- Finish Low-Stretch Trees
- Start Negative-Weight SSSP [Bernstein- Nanongkai- Wulff-Nilsen FOCS'22 Best Paper]

Simplified: [BCF FOCS'23]

- uses LDD on directed graphs
- "modern" graph algo

Johnson's Potential Reweighting

Let $G=(V,E)$ graph with possibly negative weights.

Goal: reweight graph to ① preserve shortest paths
② make weights nonnegative.

Let $\phi(v) : v \in V$ be potentials, to be computed.

Reweight edge $w'(u,v) \leftarrow w(u,v) + \phi(u) - \phi(v)$.

Property ①: for any path $u = v_0 - v_1 - v_2 - \cdots - v_k = v$
new weight of path is

$$w'(v_0,v_1) + w'(v_1,v_2) + \cdots + w'(v_{k-1},v_k)$$

$$= \big(w(v_0,v_1) + \phi(v_0) - \phi(v_1)\big)$$
$$+ \big(w(v_1,v_2) + \phi(v_1) - \phi(v_2)\big)$$
$$+ \cdots$$
$$+ \big(w(v_{k-1},v_k) + \phi(v_{k-1}) + \phi(v_k)\big)$$

$$= w(v_0,v_1) + w(v_1,v_2) + \cdots + w(v_{k-1},v_k) + \phi(v_0) - \phi(v_k)$$

So all $u \sim v$ paths change by $+\phi(v_0) - \phi(v_k)$.

Property ②: Add a source vertex s
($w'(u,v) \geq 0$)

edges (s,v) weight 0 $\forall v$

Let $\phi(s,v) = d_{G'}(s,v)$.

$$w'(u,v) = w(u,v) + \phi(u) - \phi(v)$$
$$= w(u,v) + d_{G'}(s,u) - d_{G'}(s,v) \overset{?}{\geq} 0$$
$$\Updownarrow$$
$$w(u,v) + d_{G'}(s,u) \geq d_{G'}(s,v)$$

If can compute $d_{G'}(s,u)$, then done. But this is neg-SSSP...

Suppose $w(u,v) \geq -2B$ for some integer B.

Goal: obtain $w'(u,v) \geq -B$.

(Repeat $O(\log n)$ times until $w'(u,v) \geq 0$.)

Let G^{+B} = graph with $+B$ added to each edge weight.
and vertex s with (s,v) of weight 0 $\forall v$

Main algorithm: SSSP on G^{+B}

Negative cycle in $G \iff$ cycle in G^{+B} with average weight $< B$.

So "easier".

So "easier":

v

Let $\phi(s,v) = d_{G+B}(s,v)$.

$$w'(u,v) = w(u,v) + \phi(u) - \phi(v)$$
$$= w(u,v) + d_{G+B}(s,u) - d_{G+B}(s,v) \overset{?}{\geq} -B$$
$$\Updownarrow$$
$$\underbrace{w(u,v)+B}_{= w_{G+B}(u,v)} + d_{G+B}(s,u) \overset{?}{\geq} d_{G+}(s,v)$$

Warm-up

Two "Base Cases"

① Graph is a DAG:

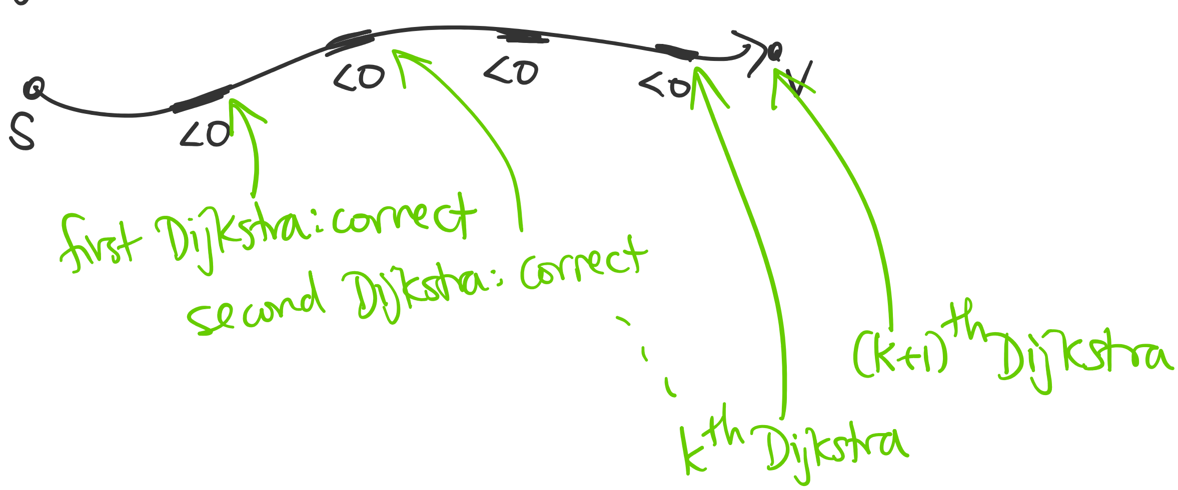
Topological sort $s = v_1 < v_2 < \cdots < v_n$ s.t.

$u < v$ for each arc (u,v).

$$d(s,v_i) = \min_{(u,v_i)} d(s,u) + w(u,v)$$

② All Shortest paths have $\leq k$ negative edges

Run Dijkstra $k+1$ times (HW1 problem)



Problem: Given graph $G=(V,E)$ with weights $\geq -B$ and source s with edges (s,v) weight 0 $\forall v \in V$,

① compute $d(s,v)$ $\forall v \in V$, or

② find a cycle with mean weight $< B$

Given input (G,s), let $K(G) =$ max # edges of any $s \to v$ path of weight ≤ 0

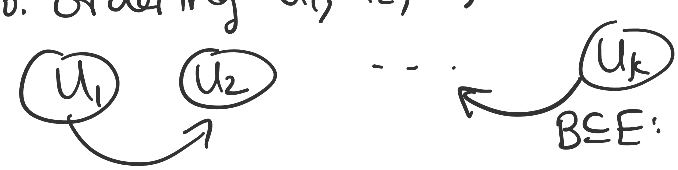
Algo: ① Decompose:

Partition V into

① small clusters with $\leq |V|/2$ vertices

② large cluster U_{large} with $K(G[U]) \leq K(G)/2$

③ Topo. ordering $U_1, U_2, \ldots, U_k$ of clusters s.t.



$B \subseteq E$: incorrect

$D \subseteq E$: intercluster edges in correct direction

for all $v \in V$, shortest s–v path has $O(\log n)$ edges in B in expectation.

② Recurse on clusters

Recursively solve $(G[U], s)$ for each cluster U, get $d_U(s,v)$.

For each U, let $\phi(v) = d(s,v)$ for all $v \in U$.

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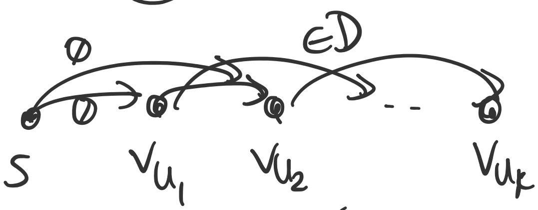
For each U, let $\phi(v) = d(s,v)$ for all $v \in U$.
Reweight $w'(u,v) = w(u,v) + \phi_U(u) - \phi_U(v) \geq 0$ $\forall (u,v) \in E$
Assert: all edges within clusters have weight ≥ 0

③ Fix DAG edges

Starting with subgraph $D \subseteq E$, contract each cluster U into v_U. U_1 U_2 -- U_k



Compute SSSP on DAG (with source s), get $d(s, v_U)$
For each cluster U, set $\phi(v) = d(s, v_U)$ $\forall v \in U$
Reweight $w'(u,v) = w(u,v) + \phi(u) - \phi(v)$ $\forall (u,v) \in E$
Assert: all edges $E \setminus B$ have weight ≥ 0

④ Fix bad edges B

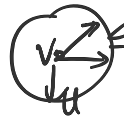
Solve SSSP when $\forall v \in V$, shortest s–v path has $O(\log n)$ edges in expectation.

