Expander and High Dimensional Expanders

\[ G = (V, E) \text{ undirected, unweighted} \]
\[ \Rightarrow \text{ can generalize to weighted, often d-regular} \]
\[ \Rightarrow \text{ can generalize, but math simpler} \]

\[ A = \text{adjacency matrix } \in \mathbb{R}^{n \times n} \]

The edge expansion of set \( S \subseteq V \) is
\[ \phi(S) = \frac{|E(S, \bar{S})|}{|S|} \quad \text{and a graph } \Phi(G) = \min_{S: \text{vol}(S)} \phi(S). \]

The conductance
\[ \phi(S) = \frac{|E(S, \bar{S})|}{\sum_{v \in S} \deg(v)} \quad \text{and a graph } \phi(G) = \min_{S: \text{vol}(S)} \phi(S) \leq 1. \]

Note:
\[ \Phi(G) \in [0, d_{\text{max}}] \]
\[ \phi(G) \in [0, 1]. \text{ It's a ratio.} \]
\[ \Phi(G) \text{ for d-regular graphs, } \phi = \frac{\Phi}{d}. \]

Combinatorial:
A graph is an \( \alpha \)-expander if \( \phi(S) \geq \alpha \).

The best we can hope for is \( \alpha \) being an absolute constant, independent of \( n \).

Also: spectral def.
\[ A = \text{adjacency matrix} \quad L = D - A \]
\[ \Rightarrow \text{ both are symmetric, have real eigenvalues} \]
\[ L = BB^T \text{ is PSD and hence has non-negative eigenvalues.} \]
For the rest of today, let $G$ be a regular and connected graph.

Then the eigenvalues of $L$ and $A$ are related by $\lambda_i(L) = d - \lambda_i(A)$.

We talk about these interchanges.

For $L$:

$\lambda_1(A) = d > \lambda_2(A) > \cdots > \lambda_n(A)$.

The eigenvectors $\frac{f_i}{\|f_i\|}$ form an orthonormal basis of $\mathbb{R}^n$.

Consider a scaled version $\bar{A} = \frac{1}{d} A$.

Then if we do a random walk on the graph:

$p^t(u) = \frac{1}{\|f_i\|} \sum_{v \in G} p^{t-1}(v) Av = (\bar{A}p^t)(u) = \lambda_i(A) p^t$.

So suppose $p^0 = \sum_i c_i f_i$.

Then $p^t = \sum_i c_i (\lambda_i f_i)$.

Now if $\lambda_2, 1, \lambda_n \leq 1 - g = \lambda_1 - \lambda_2$.

A graph is a spectral expander if

there is a spectral gap $\gamma$.$\gamma$

A graph has a (two-sided) spectral gap $\gamma$ if $\max(\lambda_2, 1 - \lambda_n) > \gamma$.

A graph is a spectral expander if it has a constant spectral gap $\gamma$.
Cheeger's Inequality. \( G \) be \( d \)-regular, with normalized adjacency \( \overline{A} \) having eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \) and \( \phi(G) \) be the conductance. Then
\[
\frac{1}{2} \left( 1 - \lambda_2 \right) \leq \phi(G) \leq \sqrt{2(1 - \lambda_2)}
\]

"easy" \quad "hard"

So spectral expansion \( \Leftrightarrow \) combinatorial expansion, at least qualitatively.

Note: \( 1 - \lambda_2 = \) second eigenvalue of normalized Laplacian = \( \frac{1}{d} (D - A) \).

\[
= \min_{f: \mathbb{R}^n \rightarrow \mathbb{R}} \frac{\sum_{j \neq i} (f_i - f_j)^2}{d \sum_i f_i^2} \leq \text{same thing for vector } f_S = \begin{cases} \frac{1}{\sqrt{d}} & \text{for } i \in S \\ -\frac{1}{\sqrt{d}} & \text{for } i \notin S \end{cases}
\]

where \( S \) is the minimizer of \( \phi(G) \).

