OK, now to HDXs.

\[ X = (U, F) \]

- Simplicial Complex = down and closed set-system \( (sc) \)
- dim of a set \( S \) = \( |S| - 1 \), \( \alpha \) "facet"
- dim of \( SC \) = dim of largest set w/it.
- pure \( SC \) = all facets have same size (also as in matroid).

So matroids are pure \( SC \) with exchange property

\[ X(i) = \text{set of size } i+1. \quad (\text{dim } i) \]
\[ X = X(0) \cup X(1) \cup \ldots \cup X(d) \]

"vertices" "edges" "faces" \( \quad \phi \rightarrow \text{dim } -1. \)

- 1-skeleton of \( X = \text{graph } (X(0), X(1)) \) called \( G(X) \)

- Links of \( X \).
  - given \( \alpha \), \( X_\alpha = \{ \beta \in F \mid \beta \subseteq \alpha, \beta \neq \phi \} \) \( \leftrightarrow \text{"umbachin" of } \alpha \)

If \( X \) in pure \( d \)-dim \( SC \) \( \Rightarrow X_\alpha = \text{pure } (d-1) \text{-dim } SC. \)

Local Spectral Expanders: \( X \) is pure \( d \)-dim \( SC \). \( X \) is \( \delta \)-(local)-spectral expander

- if \( \alpha \) faces \( \alpha \), \( G(X_\alpha) \) has \( \delta_2 \) if the random walk matrix \( \leq \delta \).
Moreover we can define a probability distribution over facets, and over each layer, as follows:

Let $X$ be a pure SC of dimension $d$. Let $\Pi_d$ be a probability over $X(d)$. We can define a distribution $X(d-1)$ by setting for each $s \in X(d-1)$:

$$\Pi_{d-1}(s) = \frac{1}{\alpha} \sum_{\beta \in X(d)} \Pi_d(\beta).$$

This gives a distribution $X(d-1)$. Now induct.

Given $(X, \Pi_d)$ be $d$-dim pure SC with weights. For $\alpha$ and $t \in X_d$:

$$\Pi^{\alpha}(t) = \frac{1}{\Pi^t(X,t)} \sum_{\beta \in X_{d-1}} \Pi^{\alpha}_{d-1}(\beta) = \frac{\Pi_d(\alpha \cup t)}{\Pi^t(X,t)} \cdot \Pi^{\alpha}(\alpha).$$

Now: given weights on the 1-skeleton (edges and vertices) of any $X$.

Call $G_\alpha$ the 1-skeleton of the guide. $\Pi^\alpha$ be distrib over edges.

$$D_\alpha = \text{adjoint matrix with } D_\alpha(x, y) = \sum_{y \in X_\alpha} \Pi^\alpha_{d-1}(x, y) = 2 \Pi^\alpha_d(x, y).$$

Now the walk is not unweighted; the weights decide the prob of going places.

$$W_\alpha = D_\alpha^{-1} A_\alpha.$$ has $\lambda_{\text{max}} = 1$, with $\text{trace} = M$. 

Thm: The SC of any matroid with the uniform distribution on facets is a 0-local expander.

Pf: Show that:
1. Every link has an attached skeleton, and
2. Every top link has a skeleton which is 0-local expander.

1. Say contract α. So left-in matroid. Can get from any vertex to any other by the exchange axiom.

2. Let's sketch the proof for graphic matroids, where d = n - 2 (since base one of size n - 1). Now any link αX corresponds to embedding edges of a forest F ≤ E with |F| = n - 3 edges. This gives 3 components.

Hence the link has skeleton whose elements correspond to three colored edges (E_1, E_2, E_3 say), and whose pairs correspond to pairs q-colored edges that can belong to some base. Note there are edges in a complete k-partite graph, since one can add any pair of elements of different colors.

It is a fact that the complete k-partite graphs have at most one positive value (and this is a characterization of complete k-partite graphs).
We discussed from Prata so let's give a proof (another to be added later).

