Wednesday, March 20

Last Time: Algorithmic Mechanism design, Sealed Bid Auction

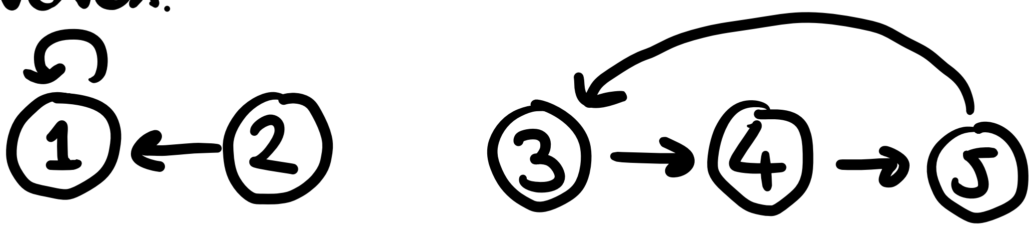
Today: Algorithms for Mechanism design w/o money.

(HAP) <u>House Allocation Problem</u> [Shapley & Scarf 1976]

- n agents, each owns a house
- every agent has a total (strict) ordering of the n houses
- q: how to reallocate houses so agents are better off?

Top Trading Cycle Algorithm (TTCA):

1. Construct a graph G s.t. each agent is a vertex.



2. while no agent remains:
   a) each remaining agent points at the owner of their favorite house
   b) find all directed cycles in G.
   c) reallocate houses as suggested by the directed cycles (each agent gives their house to the agent pointing at it)
   d) delete the agents whose houses were relocated on (c).

Claim: TTAC terminates in $O(n^2)$ time.
Proof: There will be at least one directed cycle at each round, removing at least one agent. $O(n)$ iterations, $O(n)$ time per iteration to find cycles.

Claim: TTCA is DSIC
By induction: Say N_j denote the agents whose house is relocated on iteration j.

- $i \in N_1$: everyone gets their first choice
- $i \in N_2$: everyone gets their first choice in $N \setminus N_1$. Since no agent at N_1 points at i, i cannot get a house from N_1 by misreporting
- $i \in N_j$: No agent in $N_1 \cup \ldots \cup N_{j-1}$ ever points at i, best i can do is to set its favorite remaining house.

↳ A mechanism that does nothing is also DSIC! $\square$

<u>Definition</u>: An allocation is a <u>core allocation</u> if no subset of agents can make all of its members strictly better off via internal reallocations.

<u>Claim</u>: TTCA produces a core allocation for the HAP.

<u>Proof</u>: Given any $S \subseteq N$ say $\ell = \min\{i : N_i \cap S \neq \emptyset\}$. Take $i \in S \cap N_\ell$,

since no agent in S belongs to $N_1 \ldots N_{k-1}$
no allocation from S can make i strictly
better off.

$\square$

Mechanism design w/o money
↳ Kidney exchanges

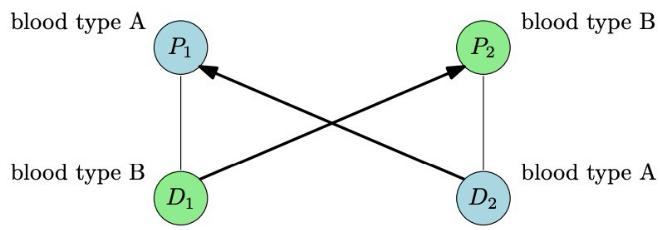


Figure 1: A kidney exchange

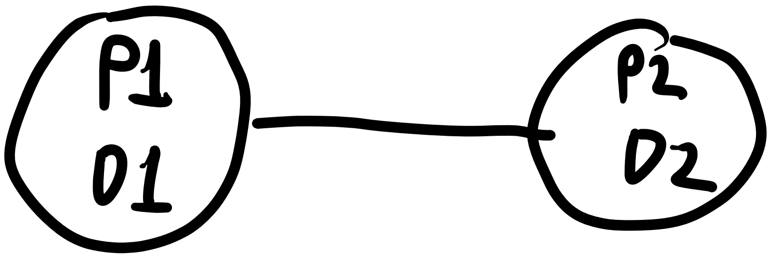
TTAC, where orders are done
according to compatibility probability

