) \quad m = \lambda_{min}(A)$

Set $\gamma = \frac{2}{M+m}$

Note $Ax = \lambda_A x$ iff $G_\gamma x = (1 - \gamma \lambda_A) x$

Suppose $GX = \lambda X$

$$\lambda = (1-\gamma\lambda_A) \leq 1-\gamma m = 1-\frac{2m}{M+m} = \frac{M-m}{M+m}$$

$$\lambda \geq 1-\gamma M = \frac{1-2M}{M+m} = \frac{m-M}{M+m} = -\left(\frac{M-m}{M+m}\right)$$

$$\sigma(G_\gamma) \leq \frac{M-m}{M+m}$$

A spd.

Def condition number of $A \equiv \frac{M_A}{m_A} = k(A)$

$$\nabla(G_\gamma) = \frac{k(A)-1}{k(A)+1} \qquad k = k(A)$$

$$1 - \frac{2}{k+1} = 1 - \frac{1}{k+1/2} \approx 1 - \frac{1}{k}$$

If we do k iterations. Error $\approx \left(1-\frac{1}{k}\right)^k \approx \frac{1}{e}$

We need k iterations per bit of accuracy.

EG $\quad A = L(P_n) \qquad k(A) \approx n^2$

n^2 iterations per bit.

Polynomial Acceleration

$Ax = b$ spd $A\bar{x} = b$

$$x^{(m+1)} = Gx^{(m)} + b \qquad G = (I - A)$$

$$\varepsilon^{(m)} = G^m \varepsilon^{(0)}$$

Idea: Use all previous $x^{(i)}$.

Compute $x^{(0)}, \dots, x^{(n)}$

pick $\alpha_i \in \mathbb{R}$ s.t. $\sum \alpha_i = 1$

$$u^{(n)} = \sum \alpha_i x^{(i)}$$

Question: How to pick the α_i

$$\tilde{\varepsilon}^{(n)} \stackrel{def}{=} u^{(n)} - \bar{x}$$

$$= \sum_{i=0}^{n-1} \alpha_i x^{(i)} - \bar{x} \qquad \sum \alpha_i = 1$$

$$= \sum \alpha_i (x^{(i)} - \bar{x})$$

$$= \sum \alpha_i \varepsilon^{(i)} = \sum \alpha_i G^i \varepsilon^{(0)}$$

$$= \left(\sum \alpha_i G^i\right) \tilde{\varepsilon}^{(0)} = Q_n(G)\, \tilde{\varepsilon}^{(0)}$$

$$Q_n(z) = \alpha_0 + \alpha_1 z + \cdots \alpha_{n-1} z^{n-1} \qquad \sum \alpha_i = 1$$

Goal: For each n pick $Q_n(z)$ to

$$\min \| Q_n(G) \| \quad \text{u} \quad \min \sigma(Q_n(G))$$

Chebyshev Acceleration

Def The Chebyshev Polys

$$T_0(w) = 1 \qquad T_1(w) = w$$

$$T_{n+1}(w) = 2w\,T_n(w) - T_{n-1}(w)$$

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eg $T_2(w) = 2w \cdot w - 1 = 2w^2 - 1$

$$T_3(w) = 2w(2w^2-1) - w$$

$$= 4w^3 - 2w - w = 4w^3 - 3w$$

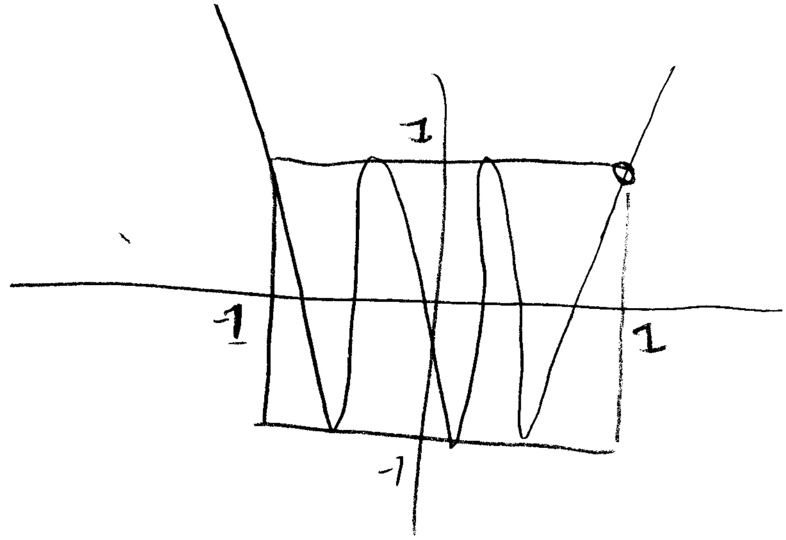
Note $T_n(1) = 1 \Rightarrow \sum \text{coef} = 1$

$$T_n(-1) = (-1)^n$$

$$T_n(w) = \frac{1}{2}\left[\left(w + \sqrt{w^2-1}\right)^n + \left(w + \sqrt{w^2-1}\right)^{-n}\right]$$

HW!

T_n even



Thm $d>1$ Let $H_n(w) = T_n(w)/T_n(d)$ then

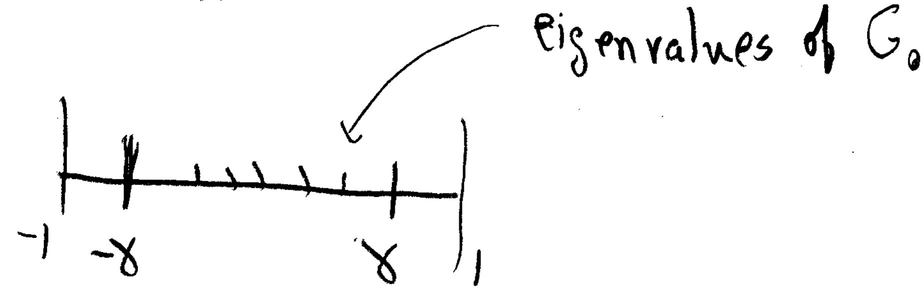
1) $\max_{-1\le w\le 1} |H_n(w)| = 1/T_n(d)$

2) $H_n(w)$ is min such poly

Chebyshev & extrapolated method.

