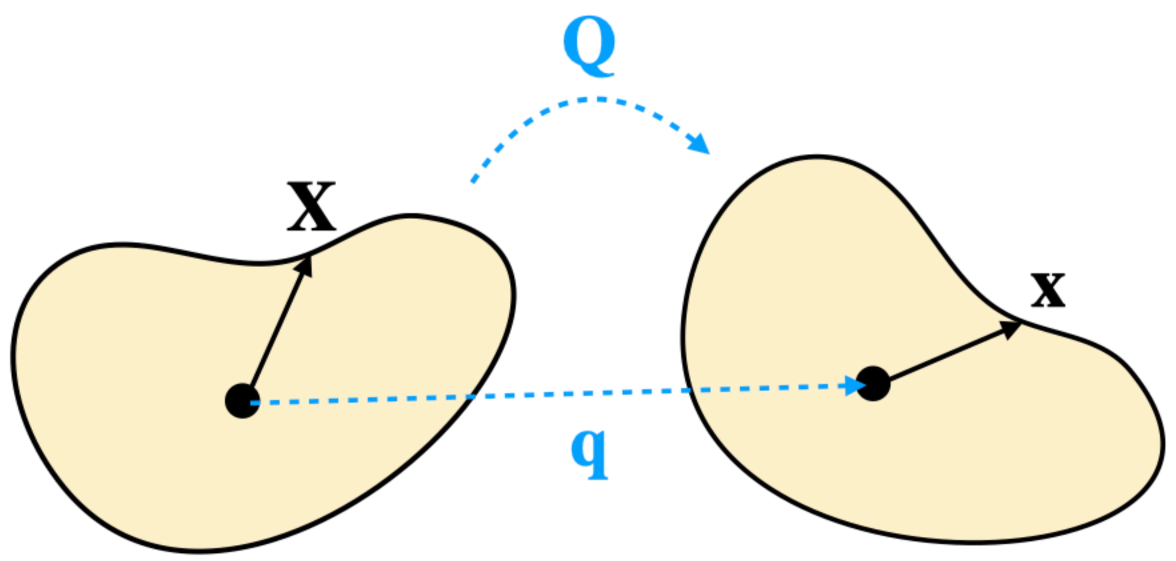
Instructor: Minchen Li

Lec 13: Codimensional Solids

15-763: Physics-based Animation of Solids and Fluids (S25)

Recap: Rigid Body Dynamics



$\mathbf{x} = \mathbf{QX} + \mathbf{q}$

Reduced order DOF:

$$\mathbf{x} = \mathbf{QX} + \mathbf{q} \in \mathbb{R}^3 \iff \begin{array}{l} x = \bar{X}Q + \bar{S}q \in \mathbb{R}^{3n} \\ Q \in \mathbb{R}^{9m},\ \bar{X} \in \mathbb{R}^{3n\times 9m} \\ q \in \mathbb{R}^{3m},\ \bar{S} \in \mathbb{R}^{3n\times 3m} \end{array}$$

Reduced order dynamics (from subspace optimization):

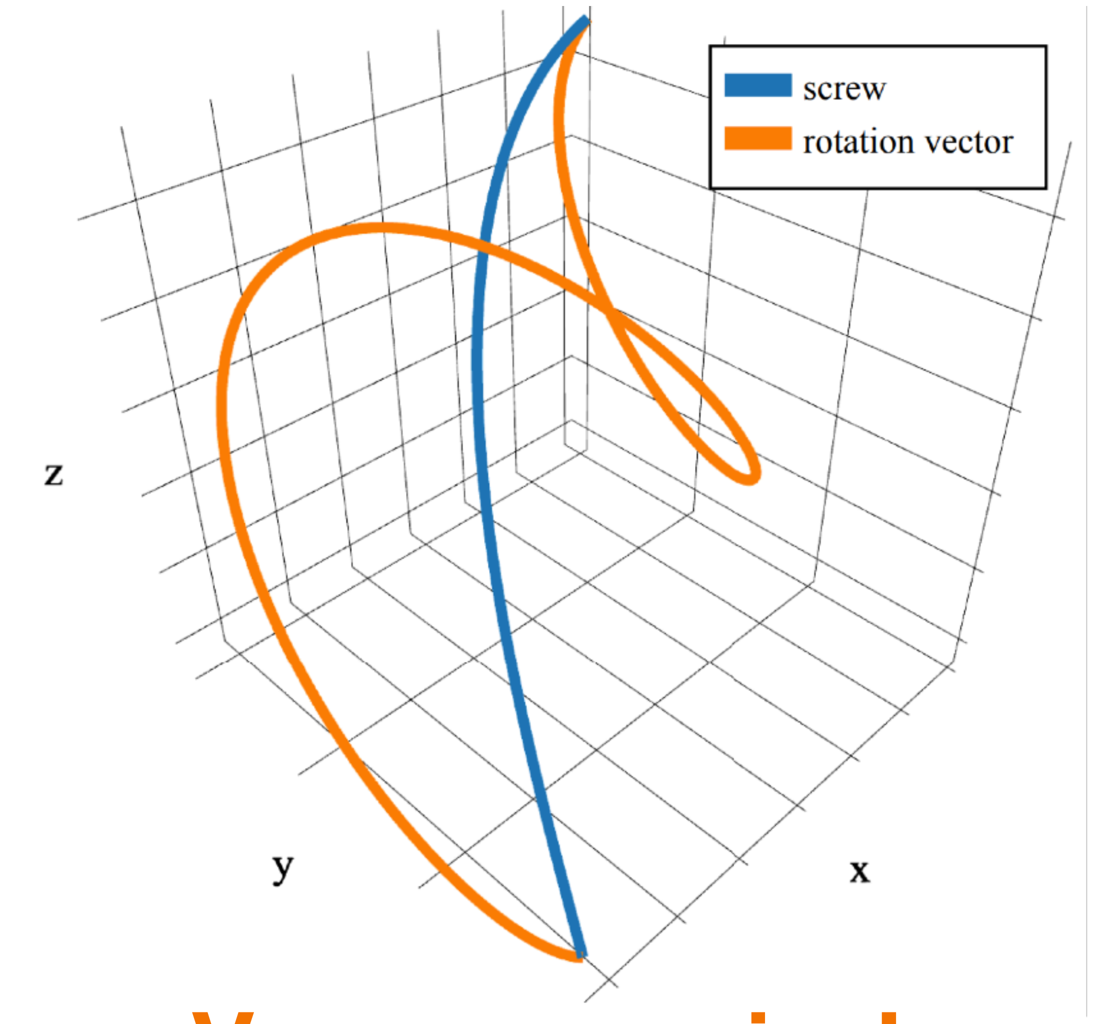
$$\min_{\theta,q} \frac{1}{2}\|\bar{X}R(\theta) + \bar{S}q - \tilde{x}^n\|_M^2 + h^2 \sum P(\bar{X}R(\theta) + \bar{S}q)$$

Rodrigues' Rotation Formula:

$$\mathcal{R}(\boldsymbol{\theta}) = \mathrm{Id} + \sin(\|\boldsymbol{\theta}\|)\left[\frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}\right] + (1 - \cos(\|\boldsymbol{\theta}\|))\left[\frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|}\right]^2$$

Just include IPC energies here

CCD is on nonlinear trajectories:



Very expensive!

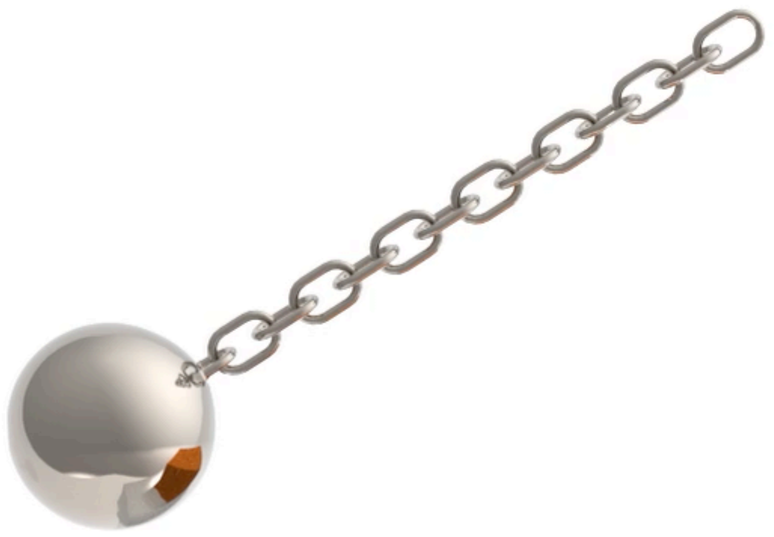
Zachary Ferguson, Minchen Li, Teseo Schneider, Francisca Gil-Ureta, Timothy Langlois,Chenfanfu Jiang, Denis Zorin, Danny M. Kaufman, Daniele Panozzo. Intersection-free Rigid Body Dynamics. SIGGRAPH 2021.

Recap: Affine Body Dynamics (ABD)

Reduced order dynamics with penalty method:

$$\min_{Q,q} \frac{1}{2}\|\bar{X}Q + \bar{S}q - \tilde{x}^n\|_M^2 + h^2 \boxed{\sum P(\bar{X}Q + \bar{S}q)}$$

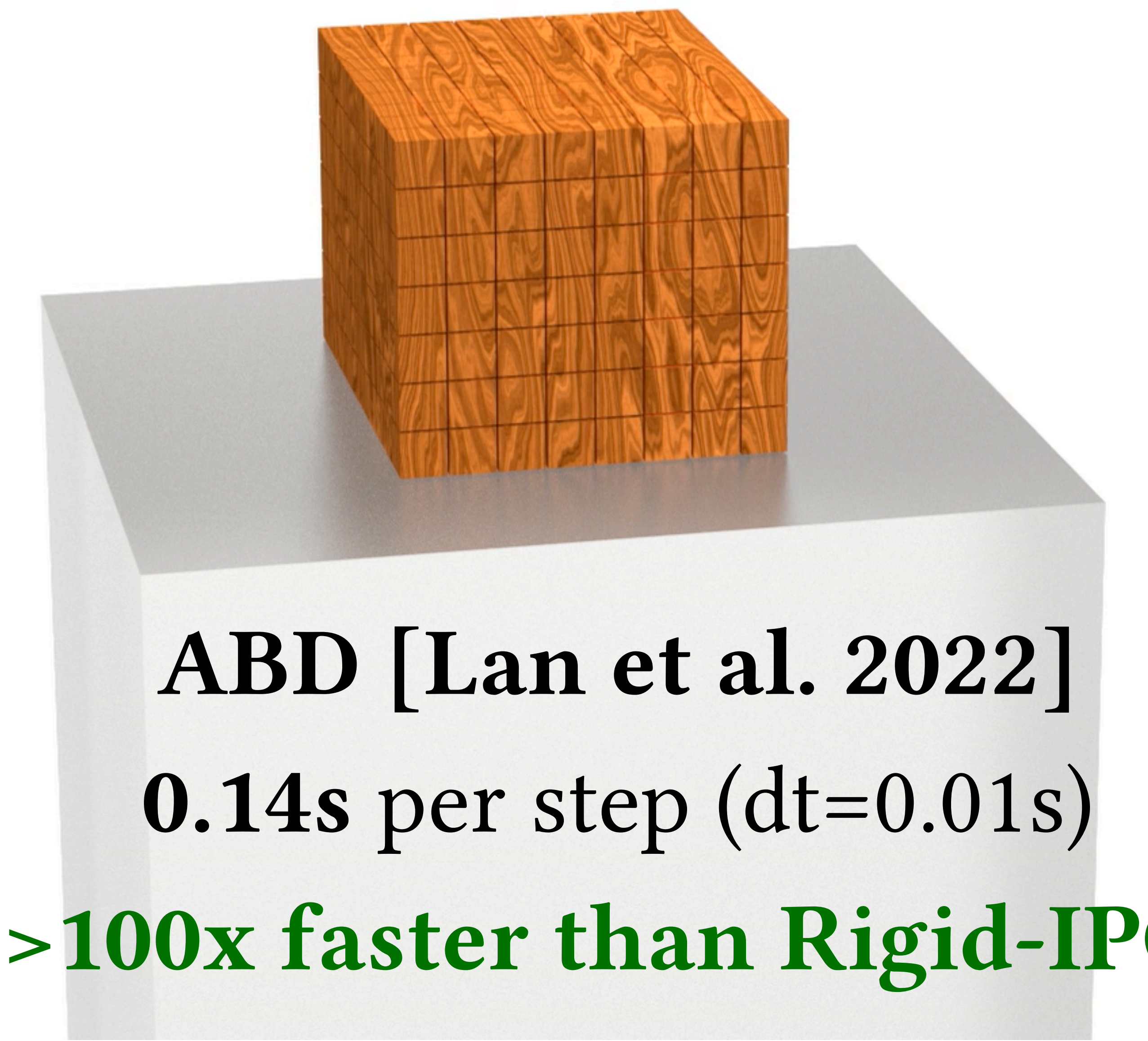
Use elasticity with large Young's modulus



12 DOF per body, still significantly reduced

$x = \bar{X}Q + \bar{S}q$ is linear w.r.t. both Q and q -> linear CCD

A stiff Ψ won't make the problem harder with stiff IPC energies



Lei Lan, Danny M. Kaufman, Minchen Li, Chenfanfu Jiang, Yin Yang. Affine Body Dynamics: Fast, Stable & Intersection-free Simulation of Stiff Materials. SIGGRAPH 2022.

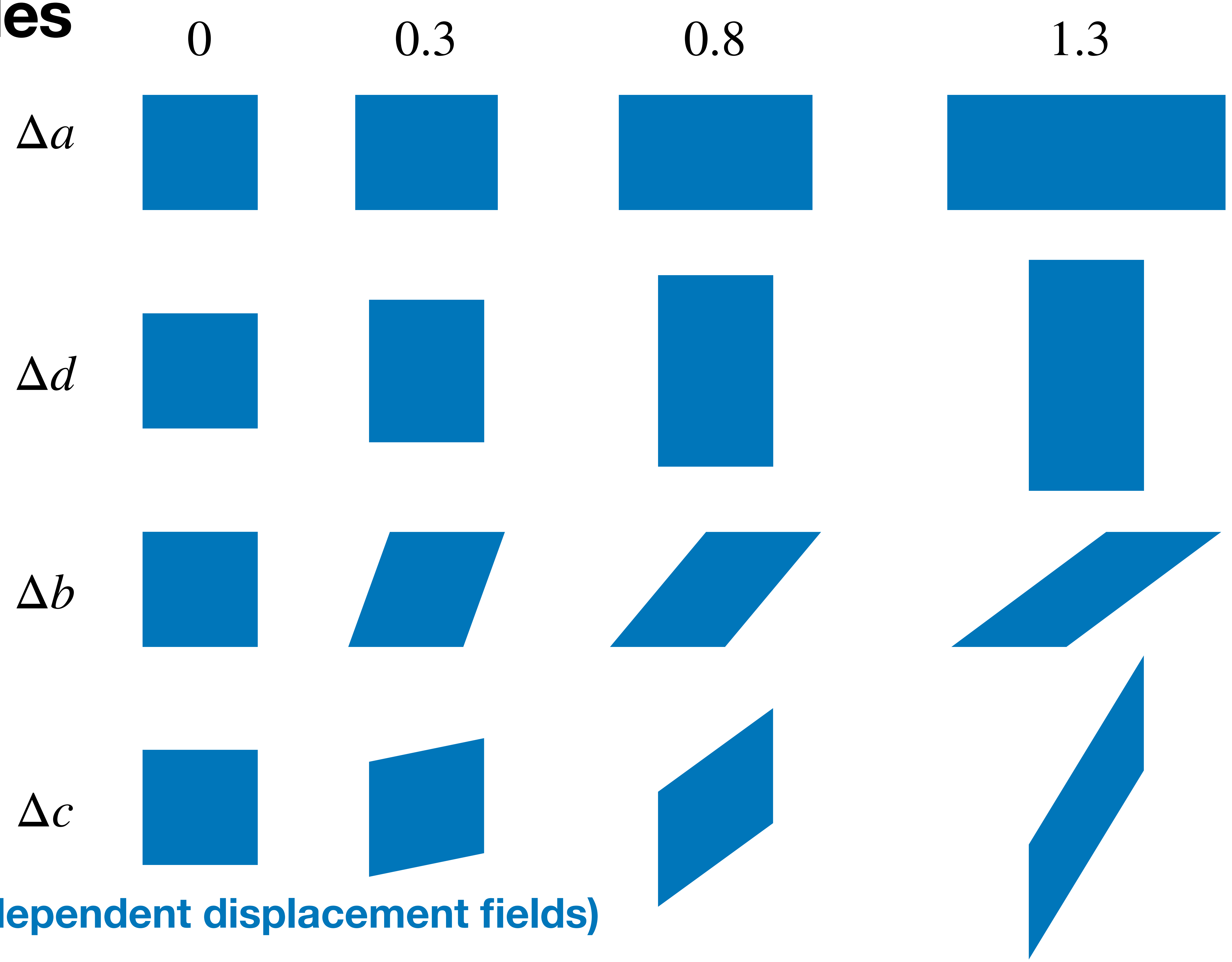
Recap: ABD in Another Perspective

Affine Deformation Modes

$$\mathbf{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{X} + \begin{bmatrix} e \\ f \end{bmatrix}$$

DOF: a, b, c, d, e, f

X: 

x: 

$$\mathbf{x} = A \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = aA_1 + bA_2 + \dots$$

Deformation modes (linearly independent displacement fields)

Recap: Reduced Order Model

Linear Modal Analysis

Assume linear elasticity problem: $M\ddot{u} + Ku = f$ **s.t.** $Sx = 0$ **(Dirichlet BC)**

Intuition: Meaningful deformation modes are those don't generate large forces

Can solve the generalized Eigenvalue problem to find them: $\bar{K}y = \lambda \bar{M}y$

(where $\bar{K}$ and $\bar{M}$ do not account for BC nodes)

(Take the Eigenvectors with smallest Eigenvalues as modes.)

Let $u = x - X = Uz$, where $z \in \mathbf{R}^k$ are the reduced DOF, $U \in \mathbb{R}^{3n\times k}$ formed by the Eigenvectors,

Plugging in $M\ddot{u} + Ku = f$, ignoring BCs for now: $\ddot{z} + \Lambda z = U^T f$ **Diagonal system! Super fast!**

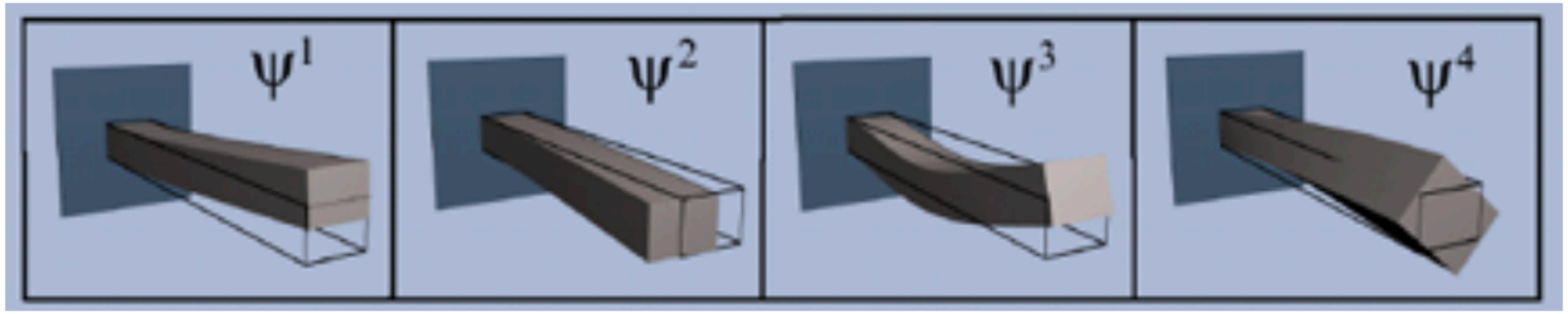


Figure 2: **Linear modes** for a cantilever beam.

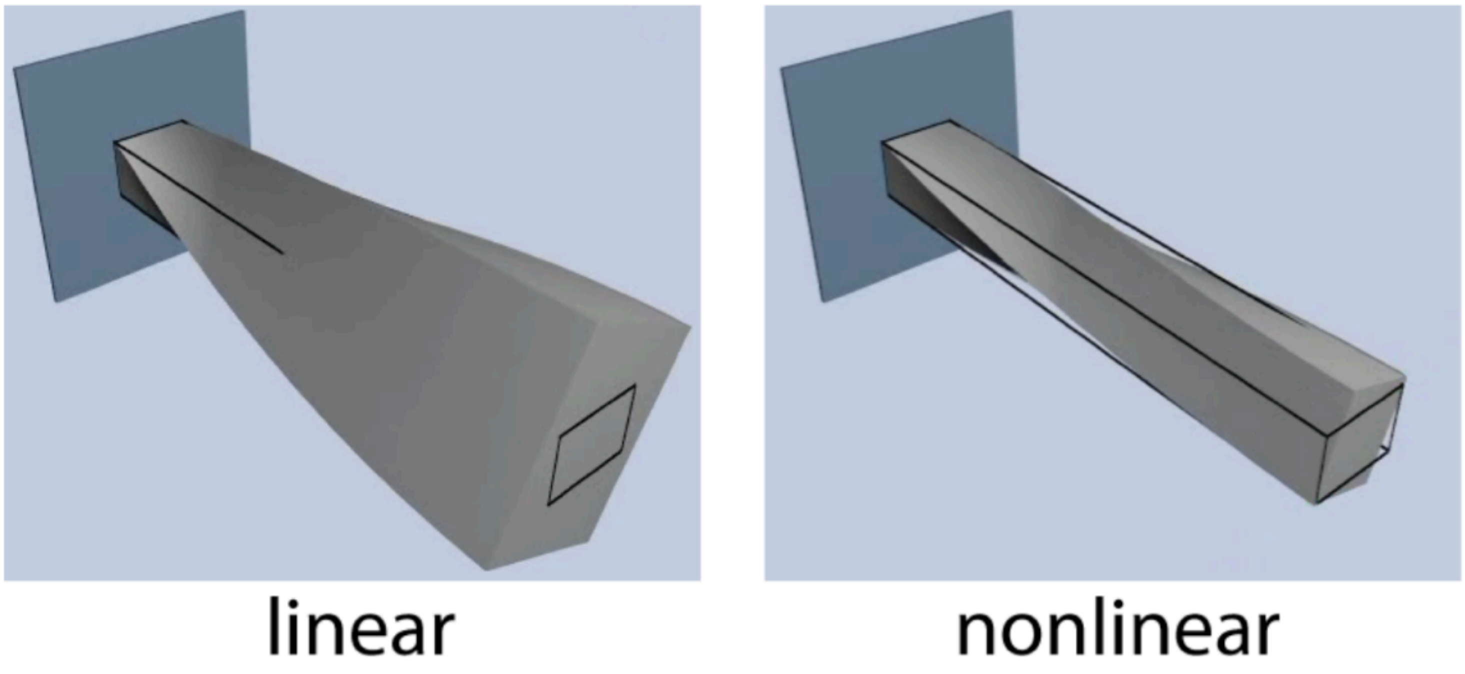


Figure 3: **Model reduction applied to a linear and nonlinear system.**

Recap: Reduced Order Model

Nonlinear Elasticity

Plugging in $u = Uz$: $\min_z \frac{1}{2}\|X + Uz - \tilde{x}^n\|_M^2 + h^2 \sum P(X + Uz)$

Gradient: $U^T M(X + Uz - \tilde{x}^n) + h^2 \sum U^T \nabla P(X + Uz)$

Hessian: $U^T MU + h^2 \sum U^T \nabla^2 P(X + Uz)U$

Can use numerical integration to approximate Gradient and Hessian, minimizing the number of quadratures [An et al. 2008]

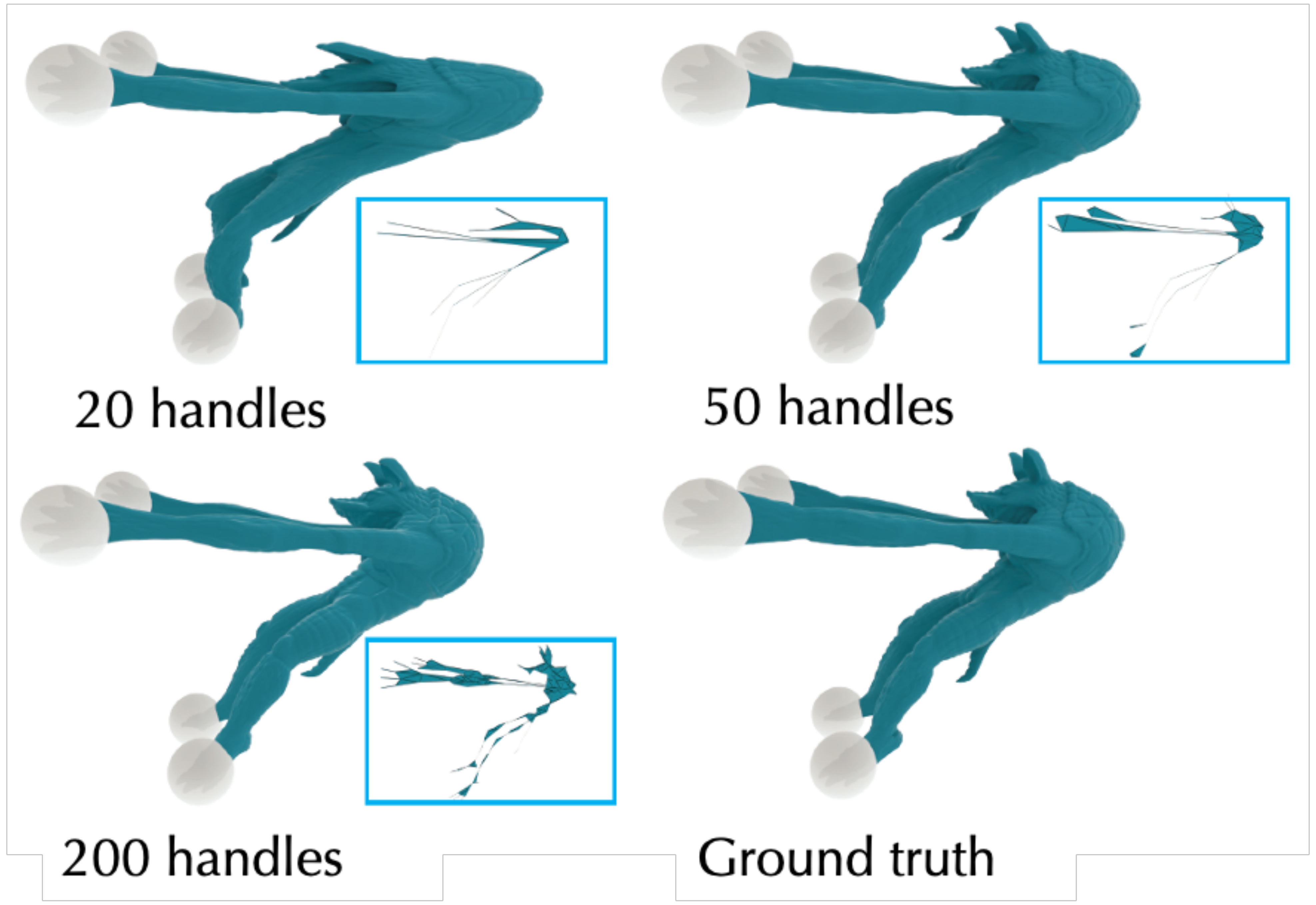
To design better deformation modes:

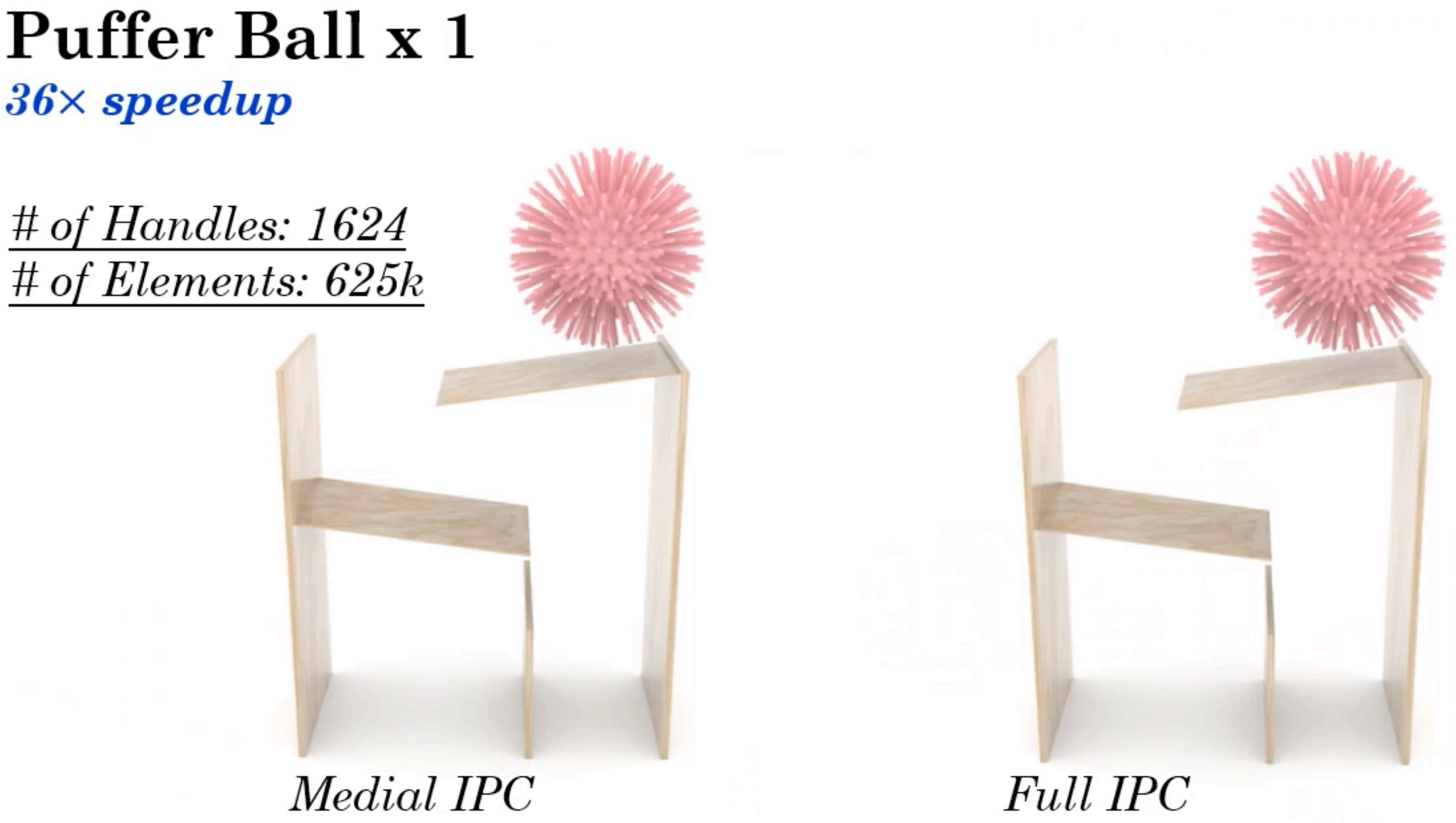
- **Use locally supported modes for sparsity, e.g. Medial Axis Mesh [Lan et al. 2021]**
- **Use simulated poses/deformed configurations as data, and perform PCA**
- **Use modal derivatives to construct a quadratic function $u = f(z)$ [*]**
- **Use neural networks to learn $u = f(z)$**

*Jernej Barbič, Doug James. Real-Time Subspace Integration for St.Venant-Kirchhoff Deformable Models. SIGGRAPH 2005.

Results: Reduced Simulation of Deformable Solids

Medial IPC [Lan et al. 2021]





Lei Lan, Yin Yang, Danny M. Kaufman, Junfeng Yao, Minchen Li, Chenfanfu Jiang. Medial IPC: Accelerated Incremental Potential Contact With Medial Elastics. SIGGRAPH 2021.

Topics Today:

- Thin Shells
  - Stretching and Bending
  - Strain Limiting and Thickness
- Rods and Particles

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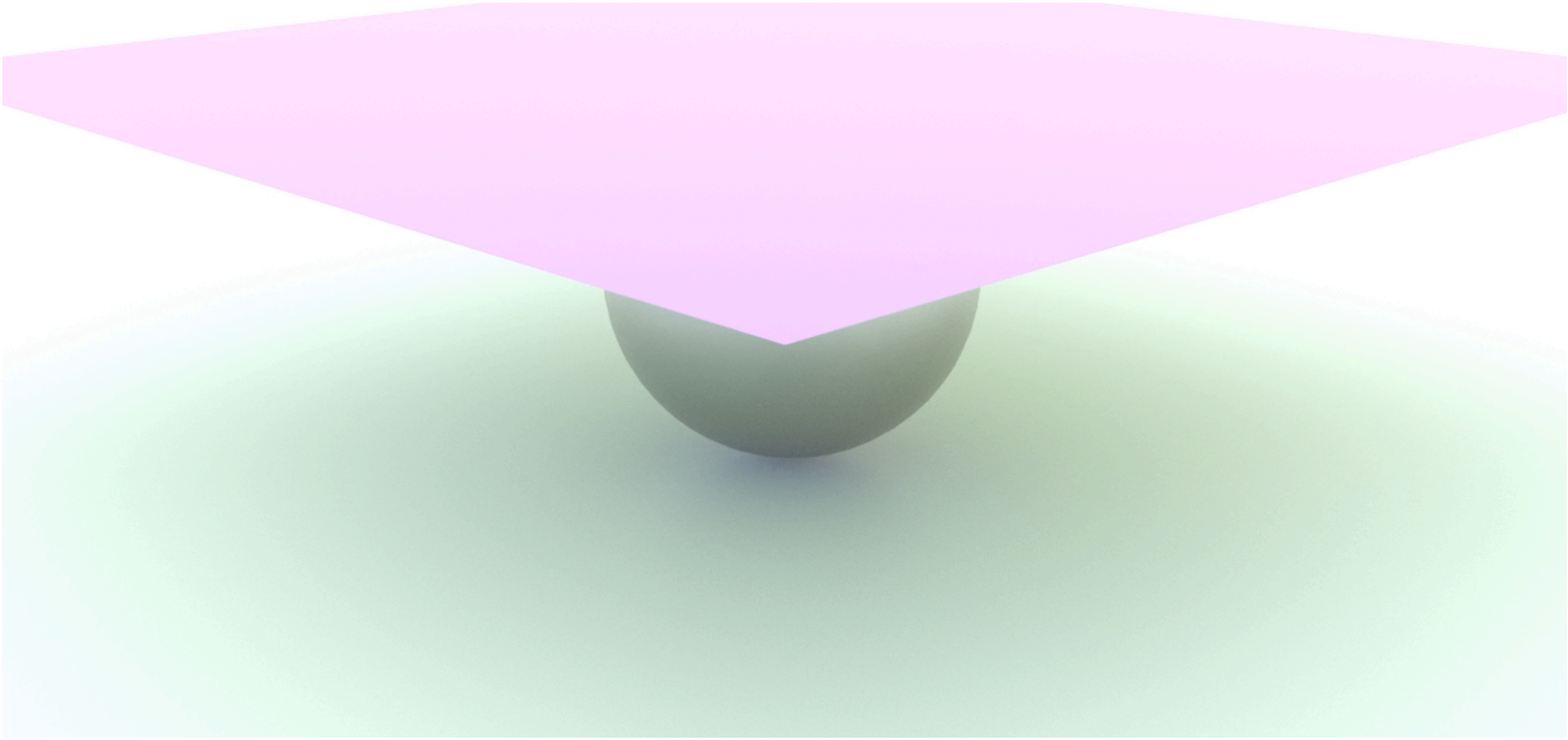
Simulating Thin Objects

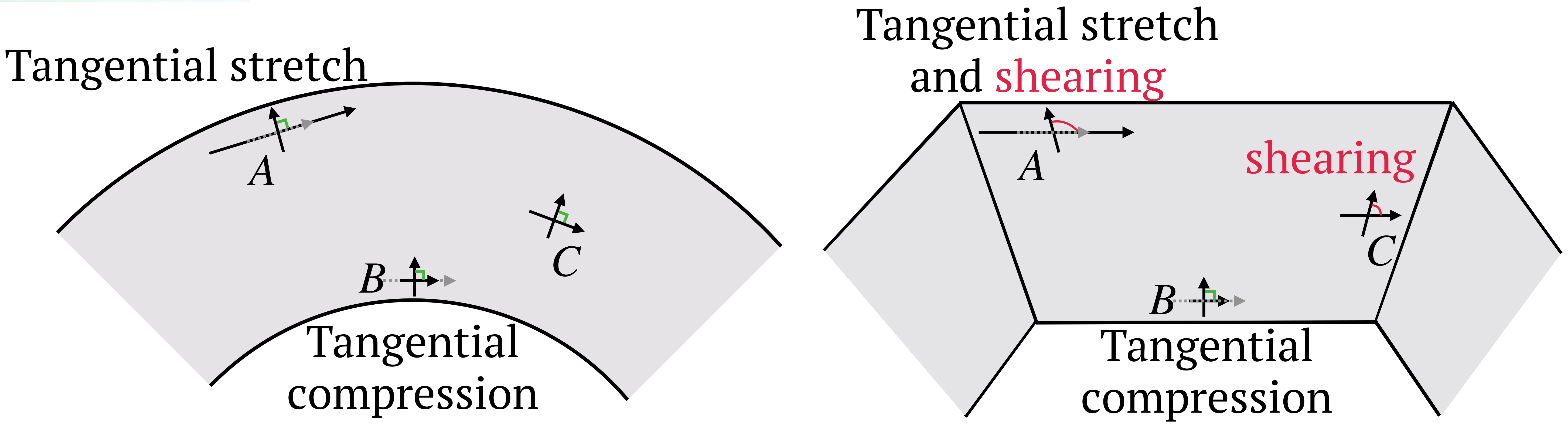
Using Volumetric Meshes?