\[
= \frac{1E(S, S)1}{d(1 + \frac{1}{\sqrt{d}})} \left( \frac{1}{\sqrt{d}} + \frac{1}{\sqrt{d} - 1} \right)^2 \leq \frac{1E(S, S)1}{d \frac{1}{\sqrt{d} - 1}} \frac{1}{\sqrt{d} - 1}
\]

\[
\leq 2 \phi(S) \quad \text{or } 2 \phi\left( \frac{1}{d} \right)
\]

Hard direction is not very hard; but another time.

\[
\text{Examples:}
\]
- for an \( n \)-cycle, \( \lambda_2 \approx 1 - \Theta\left( \frac{1}{\sqrt{n}} \right) \) but \( \phi(G) = \Theta\left( \frac{1}{n} \right) \).

\Rightarrow \text{hard inequality.}

- for complete graphs, both are constant \( \Rightarrow \) easy direction tight.

\( \text{Can extend to weighted graphs, see notes by many people (e.g. Lap Chi Lau @ Waterloo)} \)

\( \text{(survey by Henry Linial Witsenon)} \)

\( \text{(Dror Sterinson's notes)} \)
The calculation we did for showing that the random walk on a regular graph approaches the uniform distribution also gives quantitative bounds.

$$\text{def: total variation distance b/w } p, q \text{ on some set } S (\text{say})$$

$$d_{TV}(p, q) = \max_{S \subseteq S} p(S) - q(S) = \frac{1}{2} \| p - q \|_2$$

$$\leq O(\sqrt{n}) \cdot \| p - q \|_2.$$ But we wanted $$d_{TV}(p^\ast, 1/n) \leq O(\sqrt{n}) \cdot \| p^\ast - 1/n \|_2$$

$$\| . \|_2^2 \leq (1 - \frac{1}{n}) \sum_{i \geq 2} \frac{1}{i^2} \leq \frac{\pi^2}{6} \leq O(\sqrt{n}), (1 - \frac{1}{n})^b \leq \sqrt{n} \cdot e^{-\frac{b}{2}} \leq \varepsilon \quad \text{if } b > 3(\log n) \quad g$$

So if gap $g = O(1)$ then mix in $O(\log n)$ steps! ← "rapid" mixing.

Interestingly, must mean that graph has small chromatic $O(\log n)$.

And indeed follows from combinatorial def.:

--- $X$ ---

< Markov Chain Monte Carlo

Another interesting angle. Sps: want to do MCMC for sampling, matching bases, or colorings, or independence sets, etc. Define a large graph $G$ on space of all valid objects, and walk from one to the other. (Random walk on $G$). Now if $r_2(G)$ is good (has good gap) then walk mixes rapidly in $\log(N/e)$.

OK even if $N = \exp(n)$; which is basis of a lot of work in this topic.
A couple of other facts:

**Expander Mixing Lemma, or "Expanders behave like random graphs"**

- Let $G$ be $d$-regular, let $1 = \lambda_1 > \lambda_2 > \ldots > \lambda_n \geq -\delta$ be eigenvalues of normalized adj matrix.

  - Call $G$ an $(n, d, \varepsilon)$ graph if $\max \{\lambda_2, 1\lambda_n/\delta\} \leq \varepsilon$.

  - "spectral radius"

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**Expander Mixing Lemma**

- Let $G$ be an $(n, d, \varepsilon)$-graph then $\forall S, T : |S, T|

\[
|E(S, T) - \frac{d}{n} |S||T| | \leq \varepsilon d \sqrt{|S||T|}.
\]

\[
\text{"expected number from } S \leq T \text{"}
\]

\[
\text{small compared to } |S||T| \gg (\varepsilon d)^2
\]

**Proof:**

- Compute $|E(S, T)| = \sum_{i \neq j} a_i b_j A_{i,j}$

  - $a_i = \langle \chi_S, \phi_i \rangle = \sqrt{\frac{|S|}{n}}$

  - $b_i = \frac{1}{\sqrt{n}}$

  - $A = \sum \phi_i \phi_i^T$

  - $a_i \phi_i$ is $\varepsilon d \sqrt{|S||T|}$

\[
|E(S, T)| \leq \varepsilon d \sqrt{|S||T|}.
\]
Applications

- Derandomizations. Sps. have an expander graph on $2^n$ vertices so that each node represents an $r$-bit random string. Suppose have an algo that fail with prob $\frac{1}{4}$ in any fixed $r$-bit string (drawn random). Then if we want to amplify,

- either choose $t$ diff $r$-bit strings, prob of error $\leq (\frac{1}{4})^t$.