Challenge 1:
- Altruistic donors and donorless patients
  ↳ Change TTCA to include
    appropriate chains

Challenge 2:
- Long cycles make simultaneous
  surgeries difficult. → incentive problems
  if not sim.

Challenge 3:
- Binary more appropriate

⇒ Moving to <u>matching</u> (unweighted)



→ undirected edge iff mutually compatible

$G = (V, E)$ where $V = \{ i$ for each $(P_i, D_i) \}$

$(i, j) \in E$ iff

$P_i \to D_j$ ✓
$P_j \to D_i$

⇓

Restrict to two pair exchanges.

→ Each $i \in V$ has a "true set" E_i and can misreport any $F_i \subseteq E_i$ → e.g. by rejecting matchings

→ Goal: Maximal matching while preserving DSIC.

(1) Collect F_i

(2) Form edge set $E = \{(i,j) : (i,j) \in F_i \cap F_j\}$

(3) Return a max-cardinality matching of $G = (V, E)$ → how?

↳ OSIC depends on this

Say $\exists$ priority over patients $1, 2, \ldots, n$

↳ Then implement (3) as:

↓ e.g. greedy: bad

(3a) Let M_0 denote the set of maximum matchings

(3b) For $i = 1, 2, \ldots, n$:

$Z_i \subseteq M_{i-1}$ that match i

$$M_i = \begin{cases} Z_i & \text{if } Z_i \neq \emptyset \\ M_{i-1} & \text{otw} \end{cases}$$