Ill-conditioning:

Shear locking issue (linear shape functions):

Higher-order shape functions are expensive

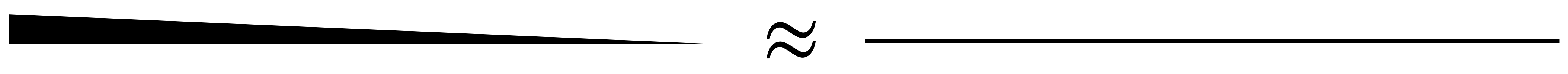


(a) Ideal Setting

(b) Coarse Tessellation with Linear Basis

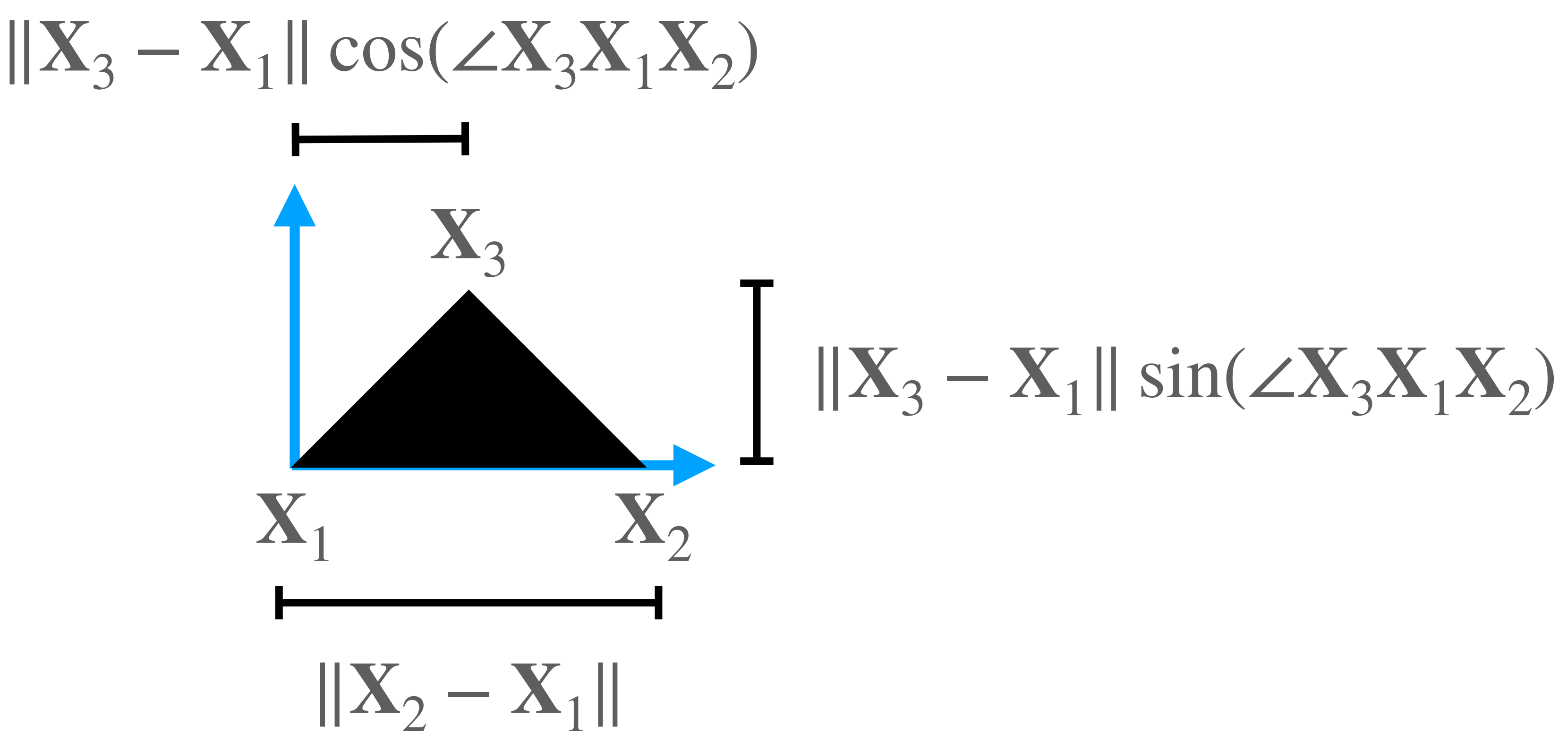
Simulating Thin Shells using Surface Meshes

Tangent Space Elasticity

Thickness is tiny and barely visible -> can ignore it: 

Use surface meshes, e.g. triangle meshes, to simulate thin shells, e.g. cloth, paper, etc

$$F = \begin{bmatrix} x_2 - x_1, x_3 - x_1 \end{bmatrix} \begin{bmatrix} ||X_2 - X_1|| & \frac{(X_3 - X_1)\cdot(X_2 - X_1)}{||X_2 - X_1||} \\ 0 & \frac{||(X_3 - X_1)\times(X_2 - X_1)||}{||X_2 - X_1||} \end{bmatrix}^{-1} \in R^{3\times 2}$$



$$\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$$

$$\hat{\Psi}_{\mathrm{NH}}(\boldsymbol{\Sigma}) = \frac{\mu}{2}\left(\sum_i^d \sigma_i^2 - d\right) - \mu \ln(J) + \frac{\lambda}{2}\ln^2(J)$$

$$\sum_i V_i \Psi(\mathbf{F}_i) \qquad V_i = \xi A_i$$

(ξ: Thickness; A_i: Triangle area)

Lame parameter λ is computed differently here for shells as the thickness approaches zero:

$$\lambda = \frac{E\nu}{1-\nu^2}$$ **(Plane stress approximation)**

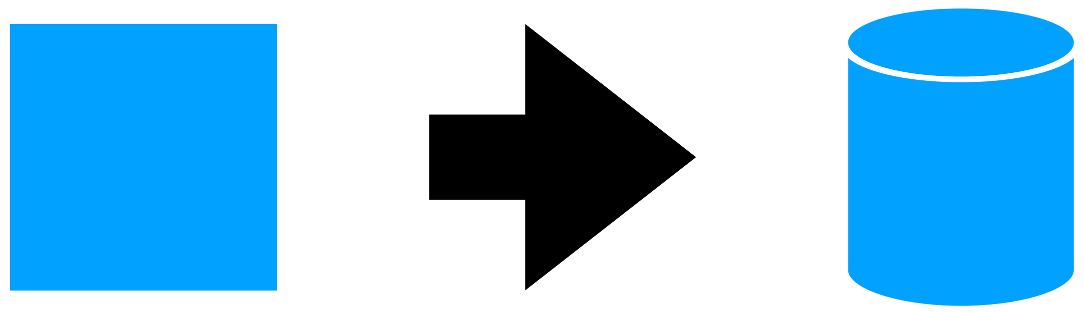
$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$ **(Volumetric)**

***May also use The First Fundamental Form to compute Ψ.**

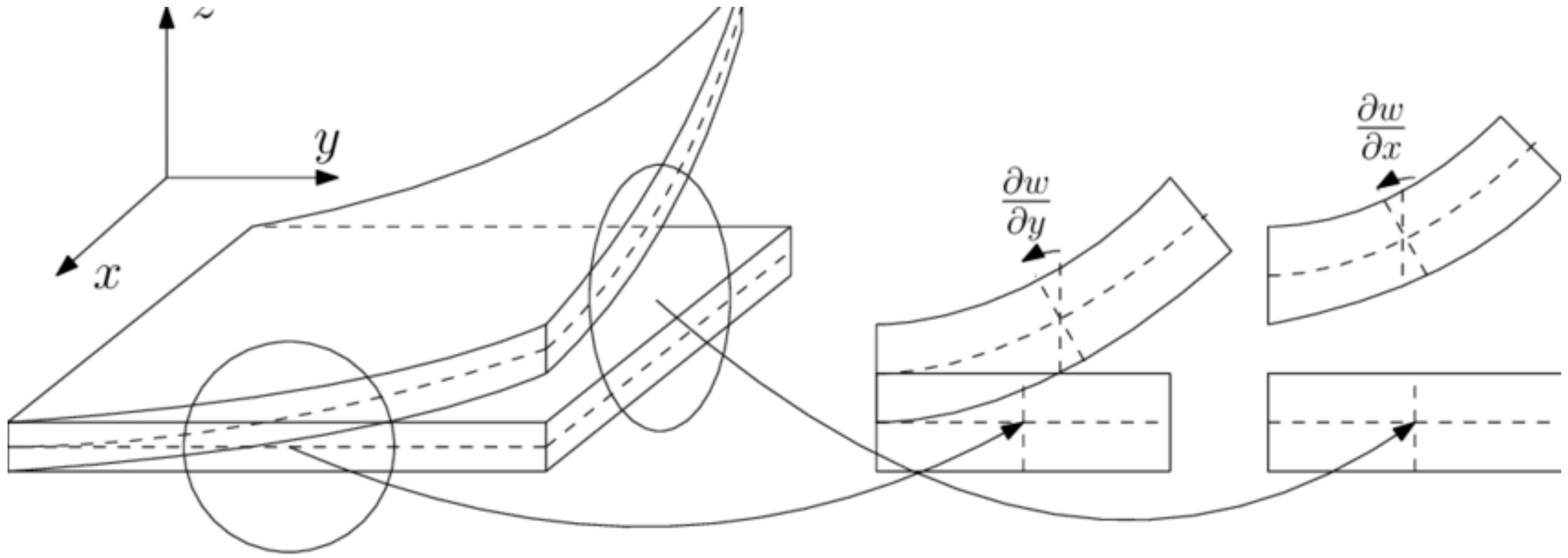
Simulating Thin Shells using Surface Meshes

Bending

With only tangent space elasticity, no force under isometric deformation:



Need to additionally model bending resistance!



The Kirchoff-Love Model

Deriving strain energies for bending from continuum mechanics, based on assumptions:

- **straight lines normal to the mid-surface remain straight and normal to the mid-surface after deformation**
- **the thickness of the plate does not change during a deformation.**

Simulating Thin Shells using Surface Meshes

Discrete Shell — Hinge Bending Model [Grinspun et al. 2003]

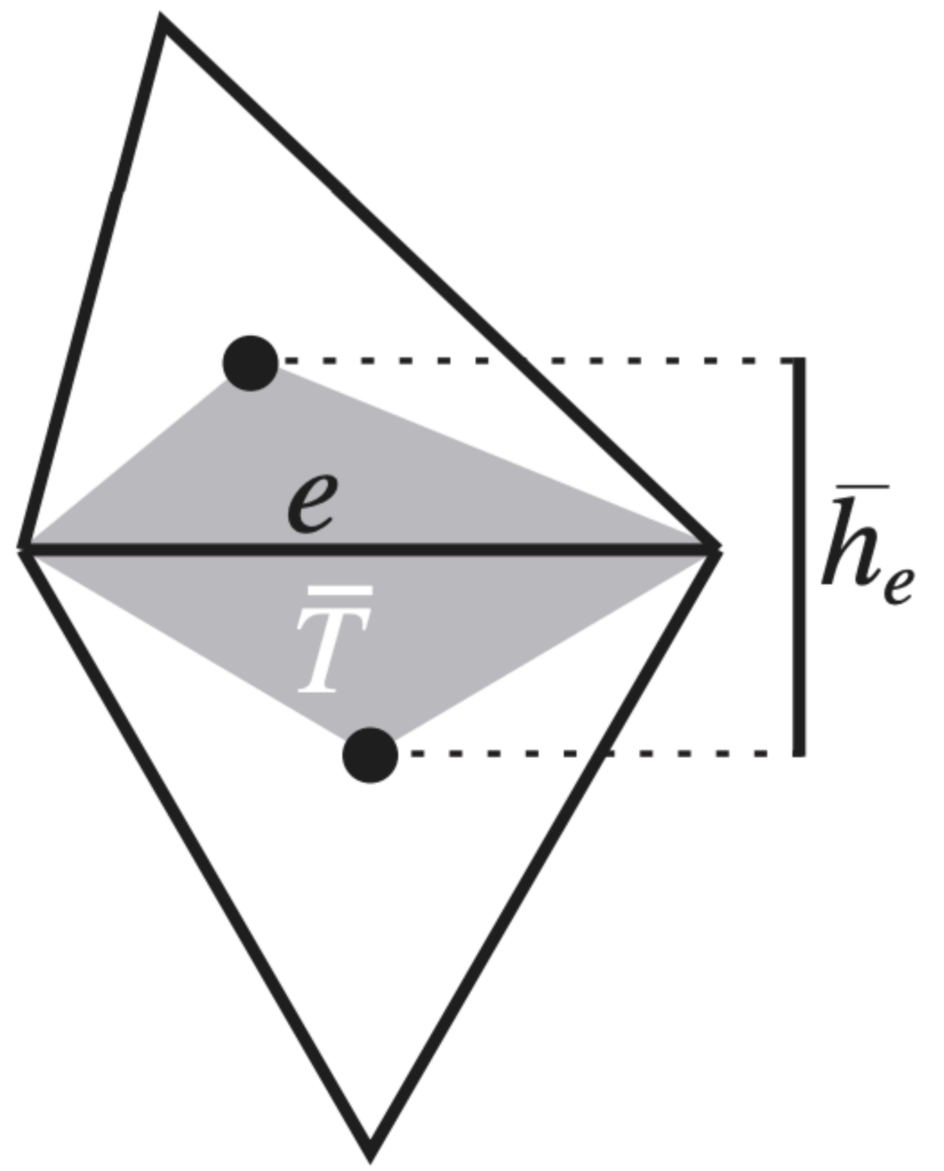
Model the strain energy for bending directly as a penalty of mean curvature changes

$$\int_{\bar{\Omega}} (H \circ \varphi - \bar{H})^2 d\bar{A}$$

Discretizing onto edges of triangle meshes:

$$W_B(\mathbf{x}) = \sum_e (\theta_e - \bar{\theta}_e)^2 \|\bar{e}\| / \bar{h}_e$$

θ **is the dihedral angle, which can be computed with** $\arctan()$



Adding stiffness and reparameterize:

$$\Psi_{\text{bend}}(x) = \sum_i k \frac{3||\bar{e}_i||^2}{\bar{A}_i} (\theta_i - \bar{\theta}_i)^2 \qquad k = \frac{E\xi^3}{24(1-\nu^2)}$$

Grinspun, Eitan, et al. "Discrete shells." Proceedings of the 2003 ACM SIGGRAPH/Eurographics symposium on Computer animation.

Simulating Thin Shells using Surface Meshes

Cubic and Quadratic Bending Energies

Garg et al. [2007]:
For isometric deformation,
A bending energy can be formulated as
a cubic polynomial of x

Bergou et al. [2006]:
For isometric deformation of plates (flat rest shapes),
A bending energy can be formulated as
a quadratic polynomial of x

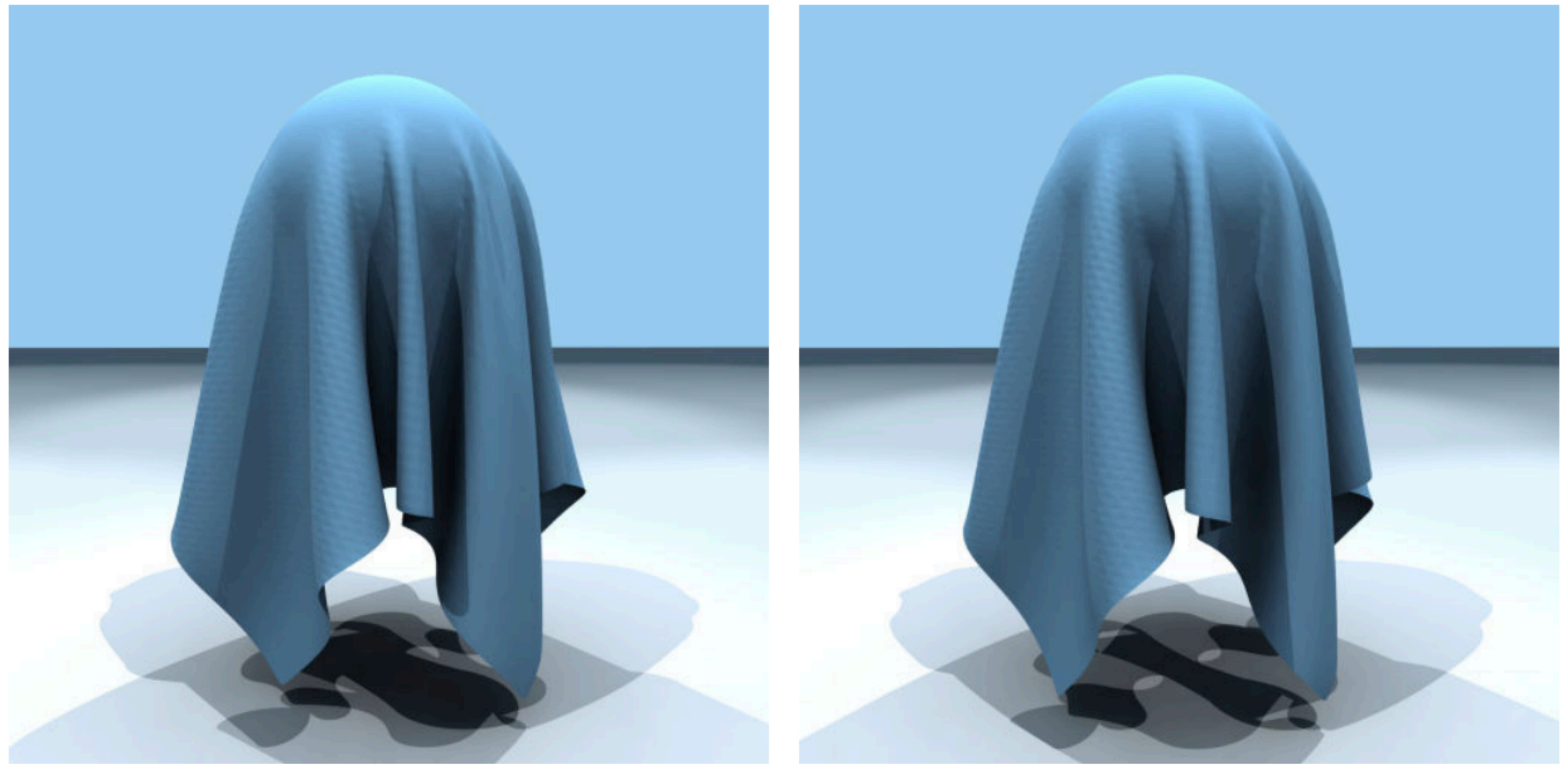


Figure 1: *Final rest state of a cloth draped over a sphere, for* (left) *the proposed isometric bending model and* (right) *the widely-adopted nonlinear hinge model.*

Bergou, Miklos, et al. "A quadratic bending model for inextensible surfaces." Symposium on Geometry Processing. Vol. 227. 2006.
Garg, Akash, et al. "Cubic shells." Proceedings of the 2007 ACM SIGGRAPH/Eurographics symposium on Computer animation. 2007.