- or pick a random node in $H$, and take an $t$-local random walk.

  
  requires $r+t$ local bits

  prob of error on all bins $\leq \left(\frac{1}{4} + \frac{1}{3}\right)^t$ if $H$ is an $(2^n, d, \varepsilon)$-graph.

**Intuition:** cannot stay in the "bad" region most of the time.

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- Error Correcting Codes: get $2^t$ longer codes from small ones.

  - take a linear code on $d$ bits.

  - take an $(n, d, \varepsilon)$-graph $G$.

    - say $[d, d, d, d]$ code

    - ensure that each vertex sees a code with round it

**Inner Code**

**Outer Code**

Given $n = \left[\frac{1}{2}(2^{d-1})^d, \delta, \varepsilon \right]$ code

- original rate better be $\frac{1}{2}$

  - $\delta \to \delta^2$ if $\varepsilon \ll \delta$.

Proof uses the Expander Mixing Lemma.
Pf: Sps linear code \( L \), then lets use a "double cover" approach instead than

1. Linear code.
   of nd bits

So just sufficient to verify the weight of
least weight code and (non-zero)

Say \( W^* \) is least Hamming wt.
uses edges \( F \) for 1s.

then let \( S, T \) be verts on \( L, R \) incident to edges \( F \).

\[ L = V \quad R = V \]

\[ \text{nd edges} \]
\[ \text{d regular, bip} \]

Both sides check if
\[ \text{ndhood in L.} \]

\[ (a) \quad \text{IF} \bigcap \text{dv} \quad \geq \text{nd} \]
\[ \implies \text{IF} \geq \text{sd} |S|, \quad \text{sd} |T| \implies \text{IF} \geq \text{sd} \sqrt{|S||T|}. \]

\[ (b) \quad \text{IF} \subseteq E(S,T) \quad \text{and} \]
\[ \text{so} \quad \text{sd} \sqrt{|S||T|} \leq |E(S,T)| \leq \frac{d}{n} |S||T| + \text{sd} \sqrt{|S||T|} \]
\[ (EML) \]
\[ \implies (d-\epsilon)n \leq \sqrt{|S||T|} \]
\[ \text{by} \[ (c) \quad \text{IF} \geq \text{sd}(d-\epsilon). \text{dn.} \implies \text{rel. dist} \geq \delta(d-\epsilon) \]

Can do linear time (and log rank) decoding as well,

the proofs are similar, show that # edges that disagree with

\[ \text{nearest codeword decrease geometrically!} \]

[Remar: decoding]
OK, now to HDXs.

$$X = (U, \mathcal{F})$$

Simplicial Complex = down ward closed set system \( (\text{sc}) \)

- \( \dim \text{ of a set } S = |S| - 1. \)
- \( \dim \text{ of sc } = \dim \text{ of largest set in it.} \)
- \( \text{pure sc } = \text{ all facets have same size } \) (also as in matroid).

So matroids are pure scs with exchange property.

$$X(i) = \text{set of size } i + 1. \quad (\dim i)$$

$$X = X(0) \cup X(1) \cup \ldots \cup X(d)$$

"Verhár" \( \leftrightarrow \) "edges" \( \leftrightarrow \) "facets"

$$\phi \in \dim - 1.$$

1-skeleton of \( X = \text{graph } (X(0), X(1)) \)

called \( G(X) \)

Local Spectral Expanders: \( X \) is pure \( d \)-dim sc. \( \Rightarrow X \) is \( \delta \)-\((\text{local})\)-spectral expander

If \( \delta > \dim sc - 2 \), \( G(X) \) has \( d_2 \) of the random node matrix \( \leq \delta \).