⑤ Undo reweightings.

The Decomposition

- Let $G_{\geq 0}$ be the graph G with all negative weights increased to $\emptyset$.
- While $\exists v$ s.t. in-ball  has $\leq \frac{|V|}{2}$ vertices

$\{u : d_{G_{\geq 0}}(u,v) \leq \frac{kB}{4}\}$

out-ball  $\le \frac{kB}{4}$ has $\le \frac{|V|}{2}$ vertices

$\{u : d_{G_{\ge 0}}(v,u) \le \frac{kB}{4}\}$

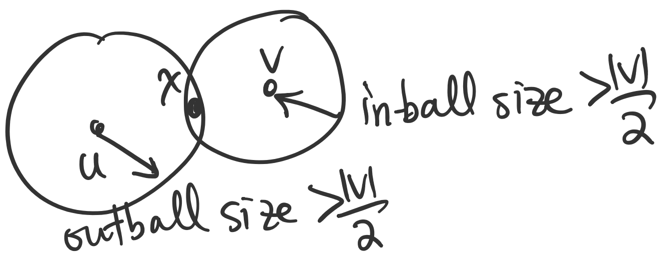
Add in/out boundary to B (bad),
out/in boundary to D (DAG)

Randomly sample $R \in \mathrm{Geom}(\Theta(\frac{\log n}{KB}))$ and take in/out-ball of radius R as new cluster. w.h.p. $R \le \frac{kB}{4}$.

- If vertices remain, let U_{large} be remainder.

Claim 1: $\forall u, v \in U_{large}$, $d(u,v) \le KB/2$
$d(v,u) \le KB/2$

Proof:



$\exists$ common vertex x.

So $d(u,v) \le d(u,x) + d(x,v) \le \frac{KB}{4} + \frac{KB}{4} = \frac{KB}{2}$.

This is in $G_{\ge 0}$, distances in G can only be smaller.

Similar for $d(v,u)$.

Claim 2: If no cycle of mean weight $<B$, then $\kappa(G[U_{large}]) \le \kappa(G)/2$.

Proof: Suppose for contradiction that $\exists s$–v path of weight ≤ 0 with $> \kappa(G)/2$ edges.

Let u be first edge. Then $u \to v$ path has weight ≤ 0

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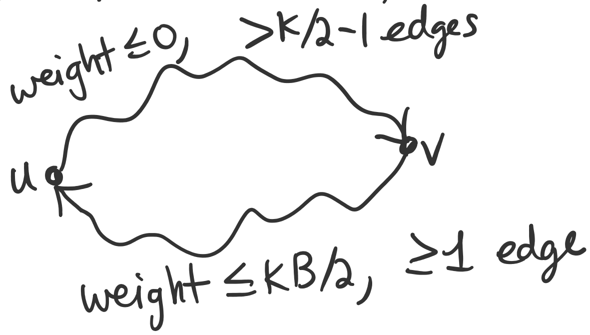
and $> k/2 - 1$ edges.

Combine with $v \to u$ path of weight $\leq kB/2$.



$\Rightarrow$ cycle weight $\leq kB/2$, $> k/2$ edges

$\Rightarrow$ mean weight $< B$, contradiction.

Claim 3: Any $s - v$ path of weight ≤ 0 has $O(\log n)$ non-DAG edges in expectation.

Proof: Since edge weights $\geq -B$ in G, they increase by $\leq B$ in $G_{\geq 0}$. There are $\leq k$ edges, so weight of path in $G_{\geq 0}$ is $\leq kB$. LDD analysis: $O(\log n)$ cut edges in expectation.