Topics Today:

- Thin Shells
  - Stretching and Bending
  - Strain Limiting and Thickness
- Rods and Particles

Simulating Cloth using Surface Meshes

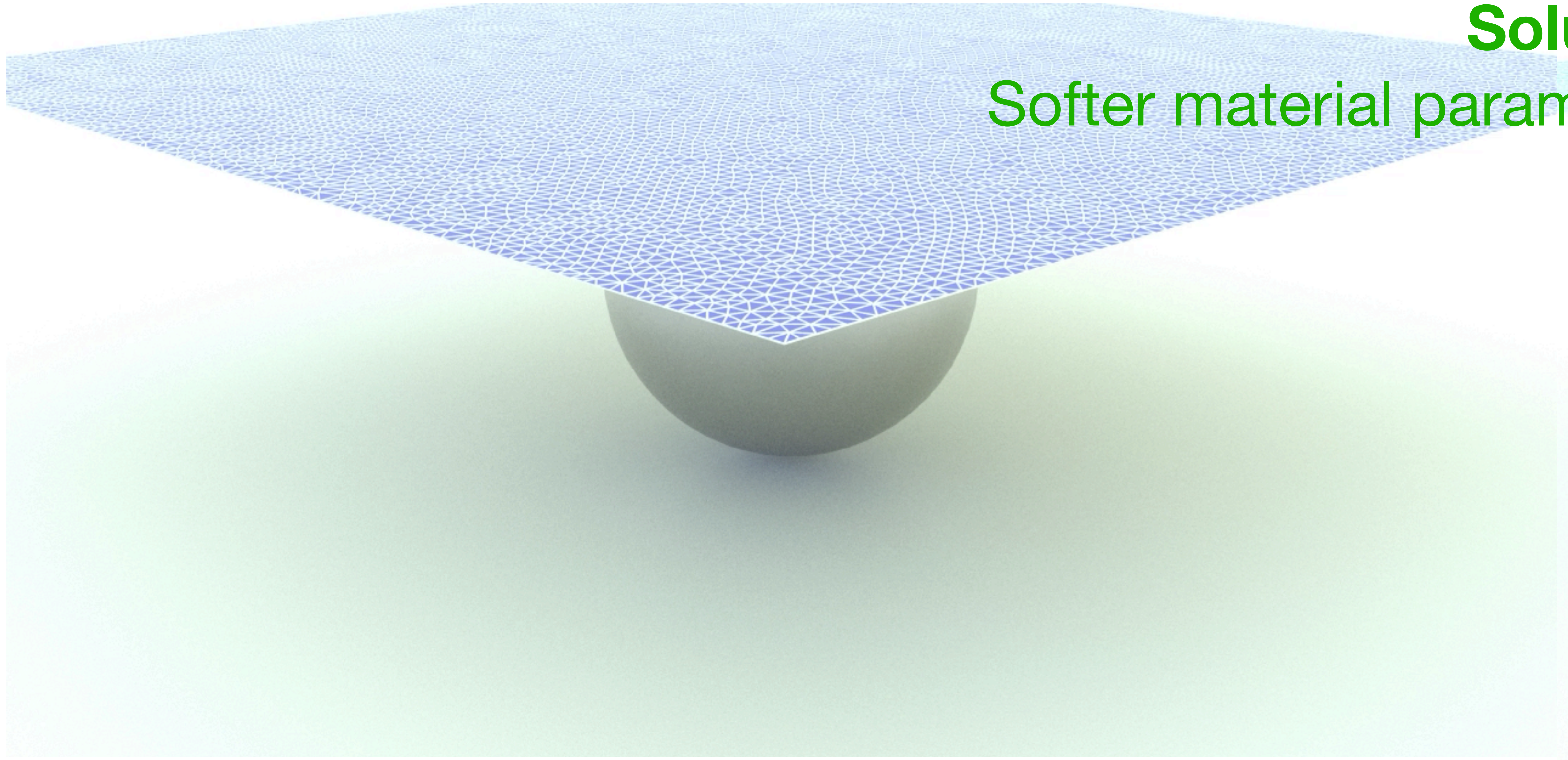
Membrane Locking

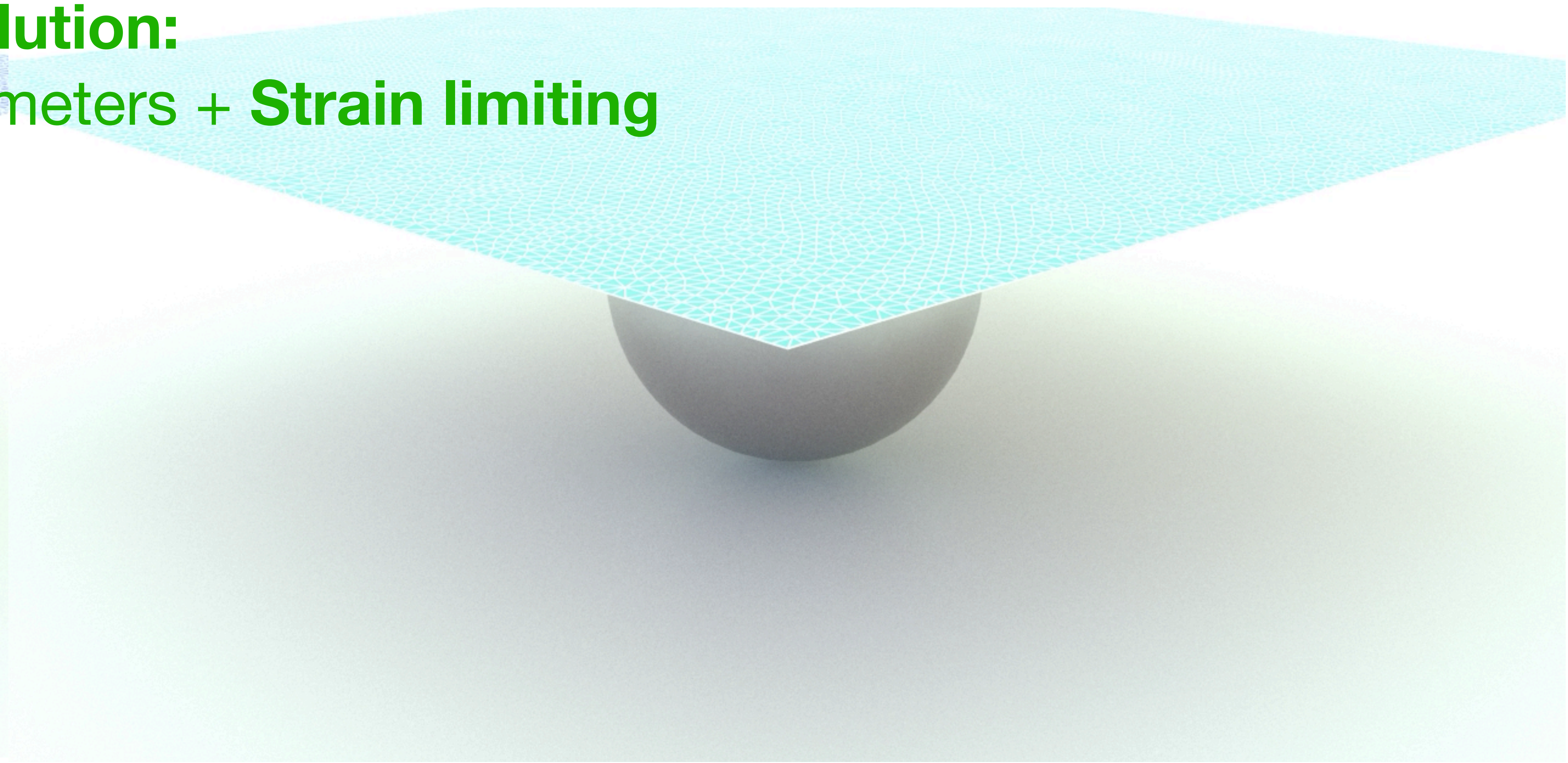
Cloth are nearly unstretchable — stiff stretch resistance, E = ~10^7 Pa

With low-res triangulation, there can be geometric lockings:

Solution:

Softer material parameters + **Strain limiting**





Stiff membrane creates extra bending penalty
(real material parameter)

Stretchy artifact
(0.01x real material parameter)

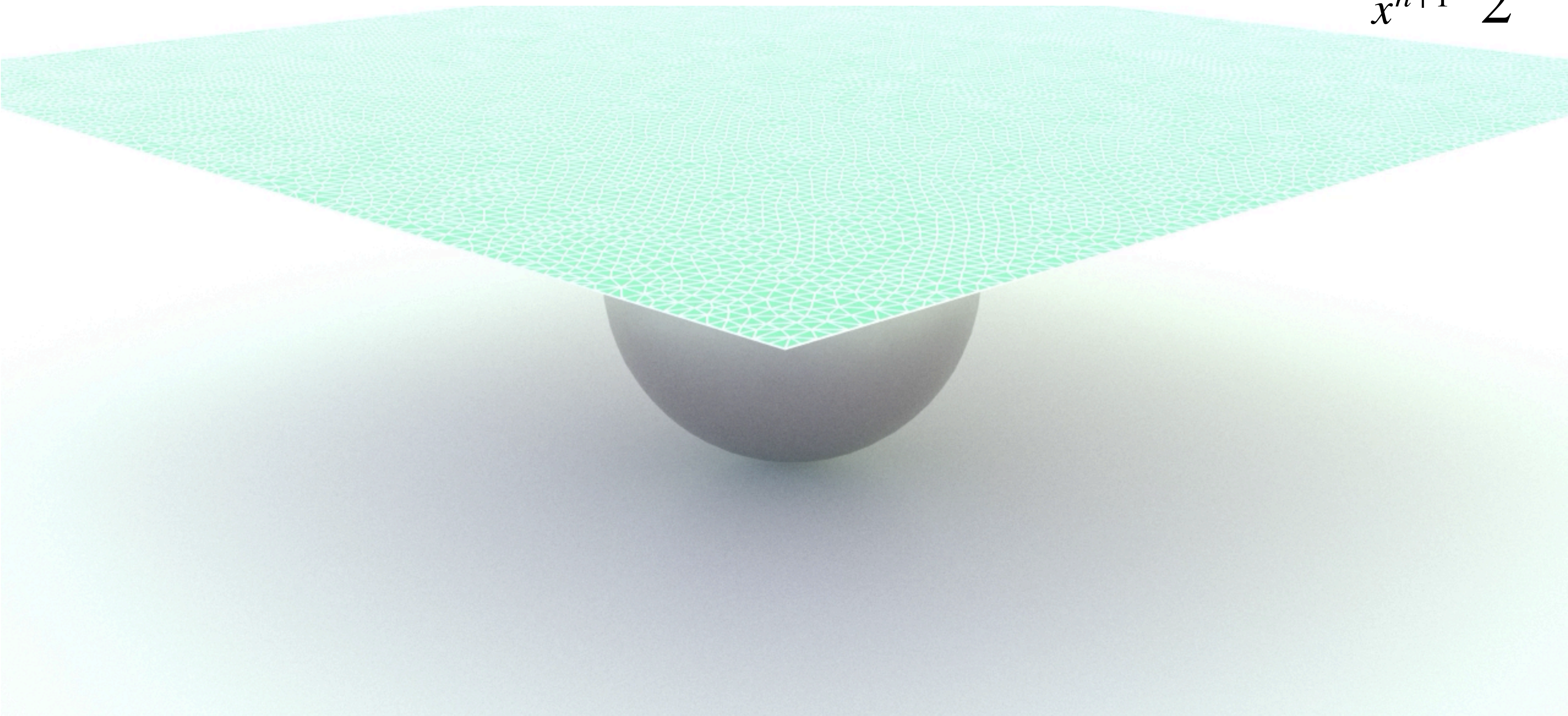
Simulating Cloth using Surface Meshes

Avoiding Membrane Locking with Strain Limiting

$\forall t, i \quad S_{t,i} < s$ $\quad$ ($F_t = U_t S_t V_t^T$ is the SVD of triangle t's deformation gradient)

Strain limit

$$\min_{x^{n+1}} \frac{1}{2}\|x^{n+1} - \tilde{x}^n\|_M^2 + h^2(\Psi(x^{n+1}) + B_s(x^{n+1}))$$



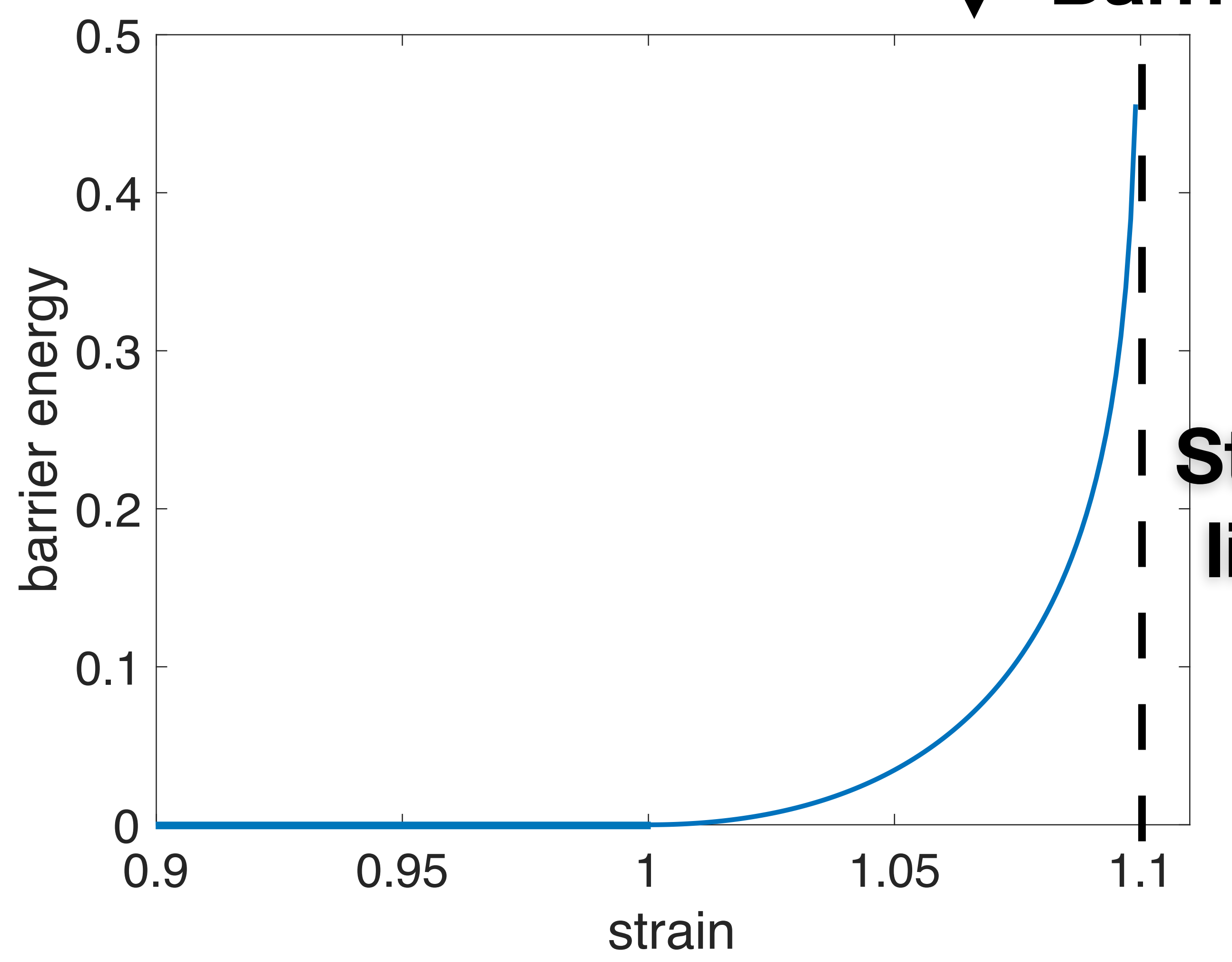
— Guarantees strain limit, but expensive.

Stiffness

Volume weighting

$$B_s(x^{n+1}) = \kappa_s \sum_{t,i} \mathcal{V}_t \, b(S_{t,i}(x^{n+1}))$$

Barrier function

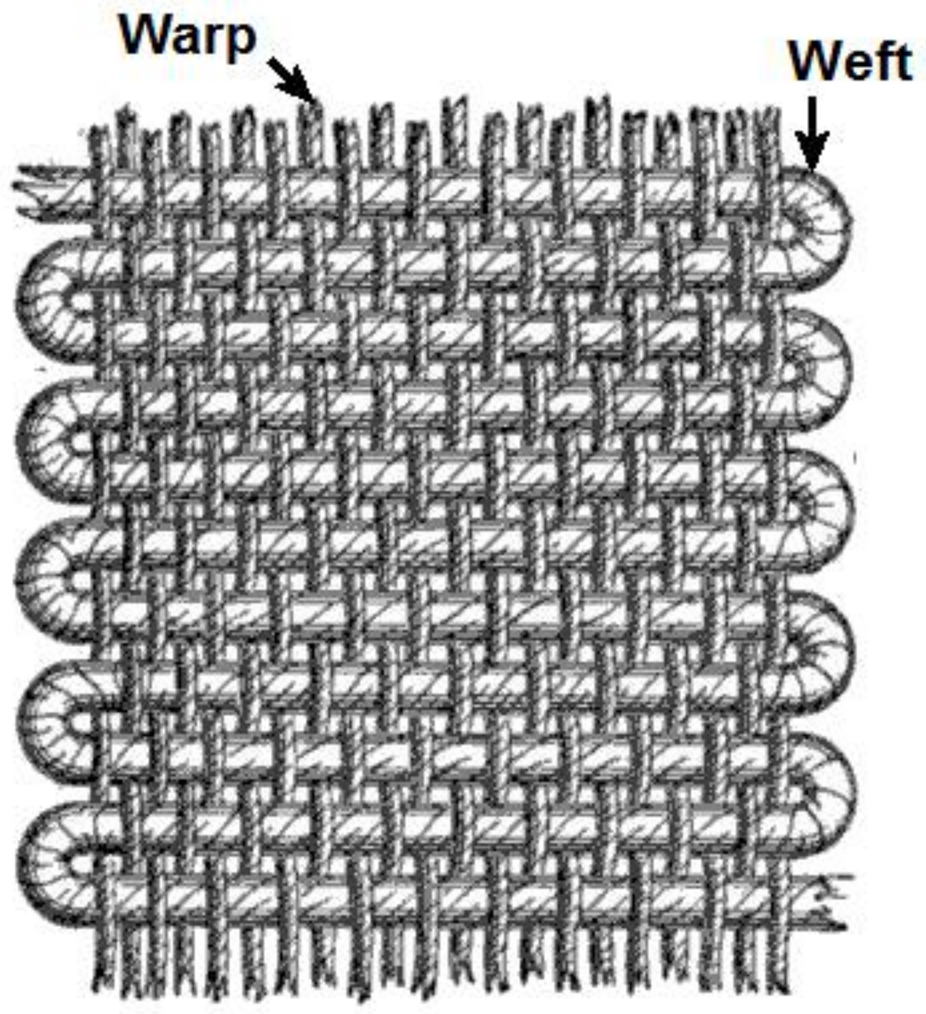


Li, Minchen, Danny M. Kaufman, and Chenfanfu Jiang. "Codimensional incremental potential contact." ACM Transactions on Graphics (TOG) 40.4 (2021): 1-24.