**Pf1:** Cauchy's interlacing theorem says that if

\[ A = B + uu^T \]

then \( \lambda_1(A) \geq \lambda_1(B) \geq \lambda_2(A) \geq \ldots \)

rank 1 (PSD)

Now, the adjacency matrix of a complete \( k \)-partite graph is

\[ A = J + (-B_1 - B_2 \ldots - B_k), \text{ where } B_i = x_i x_i^T \]

\[ = B + 11^T. \]

Now \( B \) is negative semidefinite so \( \lambda_{\text{max}}(B) \leq 0. \)

Now \( \lambda_2(A) \leq \lambda_1(B) \leq 0. \)

\[ \square \]

**Pf2:**
OK: given a distribution over facets (usually uniform), we have a natural way
to get distributions over links, and over each 1-skeleton, (and so over random
walks).

Each has a spectral gap (for the matrix $D^{-1}A$).

$(X, \pi)$ is an $8$-local-spectral expander if $\lambda_2$ for each link
matrix is at most $8$. (1st largest eigenvalue $\lambda_1 = 1$ for each, with $f_{\pi \rightarrow \pi}$.

Do we need to check the expansion for all links?

Oppenheimer gave a nice "trickle-down" theorem that says: if the top links are
local expander then the lower links are, too (with some loss).

Thm [Opp18]: $(X, \pi)$ be pure SC by dumb (weighted by $\pi$).

Sps. $G_\phi = (X(0), X(1), \pi_1)$ is connected and $\lambda_2(D^{-1}A_\phi) \leq 8$ for all $v \in X(0)$

\[ \Rightarrow \lambda_2(D^{-1}A_\phi) \leq \frac{8}{1-\pi} \]

Corollary: if top links $\lambda_2(D^{-1}A_\phi) \leq 8 \forall \alpha \in X(d-2)$

and $G_\phi$ is connected for all $\beta \in X(k), k \geq d-2$

then $\forall \beta \lambda_2(D^{-1}A_\phi) \leq \frac{\beta}{1-\pi}$

(d-1)

Pf: omitted

But note: if each $G_\phi$ is a 0-local expander

$\Rightarrow G_\phi$ is also a 0-local expander!
Lemma: Eigenvalues of $P^K$ and $P^\infty$ are the same.

And so we can talk about the spectral gap of a pair of graphs $X(k) \times X(k')$ (one for the down-up walk on $X(k')$ and one for up-down walk on $X(k)$).

Last time we saw that if $G$ was a regular n-vertex graph with

$$\max(\lambda_2, \delta n) \leq \varepsilon \Rightarrow \text{called } (n, k, \varepsilon, \text{regular graph})$$

then random walk "mixes" in $O\left(\frac{\log(n \delta)}{\varepsilon}\right)$ time (TV distance to uniform) $\leq \varepsilon$.

If graph not regular $\Rightarrow$ cannot use same proof, b/c matrices n\text{t} symmetric, but still walk on undirected graphs (perhaps with weights).

Now the mixing time (to the stationary distribution $\pi_k$) $\leq \frac{1}{\varepsilon} \log\left(\frac{\delta}{\varepsilon} \cdot \frac{\text{max degree}}{\text{min degree}}\right)$.
Hence still interesting to bound the spectral radius, by $\varepsilon$.

In fact we care about the spectral radius of $P_d^v$

But thinks the same as $P_{d-1}^A$.

Now if we could relate $P_{d-1}^A$ to $P_d^v$ etc.... Actually relate $P_{d-1}^A$ to $P_d^v$ lose $\varepsilon$

and then relate $P_{d-1}^A$ to $P_{d-1}^A$ lose a little more.

$1 - \frac{d}{d+1}$

Then [Kaufman-Openheim] if $(X,T)$ is a $d$-local spectral expander then $\forall k \leq d$.

$$\lambda_2(P_k^v) \leq 1 - \frac{1}{k+1} + k\varepsilon$$

For maximal $sc \ P=0 \Rightarrow \lambda_2(P_k^v) \leq 1 - \frac{1}{k+1} \Rightarrow \lambda_2(P_d^v) \leq \frac{d}{d+1}$$

$\Rightarrow$ spectral gap $\leq \Theta(\frac{d}{d+1})$

Now plugging into mixing bounds: $\#steps \leq O\left(\frac{1}{\varepsilon \log (N/\varepsilon)}\right)$

but $\varepsilon = \frac{1}{d}

N = n^d \rightarrow \# of bases

$\Rightarrow O(d^2 \log (n/\varepsilon))$ time!

$\rightarrow$ rank of matroid.

Intuition: