University of Michigan–Ann Arbor

Department of Electrical Engineering and Computer Science

EECS 498 004 **Advanced Graph Algorithms**, Fall 2021

Lecture 2: High Conductance = Robust Against Deletions

September 2, 2021

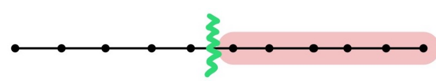
Instructor: Thatchaphol Saranurak

Scribe: Jingyi Gao

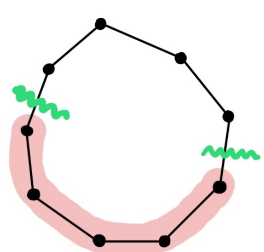
1 Motivate the Definition: Robust Against Deletions

Given a connected graph G, we could describe G is well-connected in the following sense: when we delete d edges in G, the size of the largest component can be reduces by at most $O(d)$ edges. In other words, delete a small set of edges only disconnect small parts of graph.
More formally, let $D \subseteq E$ be a set of d edges. Let $G' = G \setminus D$. Let C be the union of all small components in G'. Then, C contains at most $O(d)$ edges.
Some non-examples are as follows:

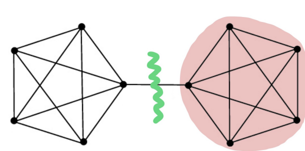
1. paths, where we can delete one edge and disconnect a large chain of vertices,



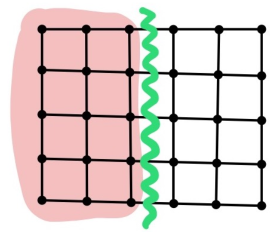
2. cycles, which is similar to a path but we can delete two edges and disconnect a large portion of the cycle,



3. dumbbells, where we can delete one edge and disconnect a large portion of the graph,

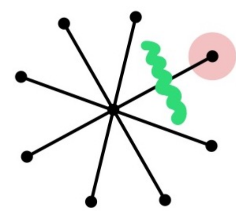


4. grids, where we delete $O(\sqrt{n})$ edges and disconnect $O(n)$ vertices.

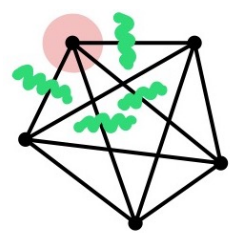


On the other hand, some examples are as follows:

1. stars, where deleting one edge always corresponds to disconnect one vertex from the graph,



2. complete graph, where disconnecting a set A of vertices requires deleting all edges that are connected to vertices outside of A.



An **expander** should satisfies the property above, namely be robust against edge deletions. This motivates the definition of conductance.

2 Formal definition of conductance

We first formalize some notations and the definition of volume

Notation 2.1 (Edges between A and B)**.** Let $G = (V, E)$ be a undirected unweighted graph. $\forall A, B \subseteq V$, $E_G(A, B) = \{(u, v) \in E \mid u \in A, b \in B\}$.

Notation 2.2 (Cut edges)**.** Let $G = (V, E)$ be a undirected unweighted graph. $\forall S \subseteq V$, denote the set of cut edges as $\partial_G S = E_G(S, V \setminus S)$.

Definition 2.3 (Volume)**.** Let $G = (V, E)$ be a undirected unweighted graph. For any $S \subseteq V$, the volume of S is $\mathrm{vol}_G(S) = \sum_{u \in S} \deg(u)$.

Observation 2.4. *Given the definition of volume, some observations are listed below:*

1. $|E(S,V)| \leq \text{vol}(S) \leq 2|E(S,V)|$, *namely the volume counts the number of edges incident to S up to a factor of 2. This is because there are two types of edges that will be counted towards* $\text{vol}(S)$*: (a) edges with two endpoints within S and (b) edges with only one endpoint within S. Edges of type (a) will be counted twice and edges of type (b) will be counted once. Hence,* $\text{vol}(S)$ *is bounded between the number of edges attached to S and twice of that number.*

2. $\text{vol}(S) + \text{vol}(V \setminus S) = \text{vol}(V) = 2|E|$.

Definition 2.5 (Conductance)**.** Let $G = (V, E)$ be a undirected unweighted graph. **Conductance of set/cut** is defined as

$$\Phi_G(S) = \frac{|\partial S|}{\min\{\text{vol}(S), \text{vol}(V \setminus S)\}}$$

Conductance of graph G is defined as

$$\Phi(G) = \min_{S:0<vol(S)<vol(V)} \Phi_G(S).$$

Intuitively, conductance describes overall the graph is sparse or not. The smaller the conductance, the sparser the graph.

Remark 2.6. Computing conductance is NP-hard.

Observation 2.7. 1. $\forall S \subset V, \Phi_G(S) \in [0,1]$. *So* $\Phi_G \in [0,1]$.

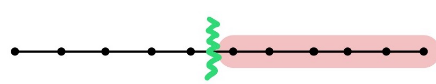
2. $\Phi(G) = 0 \iff$ *G is not connected (with the assumption that each vertex in the graph has a self loop.)*

We say

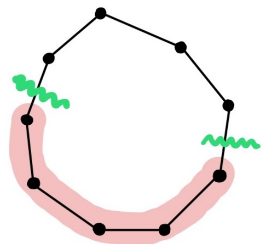
- S is a $\phi-$**sparse cut** $\iff \Phi_G(S) < \phi$ (i.e. its conductance is smaller than ϕ)
- G is a $\phi-$**expander** $\iff \Phi(G) \geq \phi$ (i.e. G has conductance at least ϕ)
- namely, G is $\phi-$expander $\iff$ G has no $\phi-$sparse cut
- larger $\phi \iff$ more well-connected.

Exercise 2.8. To internalize the definition, we look at the following examples:

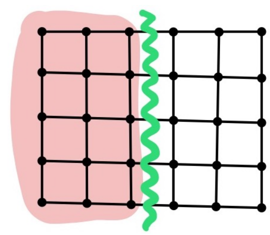
- paths, as mentioned at the beginning, if we pick the right subhalf (the red portion) as S, then $\text{vol}(S) = \text{vol}(V \setminus S) = O(n)$, and $|\partial S| = 1$. Hence $\Phi(G) = O\left(\frac{1}{n}\right)$.



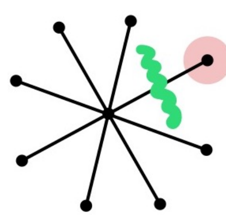
- cycles, which is similar to a path. If we pick the red portion as S, then $\mathrm{vol}(S) = \mathrm{vol}(V \setminus S) = O(n)$, and $|\partial S| = 2$. Therefore $\Phi(G) = O\left(\frac{1}{n}\right)$.



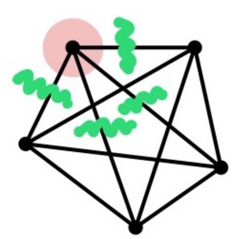
- grids, where we delete $O(\sqrt{n})$ edges and disconnect $O(n)$ vertices. Therefore $\Phi(G) = O\left(\frac{1}{\sqrt{n}}\right)$.



- stars, where disconnect one vertex from the graph always corresponds to delete the edge that attached to that vertex, whence $\Phi(G) = 1$.



- complete graph, where disconnecting a set A of vertices requires deleting all edges that are connected to vertices outside of A, whence $\Phi(G) = \Omega(1)$.



2.1 High Conductance = Robust Against Edge Deletions

We will see how the definition conductance captures the notion of "robustness against edge deletions."

Theorem 2.9. *A connected graph $G = (V, E)$ has conductance at least $\Omega(\phi)$ $\iff$ $\forall D \subseteq E$, the total volume of small components $G \setminus D$ is at most $O\left(\frac{|D|}{\phi}\right)$, where we say a connected component C in $G \setminus D$ is small if $\mathrm{vol}_G(C) \le \frac{\mathrm{vol}_G(V)}{2}$.*

Proof of Theorem. " $\Longleftarrow$ " we prove the contrapositive of the statement, namely: If $\Phi(G) < \phi$, then $\exists D \subseteq E$, s.t. the total volume of small components of $G \setminus D$ is greater than $\frac{|D|}{\phi}$.
According to the definition of conductance, $\exists S \subseteq V$, s.t. $\frac{|\partial S|}{\mathrm{vol}(S)} < \phi$, i.e. $\frac{|\partial S|}{\phi} < \mathrm{vol}(S) \le \frac{\mathrm{vol}(V)}{2}$ and the last inequality is a consequence of $\mathrm{vol}(S)$ is the minimum between $\mathrm{vol}(S)$ and $\mathrm{vol}(V \setminus S)$.
We simply pick $D = \partial S$. Clearly S is a small component of $G \setminus D$ and $\mathrm{vol}(S) > \frac{|\partial S|}{\phi} = \frac{|D|}{\phi}$.
" $\Longrightarrow$ " we again prove the contrapositive, namely: If $\exists D$ where the total volume of small components of $G \setminus D$ is greater than $|D|/\phi$, then $\Phi(G) < 3\phi$.
We break into cases:

Case 1:

The normal situation is that after we delete a batch of edges, there will be a big component in the graph. Say $\exists C$ which is not a small component of $G \setminus D$, i.e. $\mathrm{vol}(C) > \frac{\mathrm{vol}(V)}{2}$.
Let S be the union of all small components, we have $\frac{|D|}{\phi} < \mathrm{vol}(S) < \mathrm{vol}(V) - \mathrm{vol}(C) < \frac{\mathrm{vol}(V)}{2}$. We know that $\partial S \subseteq D$ since when we delete D from G, S gets disconnected. Then,

$$\Phi(G) \le \Phi_G(S) = \frac{|\partial S|}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}} \le \frac{|D|}{\mathrm{vol}(S)} < \phi.$$

Case 2:

When all connected components in $G \setminus D$ are small components of G, this is harder to tackle.

Claim 2.10. *There exists S a union of small components in $G \setminus D$ s.t. $\frac{\mathrm{vol}(V)}{3} \le \mathrm{vol}(S) \le \frac{2\mathrm{vol}(V)}{3}$.*

Proof of Claim. Prove by contradiction. Denote all small components in $G \setminus D$ as $S_1, \cdots, S_p$. Since all components in $G \setminus D$ are small components, $\mathrm{vol}(S_1 \cup \cdots \cup S_p) = \mathrm{vol}(V)$. If for all unions S of small components of $G \setminus D$, either $\mathrm{vol}(S) < \frac{\mathrm{vol}(V)}{3}$ or $\mathrm{vol}(S) > \frac{2\mathrm{vol}(V)}{3}$. Let S be union of some S_is such that $\mathrm{vol}(S) \le \frac{\mathrm{vol}(V)}{3}$, and $\forall S'$ being union of some S_is such that $\mathrm{vol}(S') \le \frac{\mathrm{vol}(V)}{3}$, $\mathrm{vol}(S') \le \mathrm{vol}(S)$. Then for all j s.t. $S_j \notin S$, $\mathrm{vol}(S_j) > \frac{\mathrm{vol}(V)}{3}$ in order that $\mathrm{vol}(S \cup S_j) > \frac{2\mathrm{vol}(V)}{3}$ as assumption. However, $\frac{\mathrm{vol}(V)}{2} \ge \mathrm{vol}(S_j) > \frac{\mathrm{vol}(V)}{3}$, which contradict with the assumption. □

Since $\partial S \subseteq D$, $|D| \le \phi \mathrm{vol}(V)$,

$$\Phi(G) \le \Phi_G(S) = \frac{|\partial S|}{\min\{\mathrm{vol}(S), \mathrm{vol}(V \setminus S)\}} \le \frac{\phi \mathrm{vol}(V)}{\mathrm{vol}(V)/3} < 3\phi$$

□

2.2 Application: Checking Connectivity under Failures in Sub-linear time

- Let $G = (V, E)$ be an undirected graph with n vertices and m edges.
- We can preprocess G in $O(m)$. So that, given any $s, t \in V$, return if s and t are connected in $O(1)$ time.
  - How: using BFS to compute all the connected components in linear time.
- What if there are failures? Consider the following dynamic settings
  - Imagine a computer/road network.
  - Some edges might fail because of ... earthquake maybe.
  - We want to know if s and t are still connected, without recomputing from scratch.

Theorem 2.11. *Suppose that $G = (V, E)$ is a ϕ-expander. Without any preprocessing on G, for any $D \subseteq E$, we can check if $s, t \in V$ are connected in $G \setminus D$ in $O\left(\frac{|D|}{\phi}\right)$ time.*

Proof of Theorem. The proof follows from Theorem 2.9. WLOG, assume that $|D| \leq \frac{\phi \mathrm{vol}(V)}{100}$, otherwise it will just trivially take $O(m)$ time which is the same as conducting BFS for the whole graph. Notice that there is a unique large component C_{large} in $G \setminus D$. Then, consider the algorithm below:

Algorithm:

- Start a BFS from s in the graph $G \setminus D$. There are two cases:
  - Once we have explored more than $10|D|/\phi$ volume, then just stop and conclude that s is in C_{large}.
  - Otherwise, we identify that is s is in a component C_s where $\mathrm{vol}_G(C_s) < 10|D|/\phi$.
  - This takes $O(|D|/\phi)$ time, because BFS takes time proportional to the volume explored.
- Do the same for t.

In this way, we can identify which connected components containing s and t respectively in $O(|D|/\phi)$ time. □

2.3 Basic Property: Expanders have Small Diameter

Definition 2.12 (Diameter)**.** The *diameter* of a graph $G = (V, E)$ is $\max_{u,v \in V} \mathrm{dist}(u, v)$.

Proposition 2.13. *The diameter of a ϕ-expander with n vertices and m edges is $O(\log(m)/\phi)$.*

Proof of Proposition. The proof uses ball-growing argument, which is also a useful tool for other proofs in the lecture.

Let $B(s, r) = \{u \mid \mathrm{dist}(s, u) \leq r\}$ be a ball of radius r around s, r_s be the threshold radius such that $\mathrm{vol}(B(s, r_s - 1)) \leq \mathrm{vol}(V)/2$ yet $\mathrm{vol}(B(s, r_s)) > \mathrm{vol}(V)/2$. Let r_t be defined similarly for t. Notice that $B(s, r_s) \cap B(t, r_t) \neq \emptyset$ since both balls occupied more than half of the edges in G. Hence $\mathrm{dist}(s, t) \leq r_s + r_t$.

We will show that $r_s, r_t \leq O(\log(m)/\phi)$.

Since all cut edges in $\partial(B(s,r-1))$ are at the boundry of $B(s,r-1)$, all paths starting from s within $B(s,r-1)$ given access to set $\partial(B(s,r-1))$ can have increment in length at most by 1, and therefore the length of all path in $B(s,r-1)\cup\partial(B(s,r-1))$ have length $\leq r$. Hence,

$$\mathrm{vol}(B(s,r)) \geq \mathrm{vol}(B(s,r-1)) + |\partial B(s,r-1)|.$$

According to the definition of G being a $\phi-$expander, $\forall r \leq r_s$

$$\frac{|\partial B(s,r-1)|}{\min\{\mathrm{vol}(B(s,r-1)),\mathrm{vol}(V\setminus B(s,r-1))\}} \geq \phi.$$

Since $\mathrm{vol}(B(s,r-1)) \leq \mathrm{vol}(B(s,r_s-1)) \leq \frac{\mathrm{vol}(V)}{2}$, $\min\{\mathrm{vol}(B(s,r-1)),\mathrm{vol}(V\setminus B(s,r-1))\} = \mathrm{vol}(B(s,r-1))$,

$$|\partial B(s,r-1)| \geq \phi\mathrm{vol}(B(s,r-1)).$$

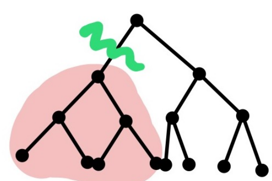
Since $\mathrm{vol}(B(s,0))$ is $\deg s \geq 1$, $\forall r \leq r_s$

$$\mathrm{vol}(B(s,r)) \geq (1+\phi)\mathrm{vol}(B(s,r-1)) \geq (1+\phi)^r.$$

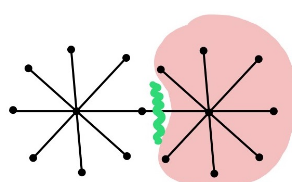
Since $e^{\phi r_s} \approx (1+\phi)^{r_s} \leq \mathrm{vol}(B(s,r_s)) \leq \mathrm{vol}(V) = 2m$, we have $r_s = O(\log(m)/\phi)$. The same holds for r_t. □

Remark 2.14. Conversely, not all low diameter graphs have high conductance. For example,

- Binary trees, where the diameter is $O(\log n)$, but the conductance is $O\left(\frac{1}{n}\right)$,



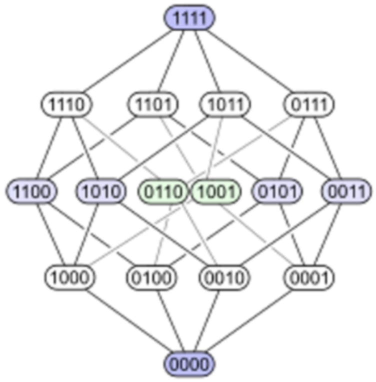
- stars connected by a short path, where the diameter is 4 but the conductance is $O\left(\frac{1}{n}\right)$.



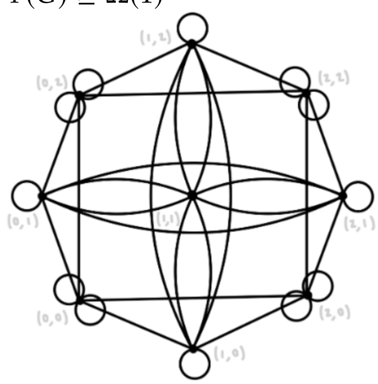
3 Non-trivial Examples: Small Degree Expanders

Intuitively, graphs containing large degree vertices can be more well-connected, whence having higher conductance. For example, the fact that complete graphs and stars have high conductance are not very surprising.
However, the existence of graphs with small maximum degree but still have large conductance is nontrivial. Namely, sparse graphs can still be very well-connected. Some examples are listed as below:

- Deterministic examples:
  - Hypercube:
    - $V = \{0,1\}^{\log n}$ and $E = \{(u, v) \mid u = v$ except that one entry$\}$.
    - $\Phi(G) \geq \Theta(1/\log n)$.

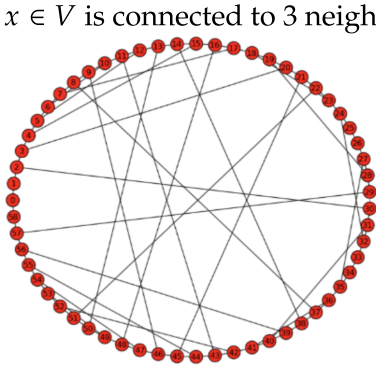


  - Margulis–Gabber–Galil
    - $V = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$
    - Each $(x, y) \in V$ has 8 neighbors: $(x \pm 2y, y), (x \pm (2y+1), y), (x, y \pm 2x), (x, y \pm (2x+1))$
    - $\Phi(G) \geq \Omega(1)$



when $n = 3$ [1]

  - Prime expander
    - $V = \mathbb{F}_p = \{0, \ldots, p-1\}$
    - $x \in V$ is connected to 3 neighbors: $x+1, x-1$ and x^{-1} (for $x \neq 0$)



when $p = 59$[2]

[1] https://web.stanford.edu/~styopa/pdfs/expanders.pdf
[2] https://lucatrevisan.github.io/teaching/expanders2016/lecture16.pdf

- Randomized examples:
  - Erdos-Renyi graph $G_{n,p}$ when $p = \Omega(\log(n)/n)$:
    * With high probability, G has maximum degree $O(\log n)$ and$\Phi(G) \geq \Omega(1)$.
    * This means that almost all graphs are actually expanders.
  - Union of random $O(1)$ matchings
    * G has maximum degree $O(1)$
    * $\Phi(G) \geq \Omega(1)$ with high probability[3]

Note 3.1. "Canonical expanders" can be depicted as: well-connected, yet with low degree. (But of course, in general graphs with high conductance can also have various other appearances.)

4 Vertex Expansion

Conductance describes the ratio between #(cut edges) and #(edges disconnected by the cut). We can also talk about the analogue of it for vertices.

Definition 4.1 (Vertex-expansion)**.** Let (L, S, R) be a vertex cut, i.e. L, S, R partition V and $E(L, R) = \emptyset$. **Vertex-expansion of a cut** (L, S, R) is

$$h_G(S) = \frac{|S|}{\min\{|L \cup S|, |R \cup S|\}} \in [0, 1]$$

Vertex-expansion of a graph G is

$$h(G) = \min_{(L,S,R)} h(L, S, R) \in [0, 1]$$

- We say that
  - S is a ϕ**-vertex-sparse cut** iff $h_G(S) < \phi$
  - G is a ϕ**-vertex-expander** iff $h(G) \geq \phi$

Exercise 4.2. (vertex expansion and conductance are not necessarily the same, but could be under certain conditions) Show that

1. If a graph G has maximum degree $O(1)$, then vertex expansion and conductance has within a constant factor $h(G) = \Theta(\Phi(G))$.

   Proof. First of all, notice that all edge cuts corresponds to a vertex cut by simply letting the endpoints of cut edges to be the vertex cut. Since $\deg G = O(1)$, there is some constant $M, s.t. \deg G \leq M$. Let the vertex cut (L, S, R) be such that $h(G) = h(L, S, R)$. If $\Phi(G) = \frac{|\partial A|}{\text{vol}(A)}$, then $\frac{|\partial A|}{M} \leq |S| \leq 2|\partial A|$, which shows that $|S| = \Theta(|\partial A|)$. Since the degree of G is bounded by a constant, this means that for any subgraphs of G, the total number of edges is also bounded by the number of vertices up to a constant factor, namely $\min\{|L \cup S|, |R \cup S|\} = \Theta(\text{vol}(A))$. Hence, $h(G) = \Theta(\Phi(G))$. ◻

[3]https://lucatrevisan.github.io/teaching/expanders2016/lecture19.pdf

2. There is a graph G where $\Phi(G) \geq \Omega(1)$ but $h(G) \leq O(1/n)$.

   Proof. A star is an example where we need to delete the same amount of edges in order to disconnect that amount of vertices, whence $\Phi(star) = 1$. On the other hand we only need to delete the central vertex of the star to disconnect all vertices in the graph, therefore $h(star) = \frac{1}{n}$. □

3. There is a graph G where $h(G) \geq \Omega(1)$ but $\Phi(G) \leq O(1/n)$.

   Proof. Consider a graph contains 2 same cliques with a perfect matching between them. To disconnect one vertex from the graph, we need to delete the whole clique that this vertex stays in plus the vertex that matches with it. Therefore, $h(G) \approx \frac{1}{2} = \Omega(1)$. On the other hand, deleting the perfect matching let the whole graph break into two parts with same volume, i.e. $\Phi(G) = \frac{n}{n^2} = \Theta(\frac{1}{n})$. □

Exercise 4.3 (Robustness of vertex-expander against vertex deletion)**.** Suppose that $G = (V, E)$ is a ϕ-vertex-expander. Without any preprocessing on G, for any $D \subseteq V$, we can check if $s, t \in V$ are connected in $G \setminus D$ in $O((|D|/\phi)^2)$ time.

Proof. The proof should be an analogue of the proof of Theorem 2.11.

Claim 4.4. *$h(G) \geq \Omega(\phi) \iff \forall D \subseteq V$, the total number of vertices of small components of $G \setminus D$ is at most $O\left(\frac{|D|}{\phi}\right)$, where we say a connected components C is small in $G \setminus D$ is the number of vertices in C is less than or equal to $\frac{|V|}{2}$.*

Proof of Claim. Suffices to show the contrapositive. We first show that if $h(G) < \phi \implies \exists D \subseteq V$, s.t. the total number of vertices of small components of $G \setminus D > O\left(\frac{|D|}{\phi}\right)$. Since $h(G) < \phi$, $\exists (L, S, R)$ a vertex cut of V s.t. $\frac{|S|}{|L \cup S|} < \phi \implies \frac{|S|}{\phi} < |L \cup S| = |L| + |S| \implies O\left(\frac{|S|}{\phi}\right) = \frac{(1-\phi)|S|}{\phi} < |L|$. Let $S = D$, then since $\min\{|L \cup S|, |R \cup S|\} = |L \cup S|, |L| \leq \frac{|V|}{2}$, and hence L is a small component of $G \setminus S$.
We next show that if $\exists D \subseteq V$, s.t. the total number of vertices of small components of $G \setminus D > O\left(\frac{|D|}{\phi}\right) \implies h(G) < 3\phi$. We break into cases:

- Case 1. $\exists C \subseteq G \setminus D$ s.t. C is not a small component in $G \setminus D$. Let A denote the union of all small components of $G \setminus D$, then $\frac{|D|}{\phi} < |A| \leq \frac{|V|}{2}$. Then

  $$h_G(D) = \frac{|D|}{\min\{|L \cup D|, |R \cup D|\}} < \frac{|D|}{|A|} < \phi$$

  Hence $h(G) \leq h_G(D) < \phi$.

- Case 2. when all elements in $G \setminus D$ are small components. Make a similar argument as in the claim in the proof of Theorem 2.11, we know that there exists A a union of small components in $G \setminus D$, s.t. $\frac{|V|}{3} \leq |A| \leq \frac{2|V|}{3}$. We also know that $|D| < \phi|V|$. Then

  $$h_G(D) = \frac{|D|}{\min\{|L \cup D|, |R \cup D|\}} \leq \frac{|D|}{|A|} < \frac{\phi|V|}{\frac{|V|}{3}} = 3\phi$$

$\square$

Knowing that for a $\phi-$vertex expander, all small components in $G \setminus D$ will have total size at most $O\left(\frac{|D|}{\phi}\right)$, we did the similar algorithm as before: we start BFS from s in $G \setminus D$, and once we have explored more than $10\frac{|D|}{\phi}$ vertices, stop and conclude that s is in C_{large}. Otherwise, we identify that s is in a small component C_s. This takes $O\left(\left(\frac{|D|}{\phi}\right)^2\right)$ time since the time complexity of BFS is proportional to the number of edges, which is bounded by the square of number of vertices. We do the same for t, whence the total time is $O\left(\left(\frac{|D|}{\phi}\right)^2\right)$. $\square$

Exercise 4.5. (vertex-expander has small diameter) Show that a ϕ-vertex-expander has diameter $O(\log(n)/\phi)$.