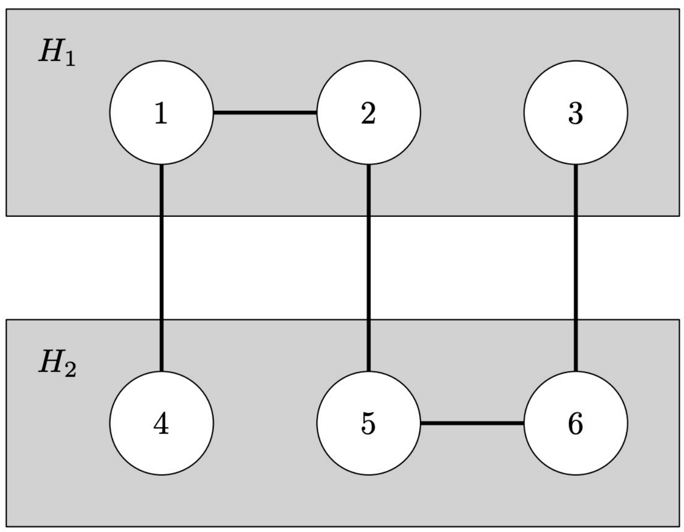
(3c) Return a matching from M_n

↳ Always matches the same set of vertices

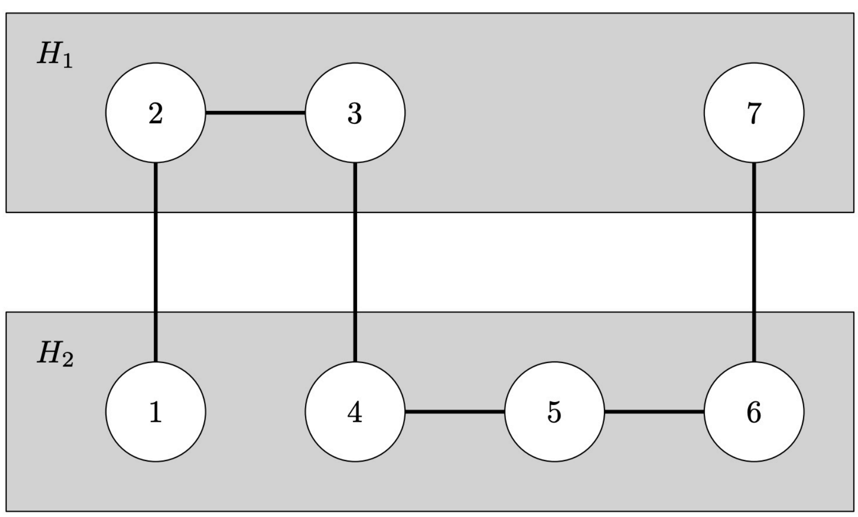
Thm $\forall$ $\{E_i\}_{i=1}^{n}$ of edge sets and every ordering of vertices, the algo above is DSIC.

What about incentives at the hospital level?

↳ Goal of hospital: Maximize its # of matched patients



Underreporting hurts



H_1 has incentive to hide 2&3 whereas H_2 has incentive to hide 5&6

Active research topic: matching patients in a way that approximately maximizes the patients for students.

Stable Matching → what if both sides have preferences?

Two sets U and V "men" and "women"
"hospitals" and "residents"

$\forall u \in U$ $\exists$ total ordering of V
and vice versa

Definition A stable matching is a bipartite matching s.t. $\nexists$ "blocking pair", i.e.
if $u \in U$ & $v \in V$ are not matched, at least one prefer their match to the other.
↳ care allocation!

Gale-Shapley Algo [1962]

while $\exists$ unattached man $u \in U$

- u propose to his favorite woman who has not rejected him yet.
- each woman $v \in V$ only entertains her best offer thus far

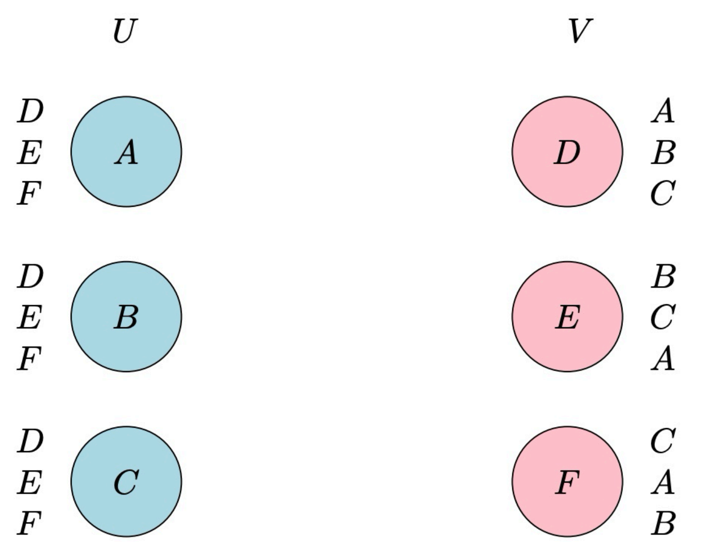


Figure 9: An instance of stable matching

Thm GS terminates after n^2 iterations w/ a stable matching

Corollary $\forall$ collections, $\exists$ stable matching

Proof of Thm 1) Each man at most makes n proposals.

2) If a man is unmatched, all women rejected him ⇒ all women matched ⇒ all men matched ⇒ perfect matching.

3) Stable, given any u, v not matched, two cases: either u never proposed or he did..

1) u never proposed ⇒ u has a match he prefers to v

2) u proposed ⇒ v eventually rejected/left u for some u' she prefers

in either case, either u or v has a match they are happier with. □

Claim: G-S assigns each $u \in U$ their fav. $v \in V$ in any stable matching, and each $v \in V$, their least fav $u \in U$ in any stable matching.

Corollary: G-S is DSIC for U but not for V.