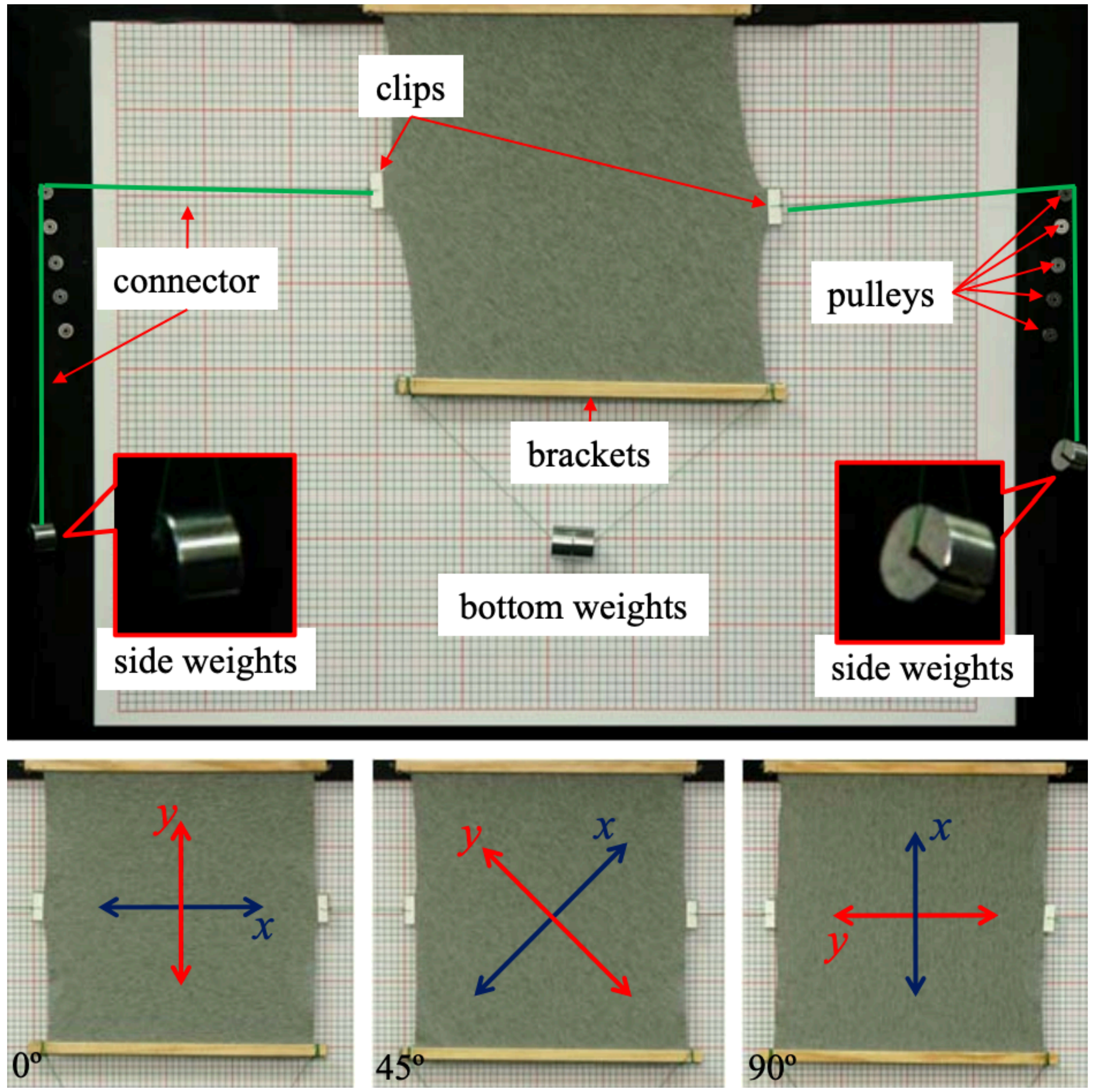
Simulating Cloth using Surface Meshes

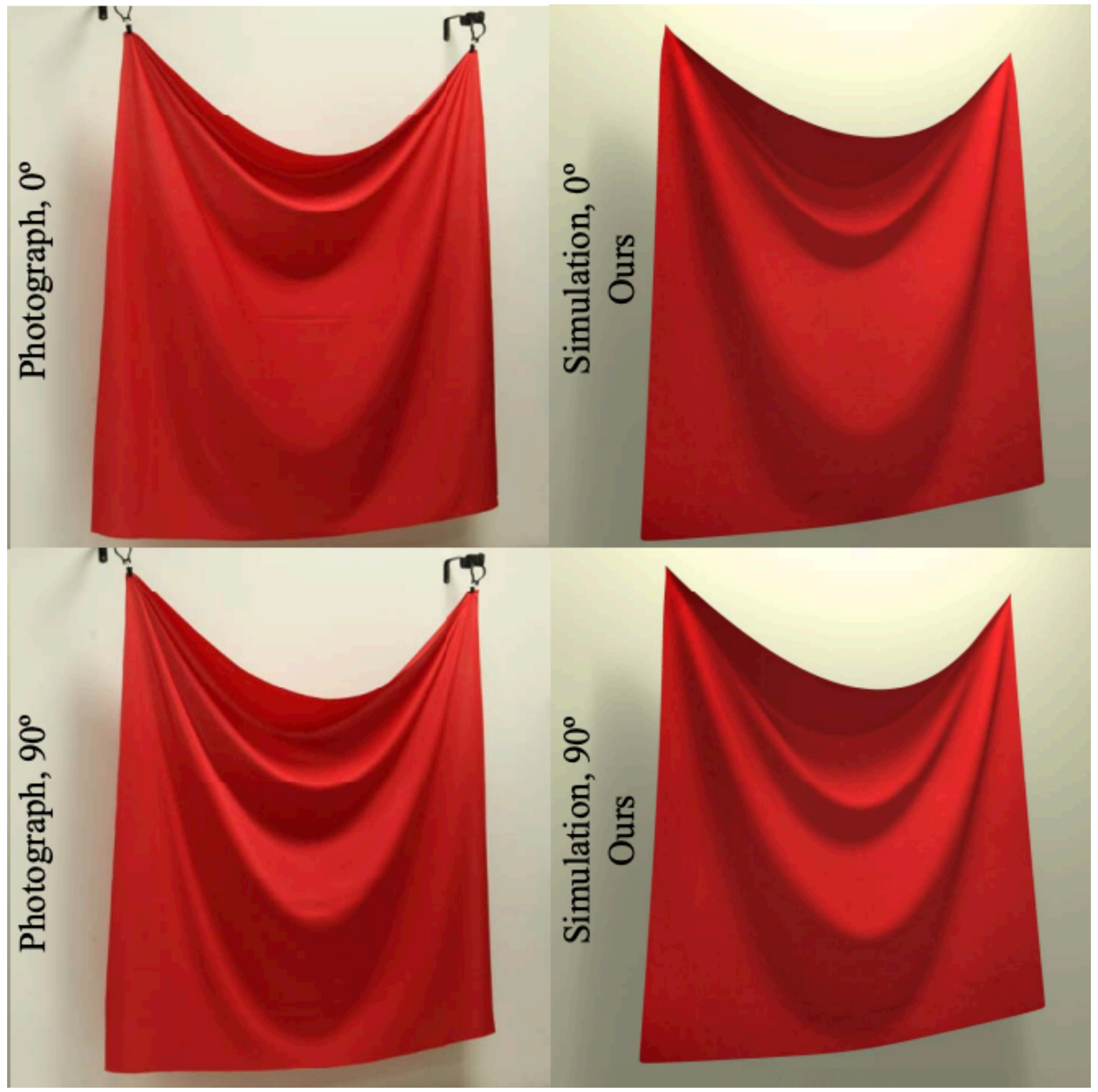
Anisotropy

Because of the knitting pattern, cloth is anisotropic



Measuring and simulating cloth based on anisotropic elasticity [Wang et al. 2011]:





Wang, Huamin, James F. O'Brien, and Ravi Ramamoorthi. "Data-driven elastic models for cloth: modeling and measurement." ACM transactions on graphics (TOG) 30.4 (2011)

Simulating Cloth using Surface Meshes

Thickness Modeling — Inelastic Thickness via Contact Gap

However, there are scenarios where thickness matters:

$$\min_{x^{n+1}} \frac{1}{2} \|x^{n+1} - \tilde{x}^n\|_M^2 + h^2 \sum P(x^{n+1}) + \kappa \sum_k b(D_k(x^{n+1}) - \xi, \hat{d})$$

elastic thickness ($\hat{d}$)

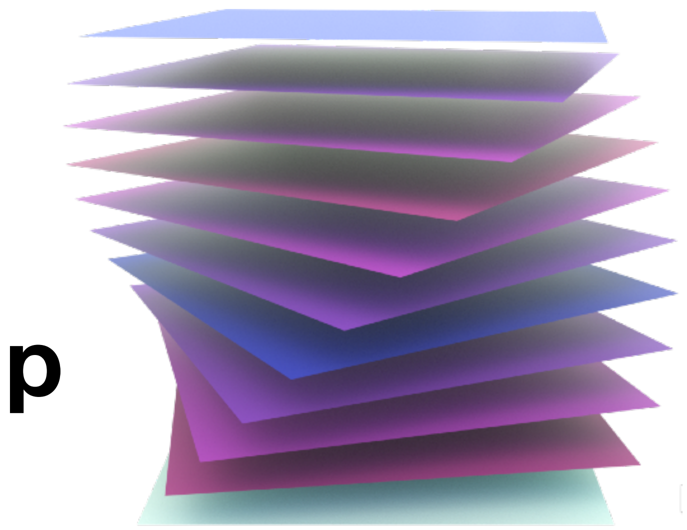
inelastic thickness (ξ)

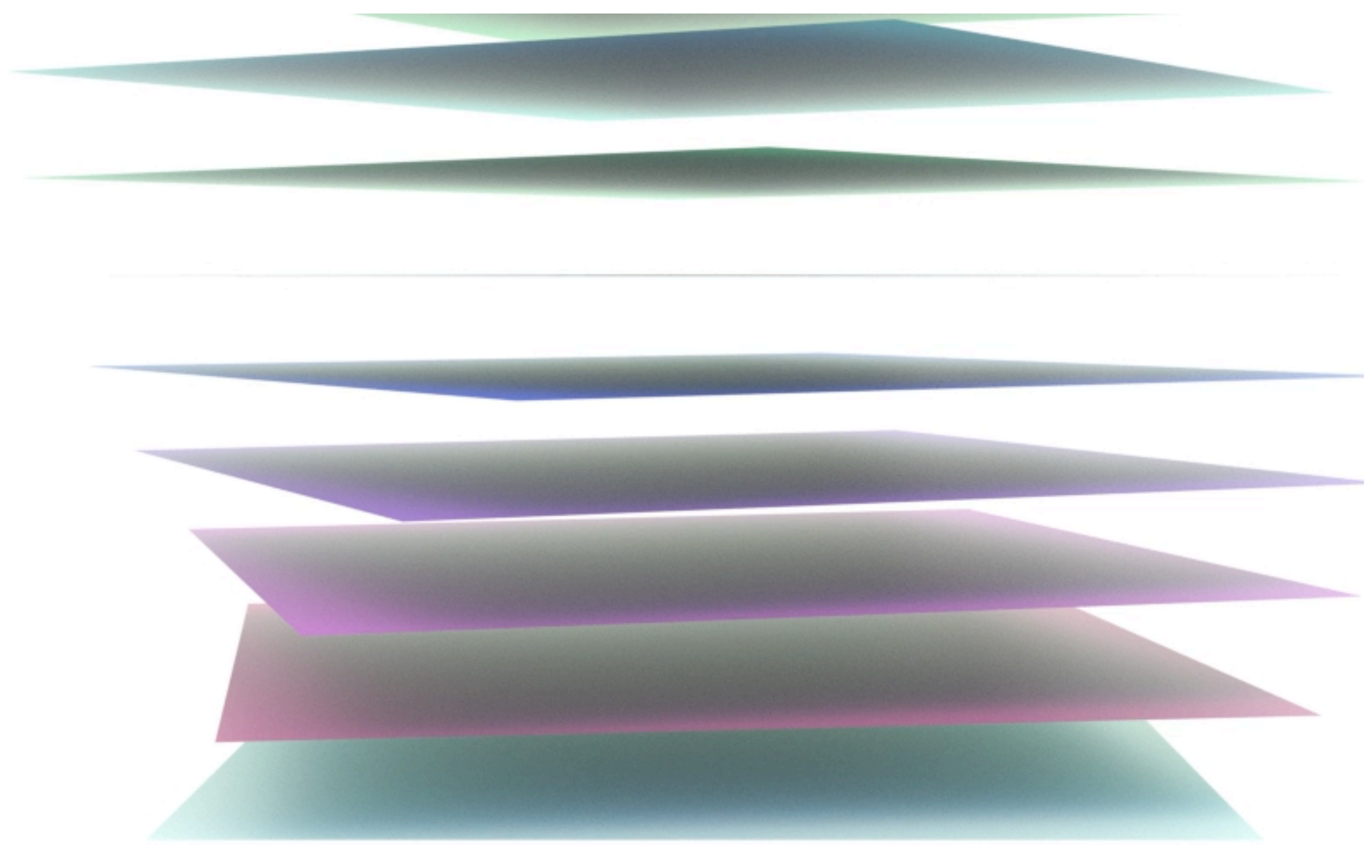
Li, Minchen, Danny M. Kaufman, and Chenfanfu Jiang. "Codimensional incremental potential contact." ACM Transactions on Graphics (TOG) 40.4 (2021): 1-24.

Simulating Cloth using Surface Meshes

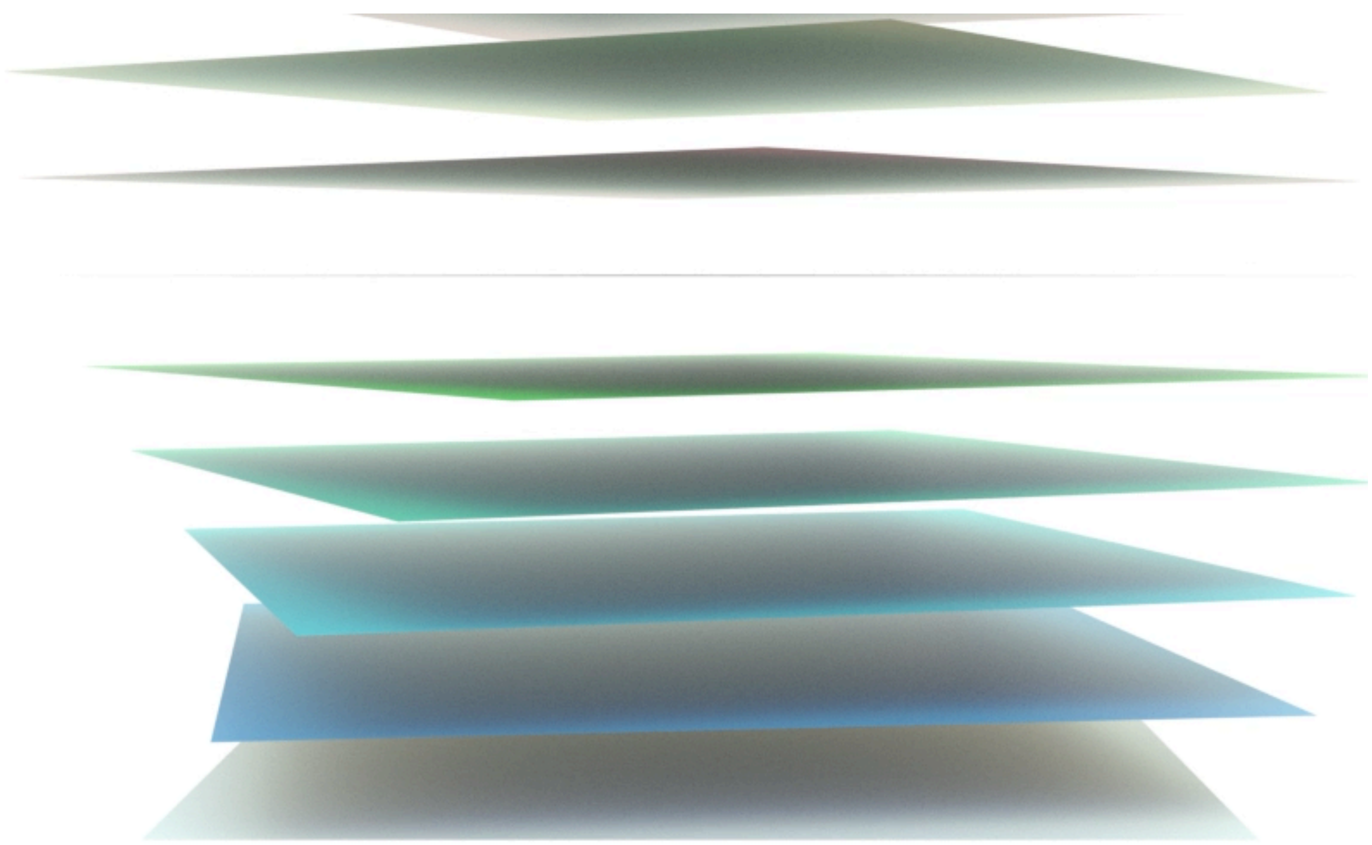
Thickness Modeling — Elastic Thickness via Barrier $\hat{d}$