Proof. We will use the ball-growing argument again. Let $B(s,r) = \{u | dist(s,u) \leq r\}$. Let r_s be a threshold s.t. $|B(s, r_s - 1)| \leq \frac{|V|}{2}$, yet $|B(s, r_s)| > \frac{|V|}{2}$. Define the analogue for t.
We know that $B(s, r_s) \cap B(t, r_t) \neq \emptyset$ since they all contains more than $\frac{|V|}{2}$ vertices and by PHP there must be at least one vertex falls into both balls. Hence, $dist(s,t) \leq r_s + r_t$.
We next show that $r_s, r_t < O\left(\frac{\log(n)}{\phi}\right)$. Indeed, since $|B(s, r_s)| \geq |B(s, r_s - 1)| + |S_r|$ where S_r is the vertex cut that separate $B(s,r)$ from G. Since G is a $\phi-$vertex expander, and $\forall r \leq r_s - 1$, $|B(s,r)| \leq \frac{|V|}{2}$, $h_G(S_r) = \frac{|S_r|}{\min\{|L \cup S_r|, |R \cup S_r|\}} = \frac{|S_r|}{|B(s,r)|} \geq \phi$ i.e. $|S_r| \geq \phi |B(s,r)|$. Hence, $|B(s, r_s)| \geq (1+\phi)|B(s, r_s - 1)| \implies n = |V| \geq |B(s, r_s)| \geq (1+\phi)^{r_s} \approx e^{\phi r_s} \implies r_s \leq \frac{\log n}{\phi}$. Similar for r_t. $\square$

5 Edge Expansion with General Demand

- The following notion generalizes conductance.
- Let $\boldsymbol{d} : V \to \mathbb{R}_{\geq 0}$ be a *demand function* of vertices.
- A **$\boldsymbol{d}$-expansion of a cut** S where $0 < \boldsymbol{d}(S) < \boldsymbol{d}(V)$ is

$$\Phi_{G,\boldsymbol{d}}(S) = \frac{\partial S}{\min\{\boldsymbol{d}(S), \boldsymbol{d}(V \setminus S)\}}$$

  where $\boldsymbol{d}(S) = \sum_{u \in S} \boldsymbol{d}(u)$.
- A **$\boldsymbol{d}$-expansion of a graph** G is

$$\Phi(G, \boldsymbol{d}) = \min_{S: 0 < \boldsymbol{d}(S) < \boldsymbol{d}(V)} \Phi_{G,\boldsymbol{d}}(S)$$

- Conductance is the same as $\boldsymbol{d}$-expansion when $\boldsymbol{d}(u) = \deg(u)$ for all u.

Exercise 5.1. (a graph with big expansion and large diameter: expansion and conductance can be different) In the literature, people often consider $\boldsymbol{d}^{unit}$-expansion where $\boldsymbol{d}^{unit}(u) = 1$ for all $u \in V$ and call this "*expansion*". This measures the ratio between #(cut edges) and #(vertices disconnected by the cut). Show that even if $\Phi(G, \boldsymbol{d}^{unit}) \geq \Omega(1)$, then diameter of G can be large.

Remark 5.2. This is unrelated to the solution, but notice that this value $\Phi(G, \boldsymbol{d}^{unit})$ is not bounded by 1 but can be as big as the max degree. For example, in a complete graph, $\Phi(G, \boldsymbol{d}^{unit}) = \frac{|S|\times|V\setminus S|}{\min\{|S|,|V\setminus S|\}} = \Omega(n)$.

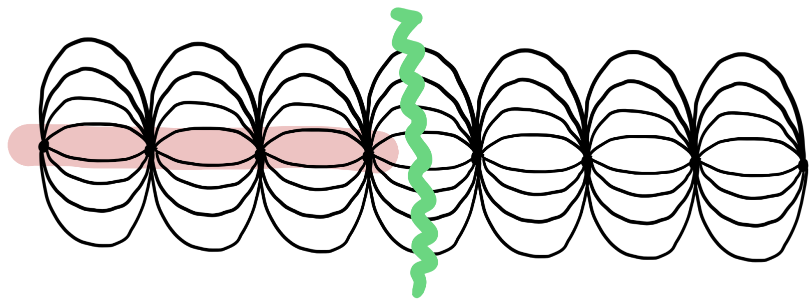


Figure 1: G_1

Solution. Consider the graph above, where $\Phi(G_1, \boldsymbol{d}^{unit}) = 2 \geq \Omega(1)$, yet the diameter is 7 which is pretty large.
Generalize to simple graph, consider changing one vertex into layer of n vertices, and connecting each layers with the layer next to it using a complete bipartite graph as below

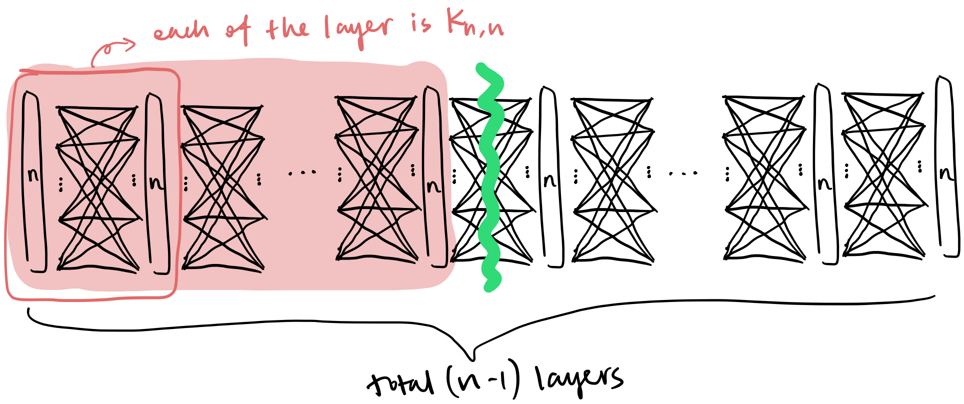


Figure 2: G_2

In this case, $\Phi(G_2, \boldsymbol{d}^{unit}) = \frac{n^2}{n^2/2} = 2 \geq \Omega(1)$, but the diameter is $O(n)$. □

Exercise 5.3. (a graph with big d-expansion but small conductance) Let $T \subseteq V$ be a subset called terminal. Let $\boldsymbol{d}(u) = \deg(u)$ if $u \in T$ otherwise $\boldsymbol{d}(u) = 0$. Give an example of a connected graph G where $\Phi(G, \boldsymbol{d}) = \Omega(1)$ but $\Phi(G)$ is very small.
Hint: Start with a graph $G_0 = (T, E_0)$ where $\Phi(G_0) = \Omega(1)$. Let $G = (V, E)$ be obtained from G_0 by subdividing each edge into a path of length n. Show that $\Phi(G, \boldsymbol{d}) = \Omega(1)$.

Solution.

1. Consider the connected graph G consist of a clique connected with a path, where the clique is T. Then, the conductance of the graph is very small since it's basically the con-

ductance of the path. However, $\Phi(G, \boldsymbol{d})$ is large since it evaluates how well-connected the subset T is, in this case the clique.

2. A more nontrivial example is as suggested in the hint. We first show the conductance of G is small. Indeed, $\Phi(G) \leq O\left(\frac{1}{n}\right)$ since we could pick any random edge in G_0, which turns into a path of length n in G, and for that path in G, we delete two edges that are attached to the original vertices in G_0, and this will make the whole chunk of path get disconnected from the graph.
   We next show that $\Phi(G, d) \geq \Omega(\Phi(G_0))$. Indeed, given a cut $S \subseteq V$ in G, we can construct a corresponding cut S_0 of G_0 by contracting G back to G_0 and take the contraction S_0 of S in G_0. Hence, $|\partial_G(S)| \geq |\partial_{G_0}(S_0)|$. Since d is defined to be $\deg(u)$ on T and 0 otherwise, $\min\{\mathrm{vol}_G(S), \mathrm{vol}_G(V \setminus S)\} = \min\{\mathrm{vol}_G(T \cap S), \mathrm{vol}_G(V \setminus S \cap T)\} = \min\{\mathrm{vol}_{G0}(S_0), \mathrm{vol}_{G_0}(T \setminus S_0)\}$. Then

$$\Phi_{(G,\boldsymbol{d})}(S) = \frac{|\partial_G(S)|}{\min\{\mathrm{vol}_G(S), \mathrm{vol}_G(V \setminus S)\}} \geq \frac{|\partial_{G_0}(S_0)|}{\min\{\mathrm{vol}_{G0}(S_0), \mathrm{vol}_{G_0}(T \setminus S_0)\}} = \Phi_{G_0}(S_0) \geq \Omega(1)$$

   Hence $\Phi(G, \boldsymbol{d}) = \Omega(1)$.

□