Today

1. Finish Simplex.

2. Duality

   - Way to show prove optimality of a solution
   - to figure out "important constraints"
   - to transform an LP into equivalent (and perhaps more malleable) LP.

3. Give intuition for other widely used solvers: Interior Point.
Duality: Consider an LP of the form
\[
\begin{align*}
\text{max } & \quad c^T x \\
\text{s.t. } & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

Say:
maximize \[2x_1 + 3x_2\]
subject to \[4x_1 + 8x_2 \leq 12\]
\[2x_1 + x_2 \leq 3\]
\[3x_1 + 2x_2 \leq 4\]
\[x_1, x_2 \geq 0.\]

Can we give an upper bound on the optimum? Can it be a million? 100? 13?

- Can’t be more than 12 because
  \[2x_1 + 3x_2 \leq 4x_1 + 8x_2 \leq 12\]
  \[\uparrow x_1, x_2 \geq 0 \quad \uparrow \text{inequality } (i)\]

- Can’t be more than 6 because
  \[2x_1 + 3x_2 \leq \frac{1}{2} (4x_1 + 8x_2) \leq \frac{1}{2} \cdot 12 = 6\]

- Can we give better bounds?
  \[\text{aha. } 2x_1 + 3x_2 \leq \frac{1}{3} ((4x_1 + 8x_2) + (2x_1 + x_2))\]
  \[\leq \frac{1}{3} (12 + 3) = \frac{15}{3} = 5\]

- Better? Hmm, now seems trickier.
  Let’s write an LP to solve this problem!
Here we are giving upper bounds on the value of LP by combining together constraints.

So let’s take \( y_1 \times 1^{st} \) constraint
\[ y_2 \times 2^{nd} \text{ constraint} \]
\[ y_3 \times 3^{rd} \text{ constraint} \]

And want this to be at least \( 2x_1 + 3x_2 \)

So
\[
\begin{align*}
4x_1 + 8x_2 & \leq 12 \quad \times y_1 \\
2x_1 + x_2 & \leq 3 \quad \times y_2 \\
3x_1 + 2x_2 & \leq 4 \quad \times y_3 \\
\end{align*}
\]

\[ 2x_1 + 3x_2 \]

So want
\[
\begin{align*}
4y_1 + 2y_2 + 3y_3 & \geq 2 \\
8y_1 + y_2 + 2y_3 & \geq 3
\end{align*}
\]

and minimize \( 12y_1 + 8y_2 + 4y_3 \)

Any solution to this new LP is an upper bound to original LP.
E.g. \( y_1 = 1 \), \( y_2 = 0 \), \( y_3 = 0 \) is \( 12 \)
\( y_1 = \frac{1}{2} \), \( y_2 = 0 \), \( y_3 = 0 \) is \( 6 \)
\( y_1 = \frac{1}{3} \), \( y_2 = \frac{1}{3} \), \( y_3 = 0 \) is \( 5 \)

etc.

Best solution is the best upper bound using these combinations.
So let's solve this problem using CVXOPT, say:-

First the original LP:

```python
cvxopt.install()  # install cvxopt if not already installed
from cvxopt import matrix, solvers
from fractions import Fraction as Frac
A = matrix([[ 4.0, 2.0, 3.0, -1.0, 0.0], [8.0, 1.0, 2.0, 0.0, -1.0]])
b = matrix([12.0, 3.0, 4.0, 0.0, 0.0])
c = matrix([-2.0, -3.0])
sol = solvers.lp(c, A, b)
x = sol['x']
print "x1 = ", Frac(x[0]).limit_denominator()
print "x2 = ", Frac(x[1]).limit_denominator()
print "value = ", Frac(2*x[0] + 3*x[1]).limit_denominator()
```

And then the "dual"

```python
cvxopt.install()  # install cvxopt if not already installed
from cvxopt import matrix, solvers
from fractions import Fraction as Frac
A = matrix([[-4.0, -8.0, -1.0, 0.0, 0.0], [-2.0, -1.0, 0.0, -1.0, 0.0], [-3.0, -2.0, 0.0, 0.0, -1.0]])
b = matrix([-2.0, -3.0, 0.0, 0.0, 0.0])
c = matrix([12.0, 3.0, 4.0])
sol = solvers.lp(c, A, b)
x = sol['x']
print "y1 = ", Frac(x[0]).limit_denominator()
print "y2 = ", Frac(x[1]).limit_denominator()
print "y3 = ", Frac(x[2]).limit_denominator()
print "value = ", Frac(12*x[0] + 3*x[1] + 4*x[2]).limit_denominator()
```

So the "primal" is \((x_1=1/2, \ x_2=5/4)\) and value = \(19/4\)

and the "dual" is \((y_1=5/16, \ y_2=0, \ y_3=1/4)\) and value = \(19/4\)!
Cool: so we can prove that max of the original LP ≤ 3/4 and indeed, that is the actual max value!

