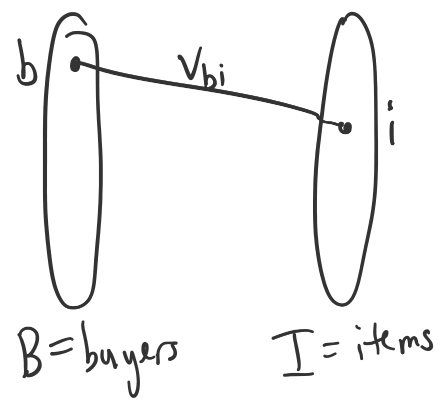
Weighted Bipartite Matching, the Hungarian algo, integrality of polytope.

Min/max-Weight Bipartite Perfect Matching

Observation: invariant under adding weights to edges, negating (= min weight)

Hungarian Algorithm



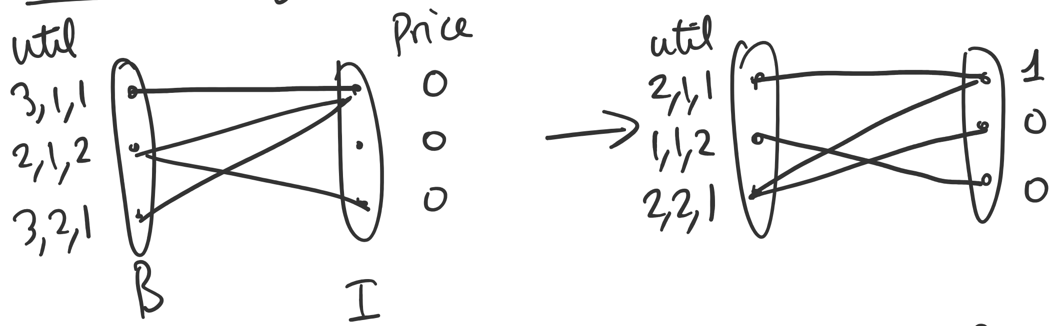
- Maintain <u>prices</u> $p = (p_1, \dots, p_n)$ of items.
- Define <u>utility</u> of item i to buyer b as

$$u_{bi}(p) = v_{bi} - p_i.$$

- Buyer b <u>prefers</u> item i if item i gives highest utility to buyer b.
- utility of buyer b is this highest utility

$$u_b(p) = \max_{i \in I} u_{bi}(p) = \max_{i \in I} (v_{bi} - p_i)$$

- <u>Preference graph</u>: edge (b,i) if buyer b prefers item i.



For any price vector p, if the preference graph contains a perfect matching M, then M is a max-value PM.
M is a max-weight PM in the "utility" graph

Proof: Clearly M is a max-weight PM in the "utility" graph

Proof: Clearly M is a max-weight PM in the utility graph with weights $u_{bi}(p)$.

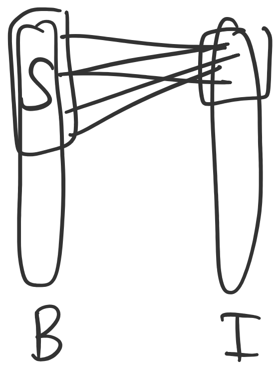
Max-weight PM in utility graph $\iff$
" " " " original graph

because any matching M' has value $\sum_{e\in M'} u_{bi}(p) = \sum_{e\in M'} (v_{bi} - p_i)$

$$= \sum_{e\in M'} v_{bi} + \underbrace{\sum_i p_i}_{\text{constant}}$$

What if no PM in preference graph?

By Hall's theorem, $\exists S$:



$N(S)$: $|N(S)| < |S|$.

Too many buyers prefer same set of items: raise their prices!

<u>Hungarian algorithm</u>

- Initialize prices to $\emptyset$.
- While $\nexists$ PM in pref. graph: (test by <u>unweighted</u> PM)
  - Compute $S \subseteq B$: $|N(S)| < |S|$
  - Raise price of each item in $N(S)$ by 1.

Potential function analysis: define $\Phi(p) = \sum_i p_i + \sum_b u_b(p)$.

Raise prices in $N(S)$ by 1: $\sum_i p_i$ increases by $|N(S)|$

$\sum u_b(p)$ decreases by $|S|$

$\sum_b u_b(p)$ decreases by $|S|$

So $\Phi(p)$ decreases by ≥ 1.

Raise price until pref graph changes: $O(n^3)$ iterations

Bipartite Perfect Matching Polytope

LP Relaxation of min-weight PM:

$$\max \sum_{bi} v_{bi} x_{bi}$$
$$\text{s.t. } \sum_b x_{bi} = 1 \ \forall i$$
$$\sum_i x_{bi} = 1 \ \forall b$$
$$x_{bi} \geq 0 \quad \forall b, i$$

Dual:

$$\min \sum_i p_i + \sum_b u_b$$
$$\text{s.t. } p_i + u_b \geq v_{bi} \ \forall b, i$$

Given p, u must satisfy

$$u_b \geq v_{bi} - p_i$$

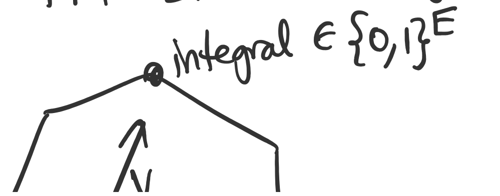
So optimal $u_b = \max_i (v_{bi} - p_i) = u_b(p)$.

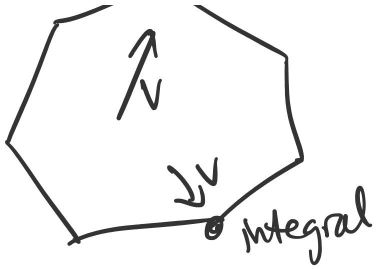
Hungarian algo outputs M with primal objective $\sum_{(b,i)\in M} v_{bi} x_{bi}$

and p, u with dual objective

$$\sum_i p_i + \sum_b u_b(p) = \sum_i p_i + \sum_{(b,i)\in M} (v_{bi} - p_i) = \sum_{(b,i)\in M} v_{bi}$$

So
- M is optimal primal and p, u are opt dual.
- min-weight PM LP is integral for any weights v.





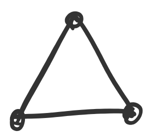
- So the min-weight PM polytope has integral extreme points, is the convex hull of all $\mathbb{1}_M \in \{0,1\}^E : M$ is PM.

General graphs

LP Relaxation: $\max \sum_e w_e x_e$

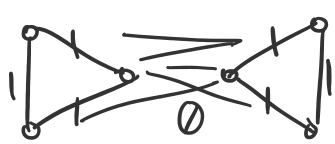
$$\text{s.t. } \sum_{e \sim v} x_e = 1 \quad \forall v \in V$$

$$x_e \geq 0 \quad \forall e \in E$$

Not integral! Consider 

$$w = (1,1,1), \quad x = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$$

$w \cdot x = \frac{3}{2}$, but no integral matching.



LP opt $= 3$, integral opt $= 2$

Let's add constraint $\sum_{e \in \partial S} x_e \geq 1 \quad \forall S \subseteq V : |S|$ odd

This polytope is integral! So solving LP $\Rightarrow$ opt PM.

This polytope is integral! So solving LP $\Rightarrow$ opt PM.

LP has exponential # constraints, but can solve fast using a separation oracle.