Max-Flow from Online Experts / Multiplicative Weight Update

Max Flow

Input: directed uncapacitated graph $G=(V,E)$ and $s,t \in V$

Output: $(1-\varepsilon)$-approximate maximum s-t flow.

LP formulation: Let $\mathcal{P}$ denote all (simple) s-t paths.

Primal

$$\max \sum_{P \in \mathcal{P}} f_P$$

$$\text{s.t.} \quad \sum_{P \ni e} f_P \le 1 \qquad \forall e \in E$$

$$f_P \ge 0 \qquad \forall P \in \mathcal{P}$$

Dual

$$\min \sum_{e \in E} l_e$$

$$\text{s.t.} \quad \sum_{e \in P} l_e \le 1 \qquad \forall P \in \mathcal{P}$$

$$l_e \ge 0 \qquad \forall e \in E$$

Let $F^* \ge 1$ be the LP optimum.

Goal: find feasible $\vec{f}$ s.t. $\sum_{P \in \mathcal{P}} f_P \ge (1-\varepsilon) F^*$.

Algorithm

① Initialize lengths $l_e^{(0)} \leftarrow \delta = m^{-1/\varepsilon}$

② For iteration $i = 1, 2, 3, \ldots$

ⓐ Compute a shortest s-t path $P^{(i)}$ under lengths l

ⓑ Update lengths $l_e^{(i)} \leftarrow l_e^{(i-1)} \cdot (1+\varepsilon)$ if $e \in P^{(i)}$

$l_e^{(i)} \leftarrow l_e^{(i-1)}$ otherwise.

...

(b) update ...

$$l_e^{(i)} \leftarrow l_e^{(i-1)} \quad \text{otherwise.}$$

(c) If $\sum_{e\in E} l_e^{(i)} < 1$, route $\frac{1}{\log_{(1+\varepsilon)}(1/\delta)}$ flow along path $P^{(i)}$

Else, terminate.

Analysis

Claim: the output flow is capacity-respecting.

Proof: Each time $e \in P^{(i)}$, we route $\frac{1}{\log_{(1+\varepsilon)}(1/\delta)}$ flow and increase its length $l_e^{(i)}$ by factor $(1+\varepsilon)$.

Since $\sum_{e\in E} l_e^{(i)} < 1$, we have $l_e^{(i)} < 1$ in particular.

So the number of times $e \in P^{(i)}$ is $\le \log_{(1+\varepsilon)}(1/\delta)$.

Claim: the output flow has value $\ge (1-\varepsilon) F^*$.

Proof:

Let $D(\vec{l}) = \sum_{e\in E} l_e$ be the value of the dual for (not necessarily feasible) $\vec{l}$.

Let $\alpha(\vec{l})$ = length of shortest s-t path under lengths l.

Then, for any $\vec{l}$, the scaled-down $\frac{\vec{l}}{\alpha(\vec{l})}$ is feasible with value $\frac{D(\vec{l})}{\alpha(\vec{l})}$,

so $\frac{D(\vec{l})}{\alpha(\vec{l})} \ge F^*$ for all $\vec{l}$.

Suppose we replace F* by the dual optimum. Then the same proof below works.

Initially, $D(\vec{l}^{(0)}) = m\delta$.

For each $i \ge 1$, $D(\vec{l}^{(i)}) = \sum_{e\in E} l_e^{(i)}$

$$= \sum_{e \in P^{(i)}} l_e^{(i)} + \sum_{e \notin P^{(i)}} l_e^{(i)}$$

$$= \sum_{e \in P^{(i)}} l_e^{(i)} + \sum_{e \notin P^{(i)}} l_e^{(i)}$$

$$= \sum_{e \in P^{(i)}} l_e^{(i-1)}(1+\varepsilon) + \sum_{e \notin P^{(i)}} l_e^{(i-1)}$$

$$= \sum_{e \in E} l_e^{(i-1)} + \varepsilon \sum_{e \in P^{(i)}} l_e^{(i-1)}$$

$$= D(\vec{l}^{(i-1)}) + \varepsilon\, \alpha(\vec{l}^{(i-1)})$$

$$\leq D(\vec{l}^{(i-1)}) + \varepsilon \cdot \frac{D(\vec{l}^{(i-1)})}{F^*}.$$

This is a bit wrong: should be (1+ε/F*)×D(...) here.
But the general idea is the same: use 1+x≤e^x at the right place.

So $D(\vec{l}^{(i)}) \leq \dfrac{D(\vec{l}^{(i-1)})}{1-\varepsilon/F^*} \leq \dfrac{D(\vec{l}^{(i-2)})}{(1-\varepsilon/F^*)^2} \leq \cdots \leq \dfrac{D(\vec{l}^{(0)})}{(1-\varepsilon/F^*)^i} = \dfrac{m\delta}{(1-\varepsilon/F^*)^i}.$

Let T be the first time $D(\vec{l}^{(T)}) = \sum_{e \in E} l_e^{(T)} \geq 1.$

$$1 \leq D(\vec{l}^{(T)}) \leq \frac{m\delta}{(1-\varepsilon/F)^T} \leq \frac{m\delta}{1-\varepsilon/F}\left(\frac{1}{1-\varepsilon/F}\right)^{T-1}$$

Should be F* not F

$$= \frac{m\delta}{1-\varepsilon/F}\left(\frac{F}{F-\varepsilon}\right)^{T-1}$$

$$= \frac{m\delta}{1-\varepsilon/F}\left(1+\frac{\varepsilon}{F-\varepsilon}\right)^{T-1}$$

$$\leq \frac{m\delta}{1-\varepsilon/F}\, e^{\frac{\varepsilon}{F-\varepsilon}(T-1)}$$

$$\leq \frac{m\delta}{1-\varepsilon}\, e^{\frac{\varepsilon}{F(1-\varepsilon)}(T-1)} \qquad [F \geq 1]$$

$$\Longrightarrow \frac{1-\varepsilon}{m\delta} \leq e^{\frac{\varepsilon}{F(1-\varepsilon)}(T-1)}$$

$$\longrightarrow \ln\left(\frac{1-\varepsilon}{\ldots}\right) \leq \frac{\varepsilon}{F(1-\varepsilon)} \cdot (T-1)$$

$$\Rightarrow \ln\left(\frac{1-\varepsilon}{m\delta}\right) \le \frac{\varepsilon}{F(1-\varepsilon)}\cdot(T-1)$$

$$\Rightarrow T-1 \ge \frac{F(1-\varepsilon)}{\varepsilon}\ln\left(\frac{1-\varepsilon}{m\delta}\right)$$

The output flow has value

$$\frac{T-1}{\log_{(1+\varepsilon)}(1/\delta)} \ge \frac{F(1-\varepsilon)}{\varepsilon}\cdot\frac{\ln\left(\frac{1-\varepsilon}{m\delta}\right)}{\log_{(1+\varepsilon)}(1/\delta)}$$

$$= \frac{F(1-\varepsilon)}{\cancel{\varepsilon}}\cdot\frac{\overbrace{\ln(1-\varepsilon)}^{\text{Small}} - \overbrace{\ln m}^{\text{Small}} - \overbrace{\ln(1/\delta)}^{\frac{1}{\varepsilon}\ln m}}{\underbrace{\ln(1/\delta)}_{\frac{1}{\varepsilon}\ln m} \,/\, \ln\cancel{(1+\varepsilon)}}$$

$$\ge F(1-O(\varepsilon)).$$

If we replace F* by the dual optimum, then we have constructed a flow of value ≥ (1-O(ε))×(dual opt).
Taking ε→0, we have proved strong duality of the flow LP!