Scene setup

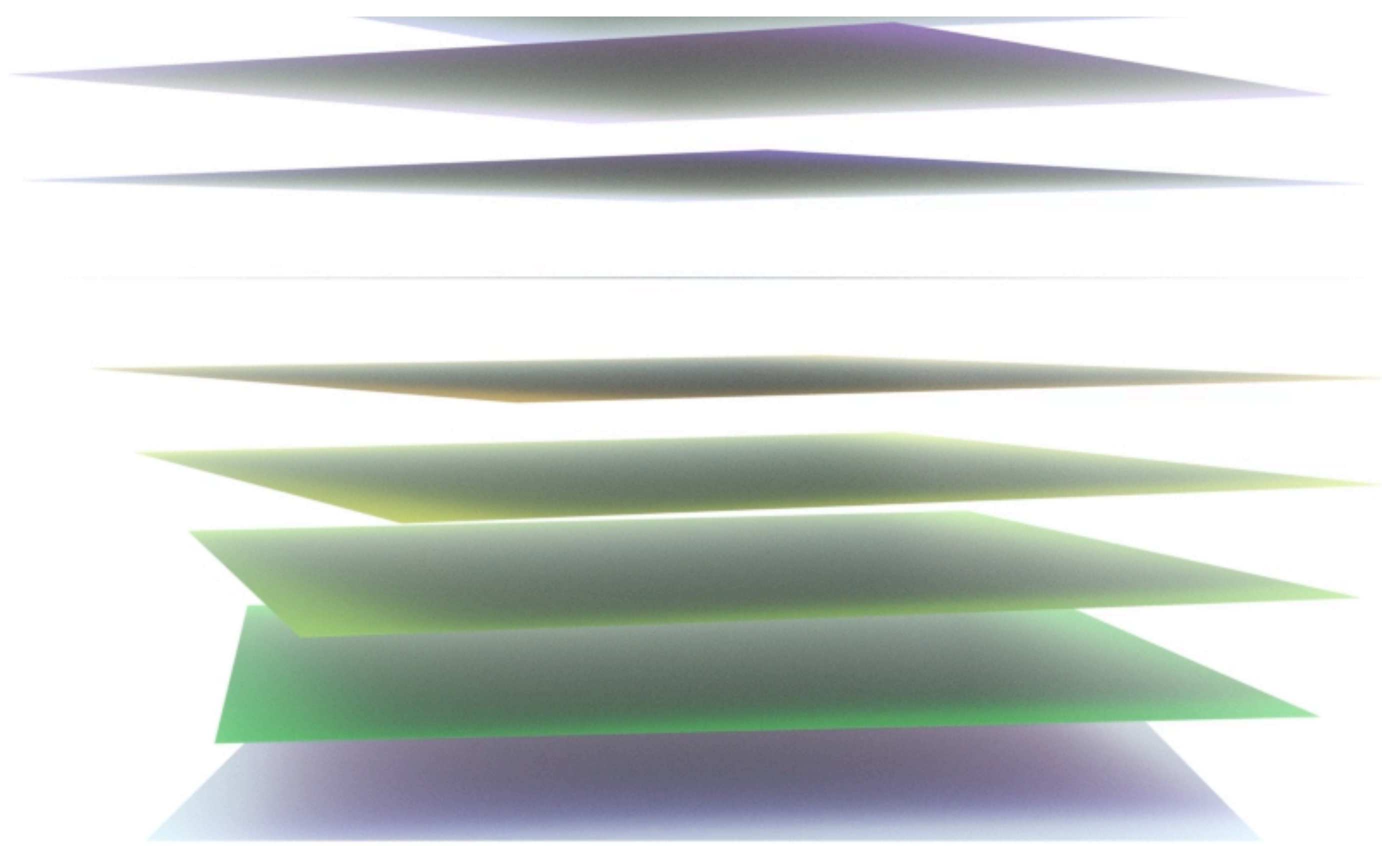




$\hat{d} = 1mm,\ h = 0.04s$



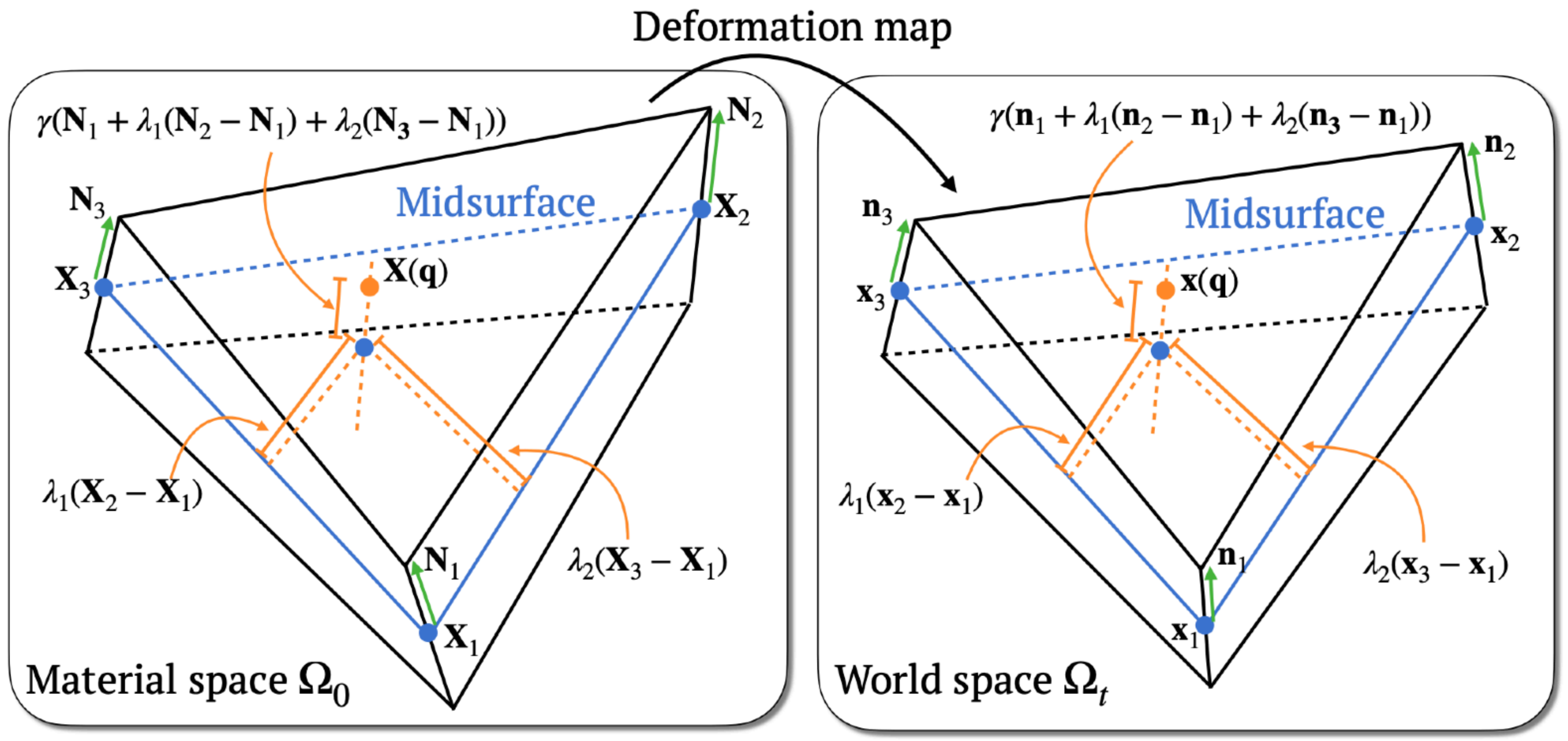
$\hat{d} = 5mm,\ h = 0.04s$



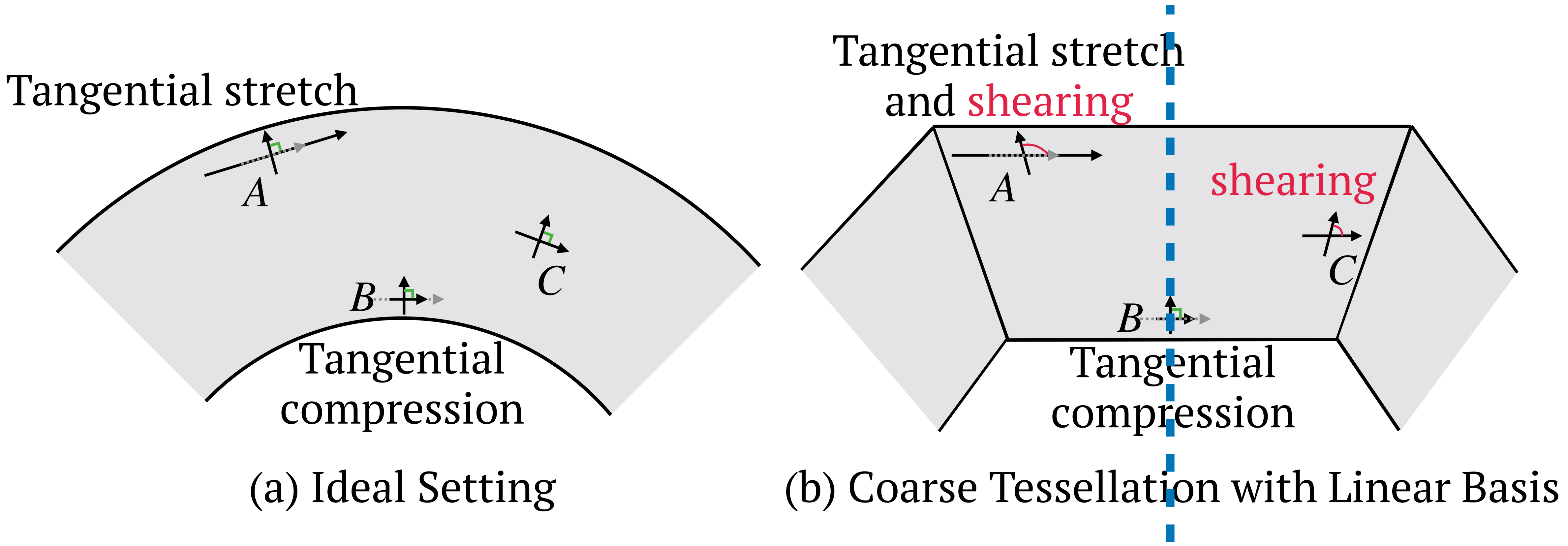
$\hat{d} = 10mm,\ h = 0.04s$

Li, Minchen, Danny M. Kaufman, and Chenfanfu Jiang. "Codimensional incremental potential contact." ACM Transactions on Graphics (TOG) 40.4 (2021): 1-24.

*Simulating Thick Shells [Chen et al. 2023]

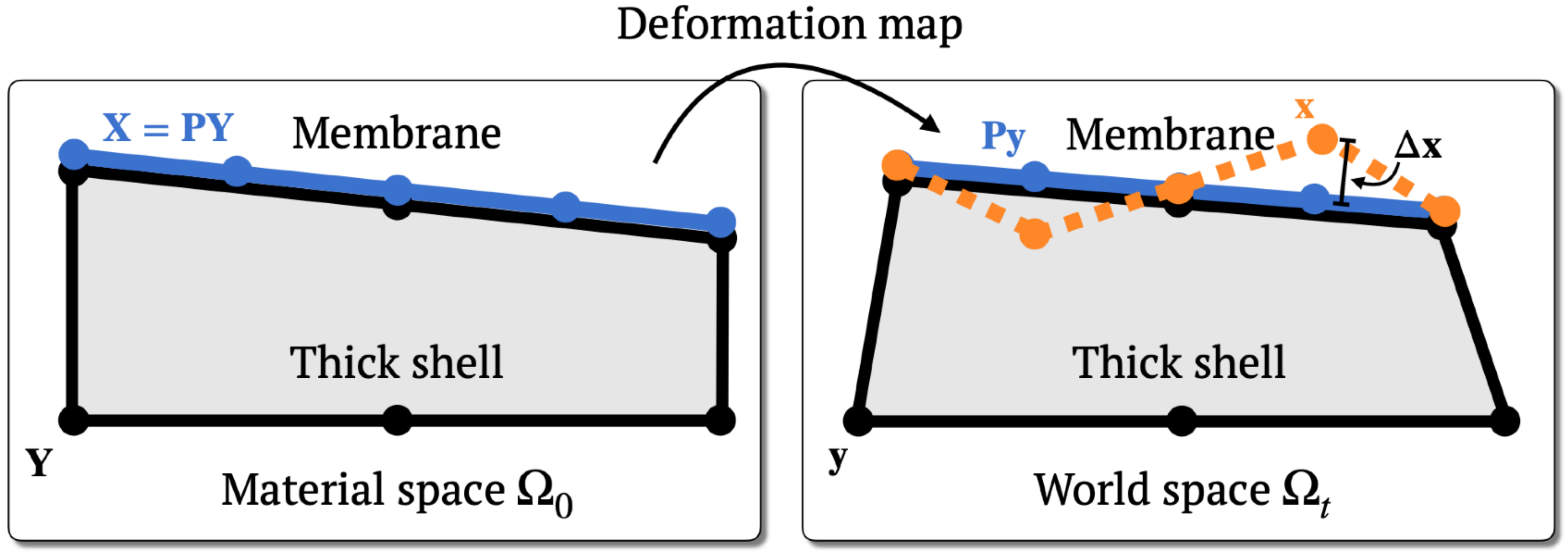


Use prism elements to explicitly track thickness



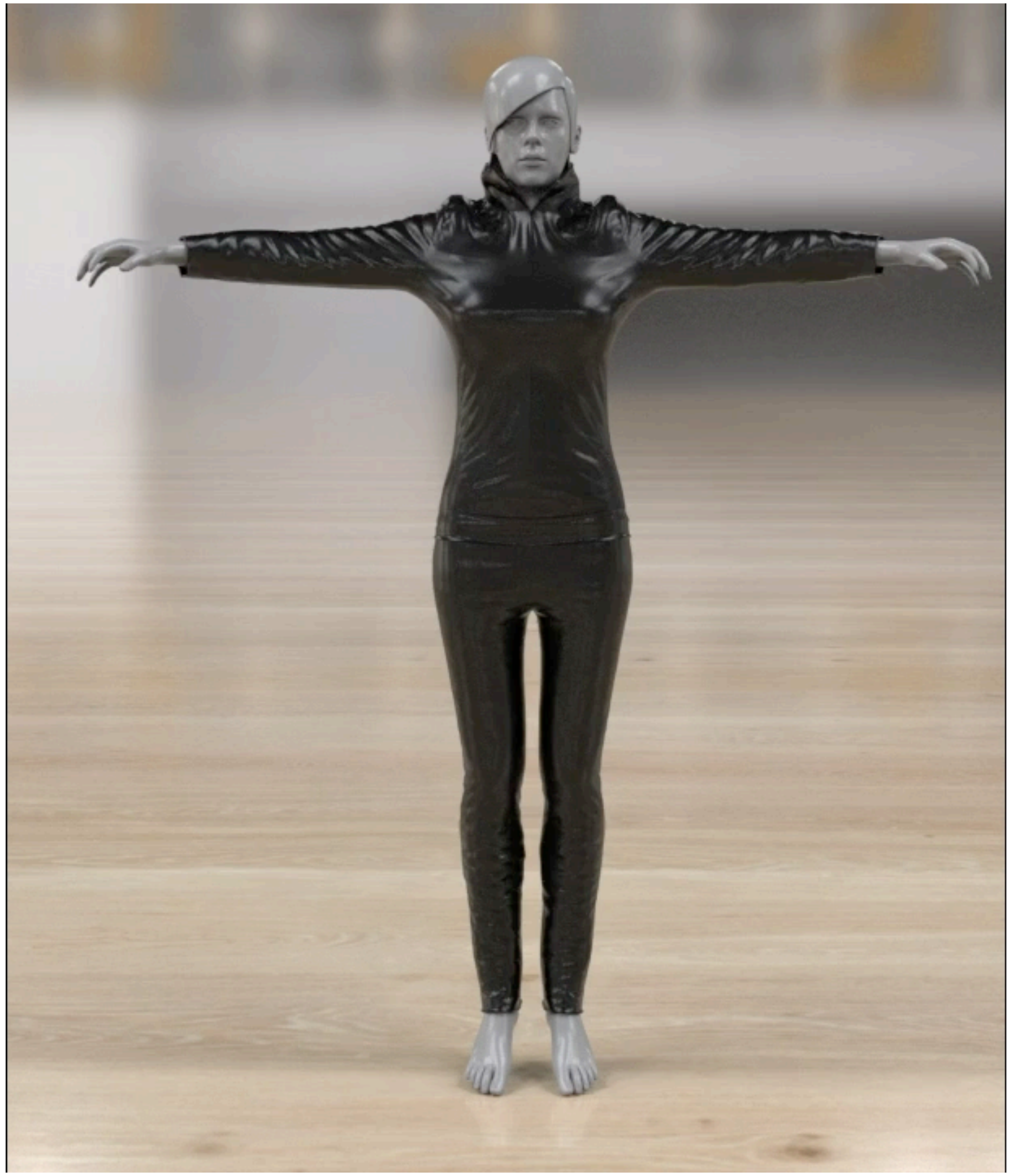
Place quadratures only on the center line to avoid shear locking

Couple with a fine thin shell to simulate wrinkles



Chen, Yunuo, et al. "Multi-layer thick shells." ACM SIGGRAPH 2023 Conference Proceedings.

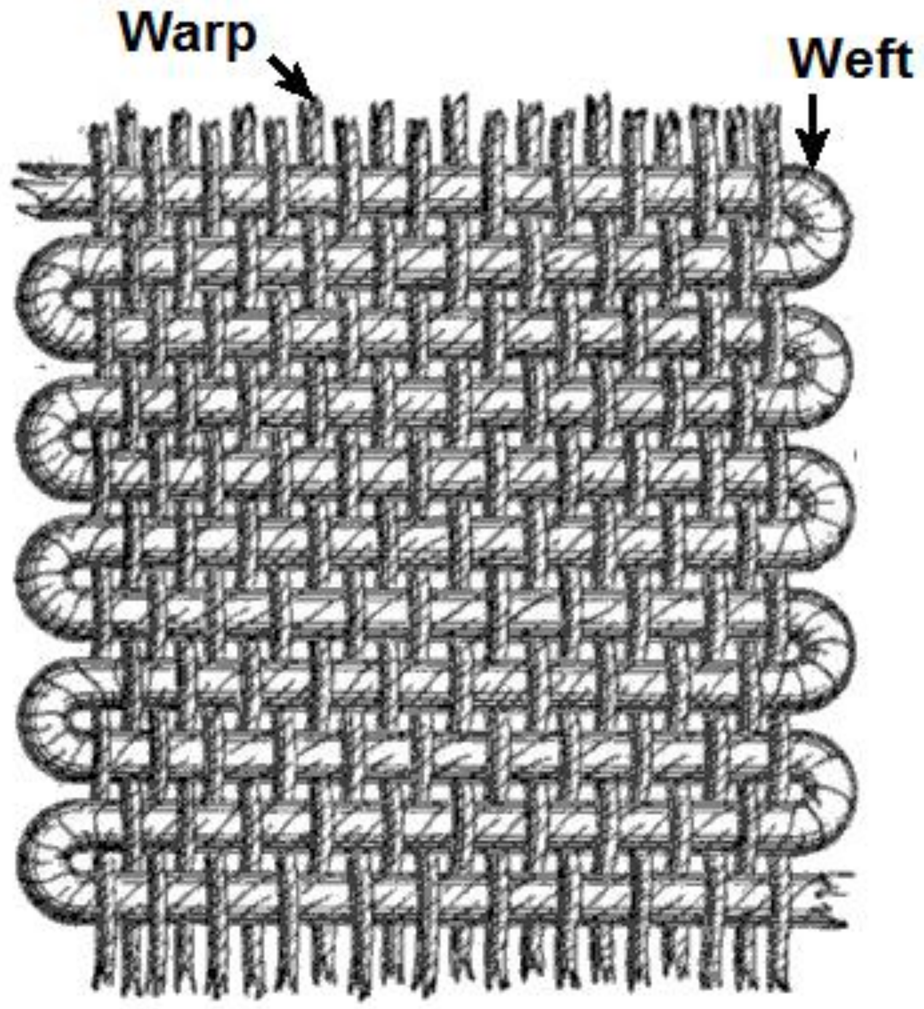
*Simulating Thick Shells [Chen et al. 2023]



Chen, Yunuo, et al. "Multi-layer thick shells." ACM SIGGRAPH 2023 Conference Proceedings.

Simulating Yarn-Level Cloth

Because of the knitting pattern, cloth is anisotropic



Can directly simulate these yarns!