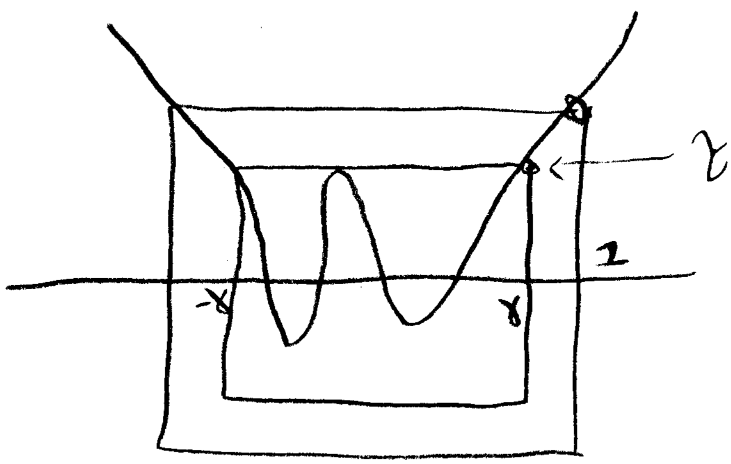
$\kappa(A) = \kappa$ & $\gamma = \frac{\kappa - 1}{\kappa + 1}$

$\lambda(G)$



Pick $\bar{T}_n = T_n(x/\gamma) / T_n(1/\gamma)$

$\bar{T}_n(1) = 1 \quad \Sigma\, \text{coef} = 1$



Using Thm: $1 = x/\gamma \Rightarrow x = \gamma$

$\tau = 1 / T_n(1/\gamma)$

<u>Goal</u> bd $T_n(1/\gamma)$

$$1/\gamma = \frac{K+1}{K-1} = 1 + \frac{2}{K-1} = 1+2\mu \quad \text{if } \mu = \frac{1}{K-1}$$

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x+\sqrt{x^2-1}\right)^{-n}\right]$$

$$\geq \frac{1}{2}\left(x+\sqrt{x^2-1}\right)^n \quad x \geq 1$$

$$T_n(1+2\mu) \geq \frac{1}{2}\left(1+2\mu+\sqrt{(1+2\mu)^2-1}\right)^n$$

$$= \frac{1}{2}\left(1+2\mu+2\sqrt{\mu(\mu+1)}\right)^n$$

$$= \frac{1}{2}\left(\sqrt{\mu}+\sqrt{\mu+1}\right)^{2n} \qquad (*)$$

$$\left(\sqrt{\mu}+\sqrt{\mu+1}\right)^2 = \left(\frac{1}{\sqrt{K-1}} + \frac{\sqrt{K}}{\sqrt{K-1}}\right)^2 = \frac{(1+\sqrt{K})^2}{K-1} = \frac{\sqrt{K}+1}{\sqrt{K}-1}$$

$$(*) = \frac{1}{2}\left(\frac{\sqrt{K}+1}{\sqrt{K}-1}\right)^n$$

$$T_n(1/\gamma) \geq \frac{1}{2}\left(\frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1}\right)^n$$

$$\tau \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^n$$

Thm Convergence Rate for Extrapolated Chebyshev is $\Theta(1/\sqrt{\kappa})$

Back to P_n $\quad \kappa(L(P_n)) = n^2$

$\therefore$ Convergence Rate for Chebyshev is

$\Theta(1/n)$

ie n iterations per bit.

This is optimal given that Chebyshev only does "local" operations!

From Polynomial Recurance to Iterative Alg

Consider a poly defined by

$$Q_0(x) = 1$$

$$Q_1(x) = \alpha_1 x + \beta_1 \qquad \alpha_1 + \beta_1 = 1$$

$$Q_{n+1}(x) = \alpha_n x\, Q_n(x) + \beta_n Q_{n-1}(x) \qquad \alpha_n + \beta_n = 1$$

Iterative Alg

$$u^{(n+1)} = \alpha_n\left(G u^{(n)} + b\right) + \beta_n u^{(n-1)}$$

<u>to show</u> $\varepsilon^{(n)} = u^{(n)} - \bar{u}$ then $\varepsilon^{(n)} = Q_n(G)\,\varepsilon^{(0)}$

$$Q_{n+1}(G)\,\varepsilon^{(0)} = \left[\alpha_n G Q_n(G) + \beta_n Q_{n-1}(G)\right]\varepsilon^{(0)} \qquad \text{induct!}$$

$$= \alpha_n G \underbrace{Q_n(G)\,\varepsilon^{(0)}} + \beta_n Q_{n-1}(G)\,\varepsilon^{(0)}$$

$$= \alpha_n G\left(\varepsilon^{(n)}\right) + \beta_n\left(\varepsilon^{(n-1)}\right)$$

$$= \alpha_n G(u^{(n)} - \bar{u}) + \beta_n (u^{(n-1)} - \bar{u})$$

$$= \alpha_n (G u^{(n)} - \bar{u} + b) + \beta_n u^{(n-1)} - \beta_n \bar{u}$$

$$= \alpha_n (G u^{(n)} + b) + \beta_n u^{(n-1)} - \bar{u}$$

$$= u^{(n+1)} - \bar{u} = \varepsilon^{(n+1)}$$