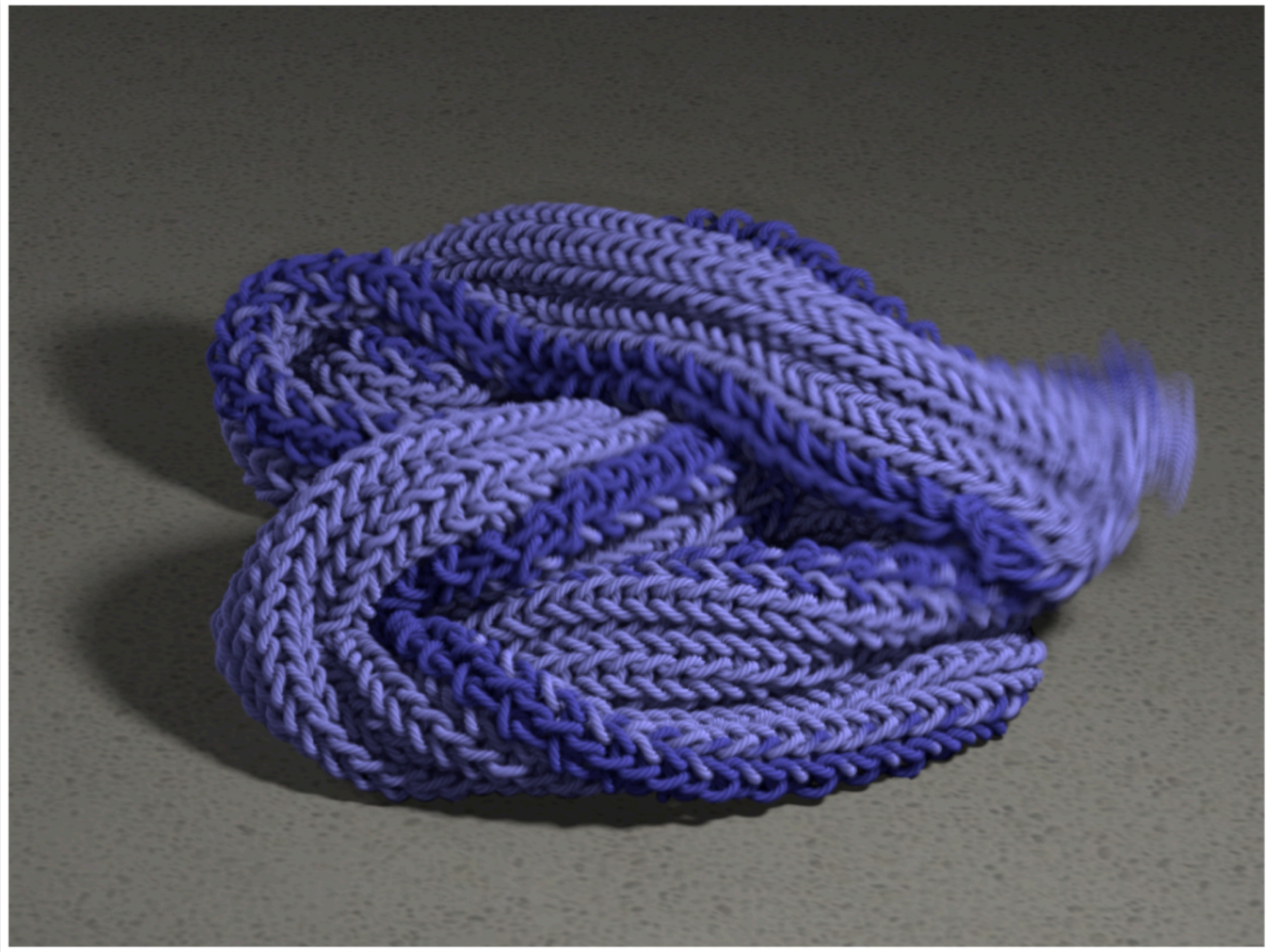
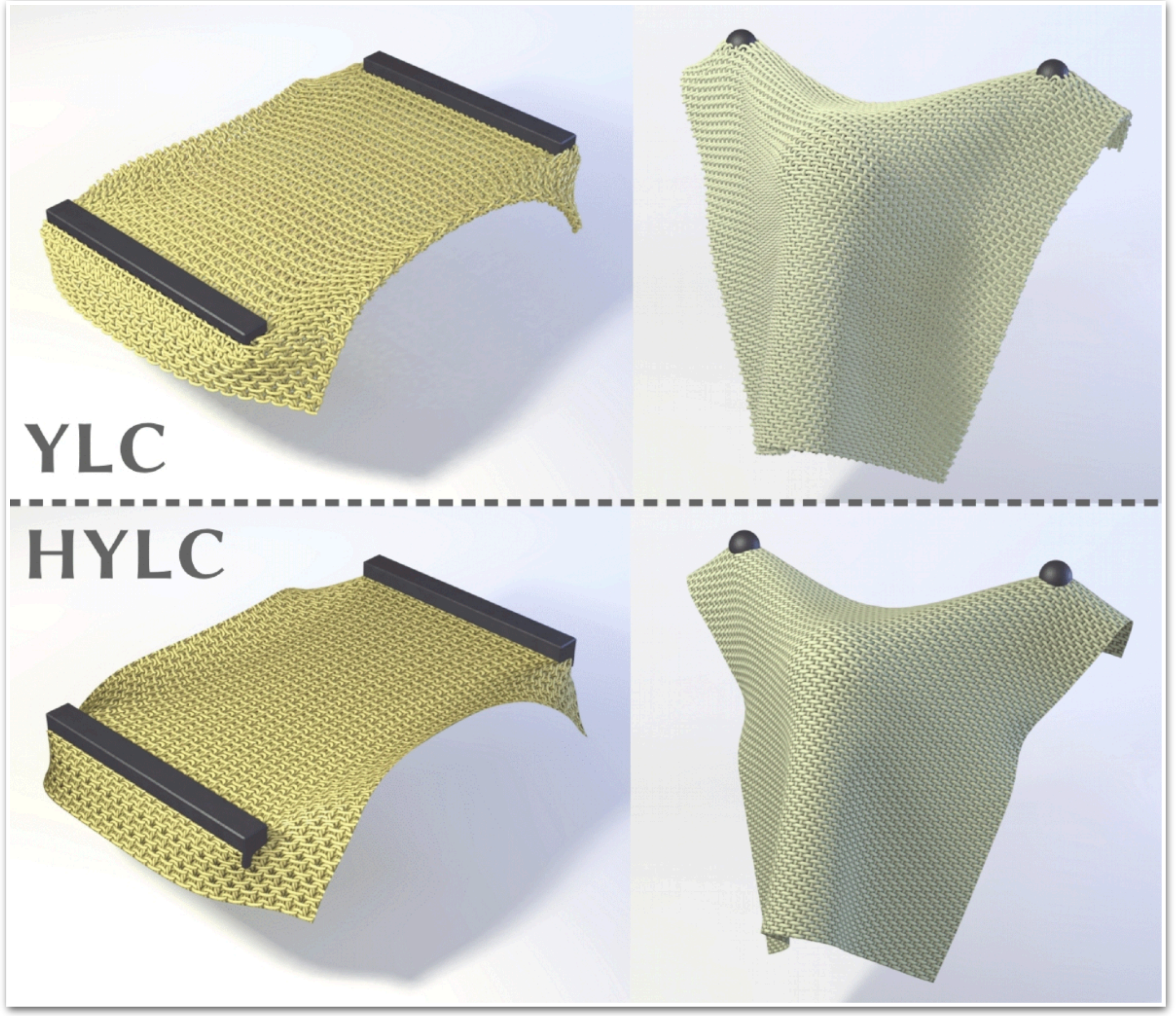


Figure 9: Scarf: *Our contact model scales to support the complex contact and folding that occurs in cloth.*

[Kaldor et al. 2008]



[Sperl et al. 2020]

Kaldor, Jonathan M., Doug L. James, and Steve Marschner. "Simulating knitted cloth at the yarn level." ACM SIGGRAPH 2008 papers. 2008. 1-9.
Sperl, Georg, Rahul Narain, and Chris Wojtan. "Homogenized yarn-level cloth." ACM Trans. Graph. 39.4 (2020): 48.

Topics Today:

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- Rods and Particles

Simulating Slender Rods using Polylines

Discrete Elastic Rods

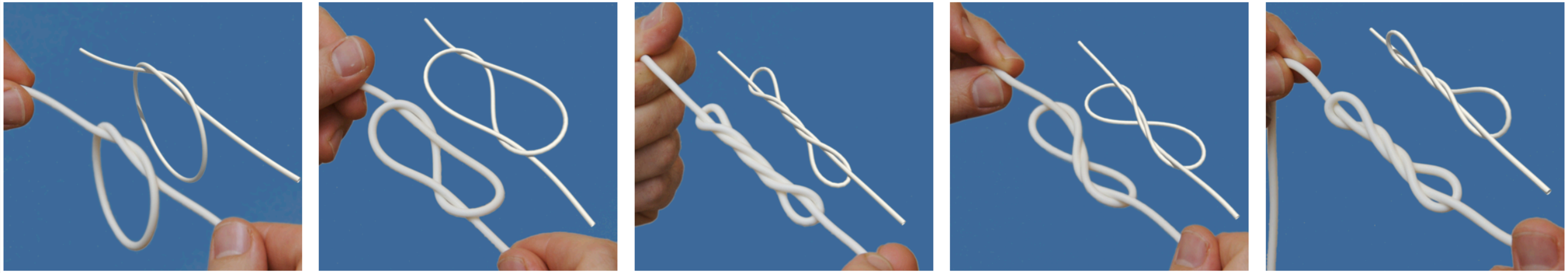


Figure 1: **Experiment and simulation:** A simple (trefoil) knot tied on an elastic rope can be turned into a number of fascinating shapes when twisted. Starting with a twist-free knot (*left*), we observe both continuous and discontinuous changes in the shape, for both directions of twist. Using our model of Discrete Elastic Rods, we are able to reproduce experiments with high accuracy.

[Bergou et al. 2008]

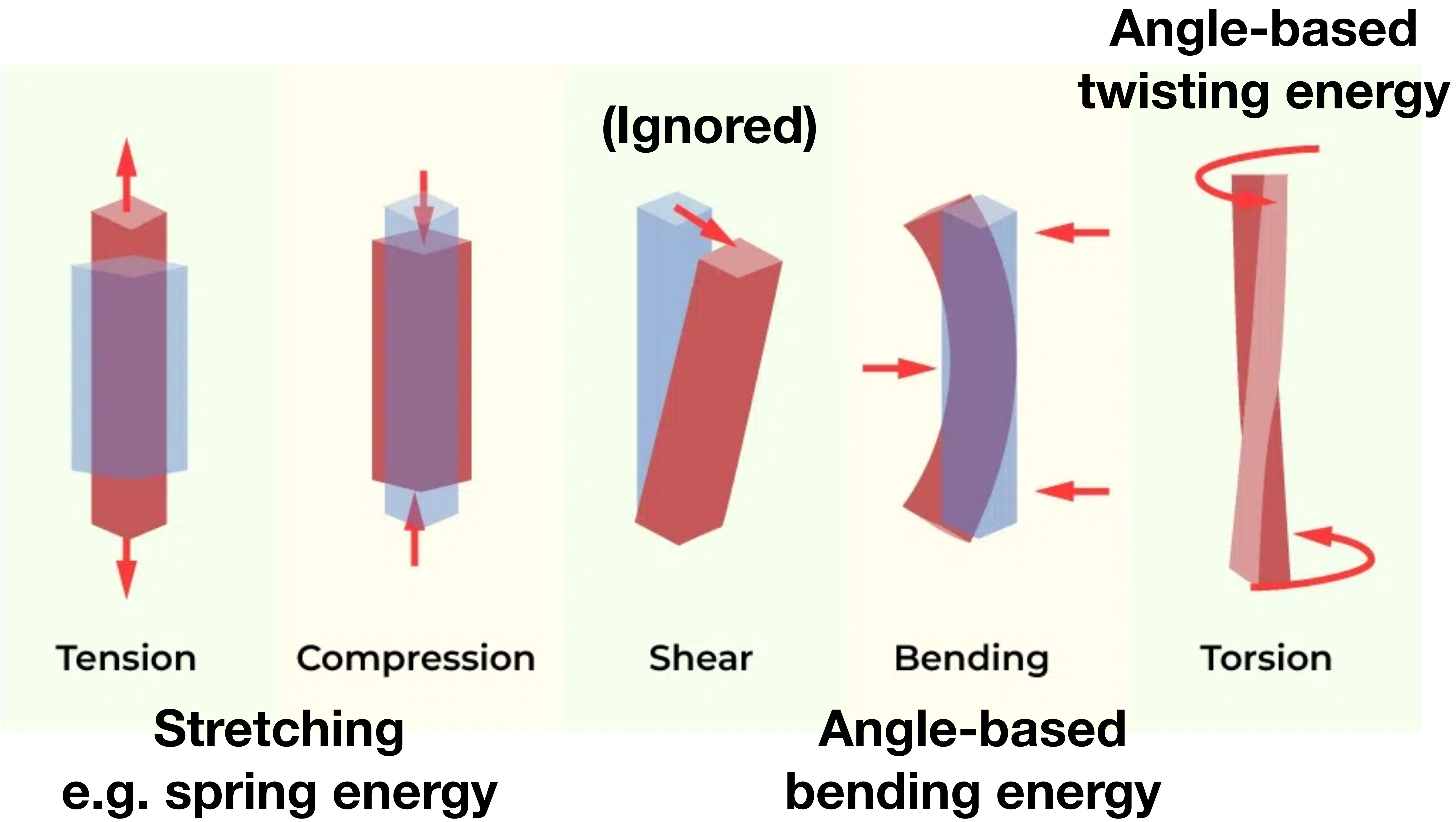
Strain energy decomposed into stretching, bending, and twisting energies.

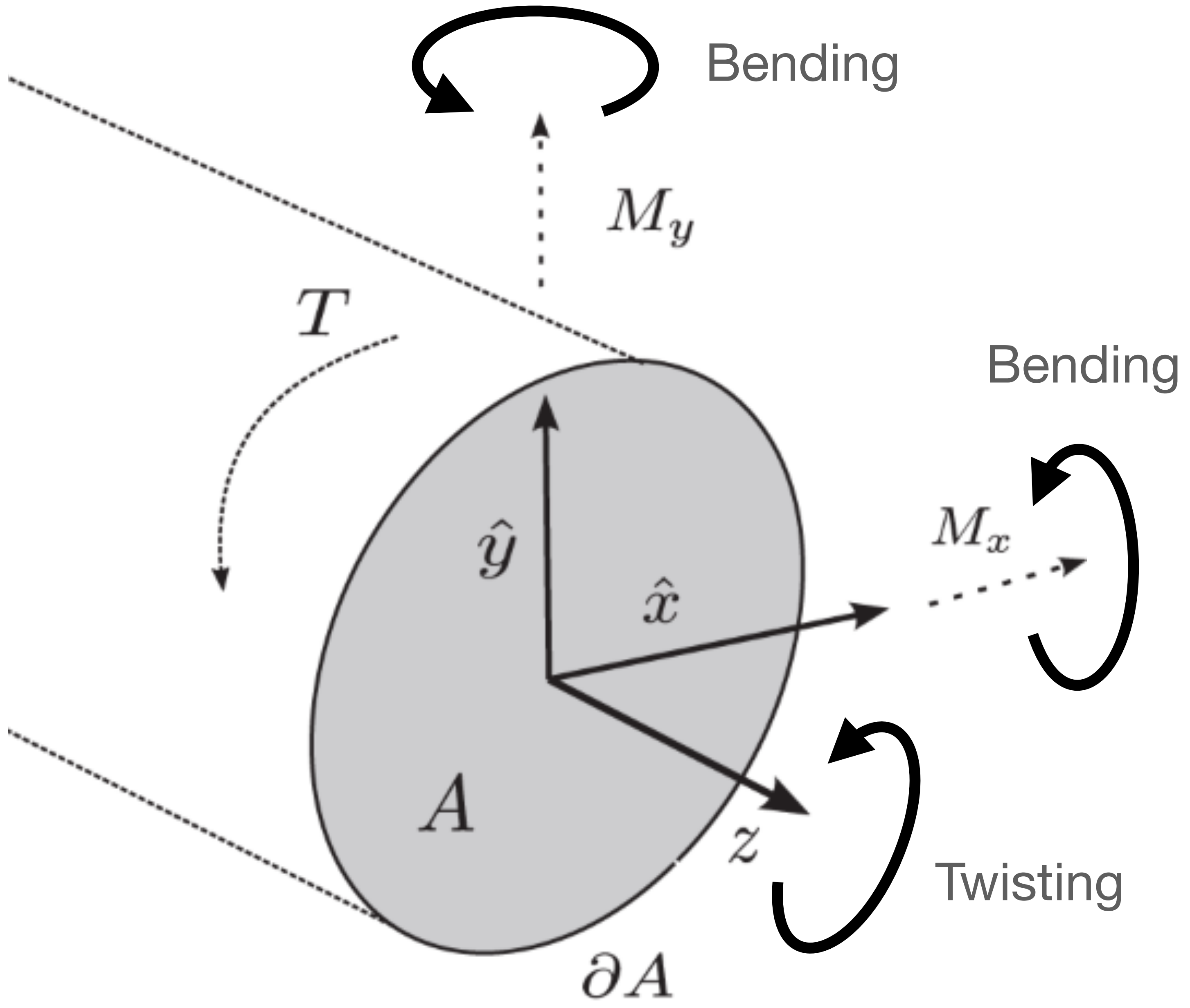
Bergou, Miklós, et al. "Discrete elastic rods." ACM SIGGRAPH 2008 papers. 2008. 1-12.

Simulating Slender Rods using Polylines

Discrete Elastic Rods — Stretching, Bending, Twisting

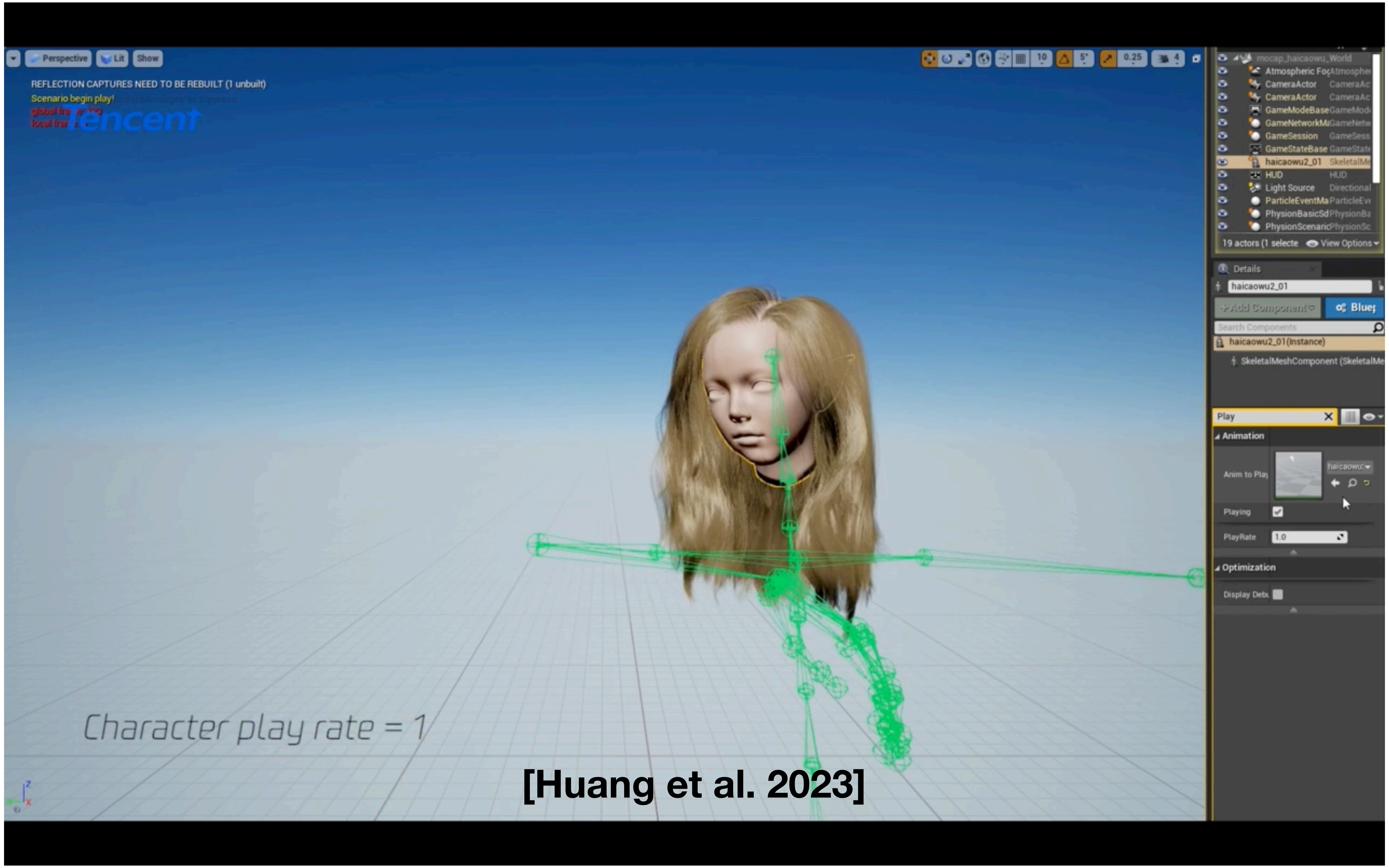
- Idea: use rotation angles of local frames to measure bending and twisting deformation:





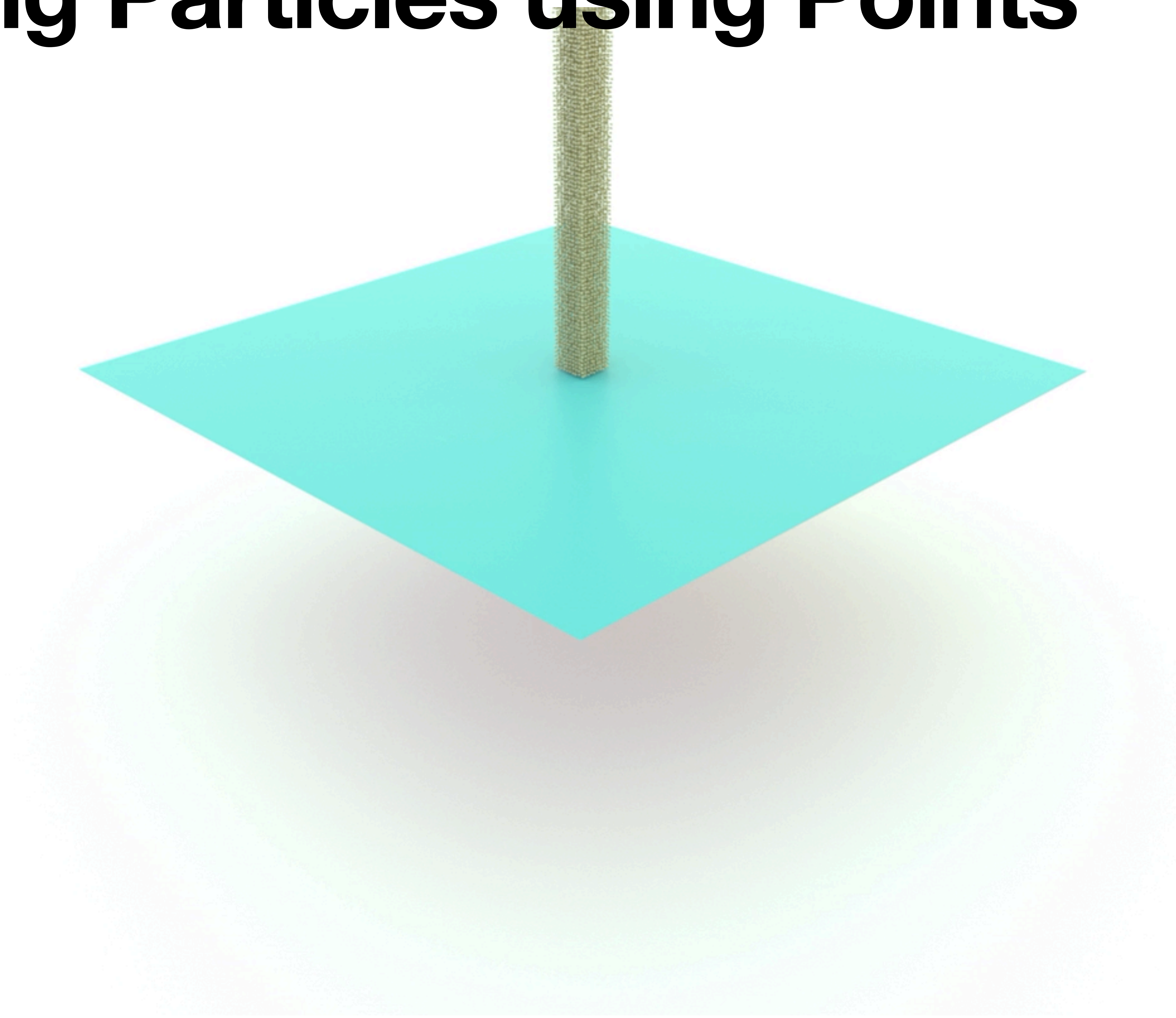
Bergou, Miklós, et al. "Discrete elastic rods." ACM SIGGRAPH 2008 papers. 2008. 1-12.

Simulating Hairs using Polylines



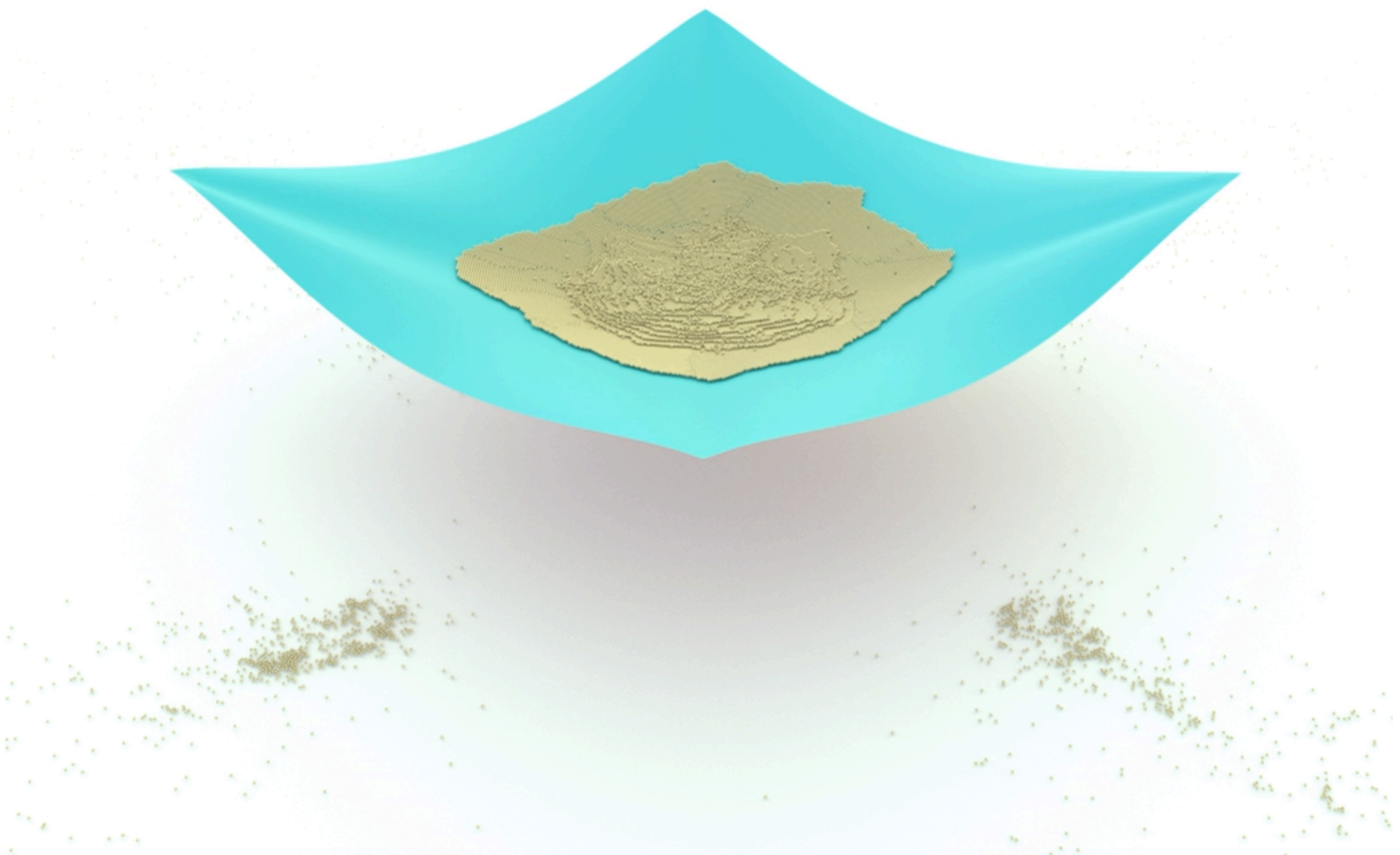
Huang, Li, et al. "Towards realtime: A hybrid physics-based method for hair animation on gpu." Proceedings of the ACM on Computer Graphics and Interactive Techniques (2023)

Simulating Particles using Points



Li, Minchen, Danny M. Kaufman, and Chenfanfu Jiang. "Codimensional incremental potential contact." ACM Transactions on Graphics (TOG) 40.4 (2021): 1-24.

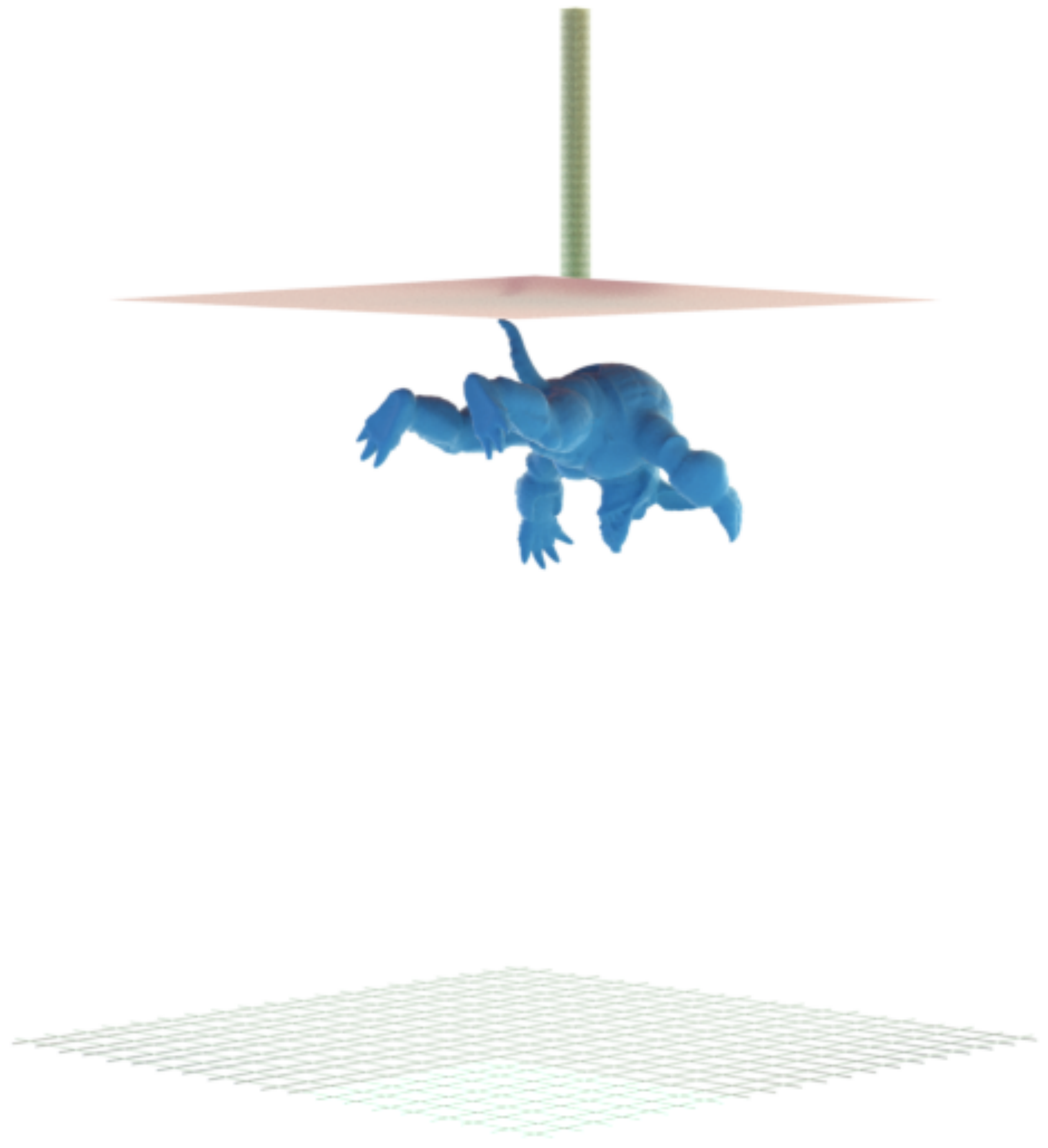
Simulating Particles using Points



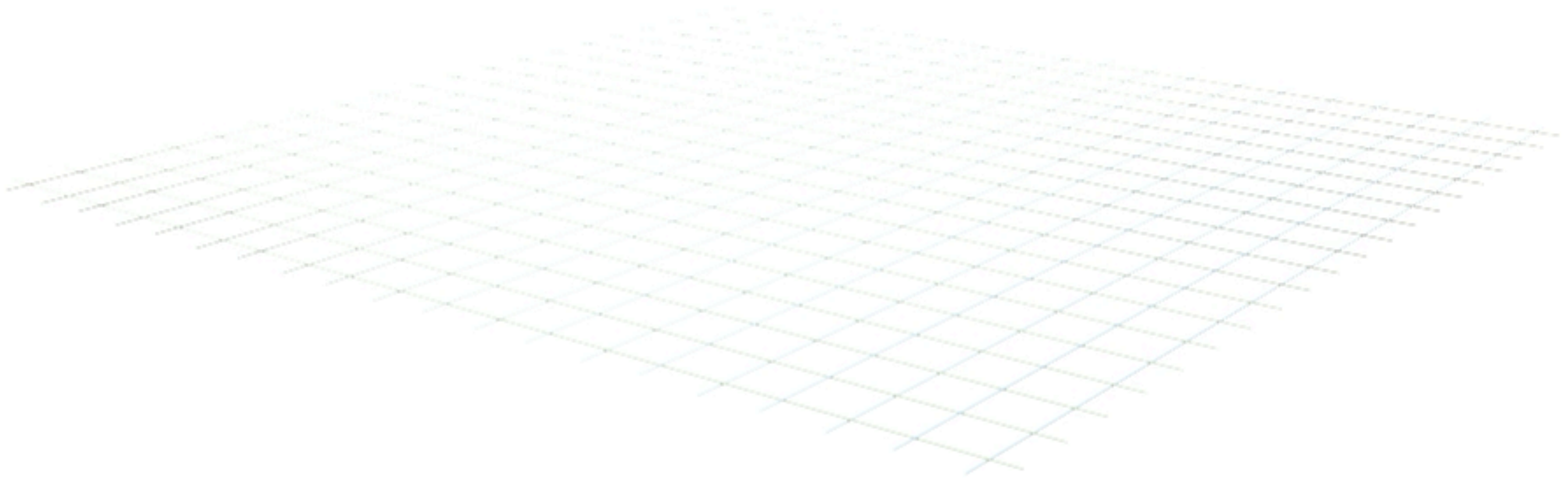
Li, Minchen, Danny M. Kaufman, and Chenfanfu Jiang. "Codimensional incremental potential contact." ACM Transactions on Graphics (TOG) 40.4 (2021): 1-24.

“All-In”

Coupled using IPC



Scene setup



Li, Minchen, Danny M. Kaufman, and Chenfanfu Jiang. "Codimensional incremental potential contact." ACM Transactions on Graphics (TOG) 40.4 (2021): 1-24.

Topics Today:

- Thin Shells
  - Stretching and Bending
  - Strain Limiting and Thickness
- Rods and Particles

Next Lecture: Fluids

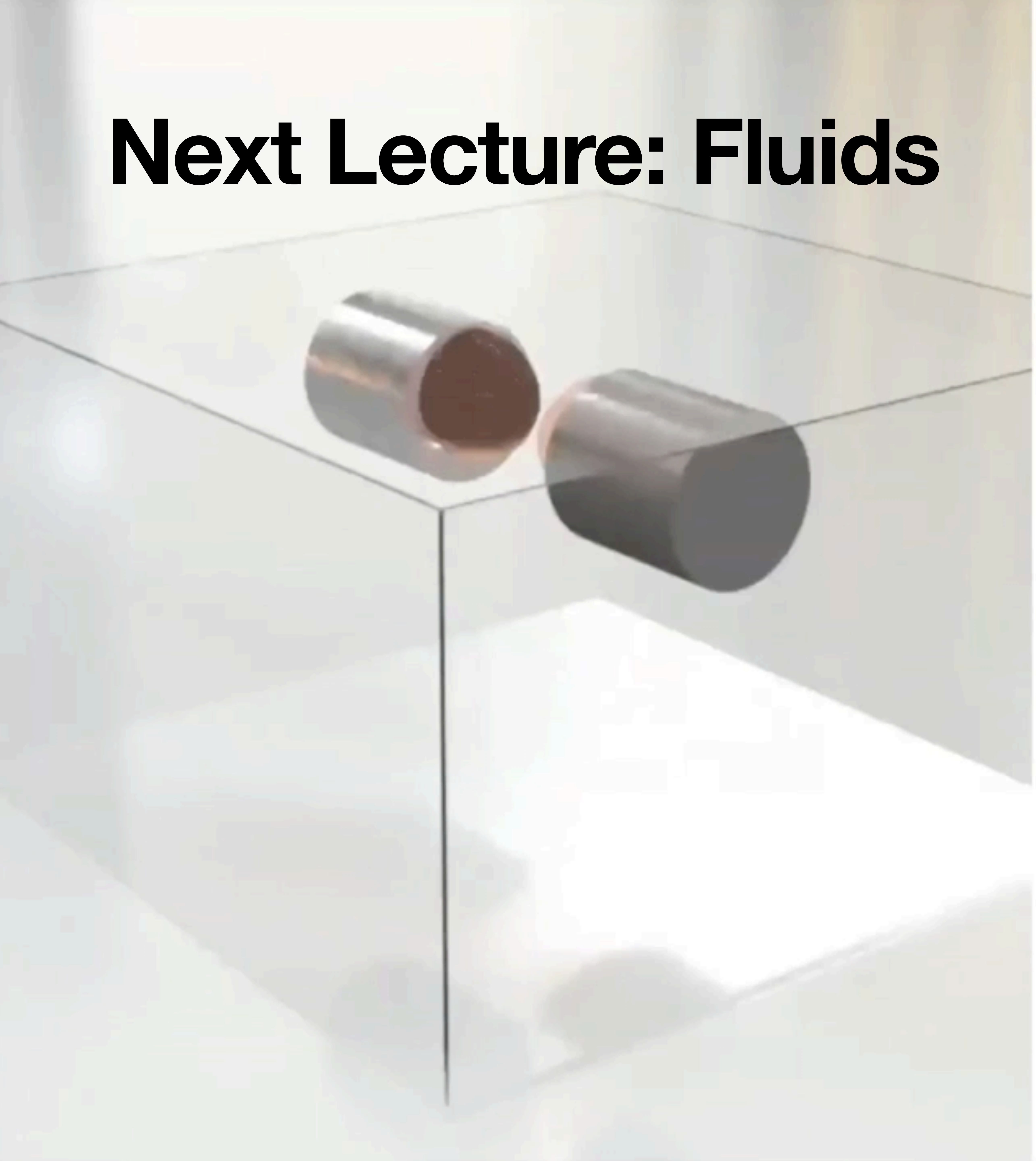


Image Sources

- http://viterbi-web.usc.edu/~jbarbic/femdefo/barbic-courseNotes-modelReduction.pdf
- https://www.researchgate.net/publication/332914440_Port-Hamiltonian_formulation_and_symplectic_discretization_of_plate_models_Part_II_Kirchhoff_model_for_thin_plates/figures?lo=1&utm_source=google&utm_medium=organic
- http://multires.caltech.edu/pubs/ds.pdf
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