Did we get lucky that we "proved" a tight bound on the LP value?
No.

**Theorem [Strong Duality]**

If an LP has a finite optimal solution then the "dual" LP is feasible and the optimal values are same.

This is like the max flow min cut theorem:

- every dual solution is an upper bound on primal
- but only one dual solution (optimal)
  
  = optimal primal solution.

In fact the following possibilities are possible

<table>
<thead>
<tr>
<th>Primal</th>
<th>Infeasible</th>
<th>Feasible &amp; Bounded</th>
<th>Feasible &amp; Unbounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infeasible</td>
<td>✓</td>
<td>✗</td>
<td>✓</td>
</tr>
<tr>
<td>Feas &amp; bad</td>
<td>✗</td>
<td>✓ (Primal = Dual)</td>
<td>✓</td>
</tr>
<tr>
<td>Feas &amp; unbd</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
</tr>
</tbody>
</table>
Couple more things

1. Let's look at the Primal & Dual again:

\[
\begin{align*}
\max & \quad 2x_1 + 3x_2 \\
\text{s.t.} & \quad 4x_1 + 8x_2 \leq 12 \\
& \quad 2x_1 + x_2 \leq 3 \\
& \quad 3x_1 + 2x_2 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\min & \quad 12y_1 + 3y_2 + 4y_3 \\
\text{s.t.} & \quad 4y_1 + 2y_2 + 3y_3 \geq 2 \\
& \quad 8y_1 + y_2 + 2y_3 \geq 3 \\
& \quad y_1, y_2, y_3 \geq 0
\end{align*}
\]

Let's write it as a matrix:

\[
\begin{align*}
\max & \quad \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\text{s.t.} & \quad \begin{bmatrix} 4 & 8 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 12 \\ 3 \\ 4 \end{bmatrix} \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\min & \quad \begin{bmatrix} 12 & 3 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
\text{s.t.} & \quad \begin{bmatrix} 4 & 2 & 3 \\ 8 & 1 & 2 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \\
& \quad y \geq 0
\end{align*}
\]

Hey, if \( \max C^T x \) \( A x \leq b \) \( x \geq 0 \) \( \Rightarrow \) dual \( \min b^T y \) \( A^T y \geq c \) \( y \geq 0 \)

 Completely mechanical transformation!!
2. In fact if you write the max flow LP and take its dual, get the min-cut LP!!
And both views of shortest path are duals of each other.

3. Advantages of duality
   - The biggest thing – gives a proof of optimality (or near optimality)
     - if primal and dual values are close, then both must be close to optimum.
   - Often the dual can give insight into problems
     - see HW problem 5
     - Where can use dual to solve problems.
   - If primal has many variables and few constraints
     => dual is other way around
     Useful if we have a good solver for one but not other.

- Dual tells you which ineqs are important
  - eg have the optimal dual was
    \[ (\frac{5}{16}, 0, \frac{7}{4}) \]
  - The zero says “2nd ineqality not important at optimal point”.
  - And indeed \( 2x_1 + x_2 \leq 3 \) is not tight at opt.
Other Algorithms

1. Interior Point Algorithms:

Diametrically opposite, in a sense, to simplex.
"Boundary is what makes polytope complicated."

So let's stay away from boundary.

Here's one way. Suppose LP is
\[
\begin{align*}
\min & \quad c^T x \\
A x &= b \\
x &\geq 0
\end{align*}
\]

Log barrier function
\[
f(x) = \sum_i \log \left( \frac{1}{x_i} \right)
\]

Note: \( f(x) \to \infty \) if some \( x_i \to 0 \)

So prevents us from going negative.

When \( \eta \) is large just want to solve
\[
\min \ f(x) \\
A x = b
\]

When \( \eta \) is small, we're close to solving \( \min \ c^T x \)

Ax = b
In pictures

\[ \text{solution of } \min f(x) \quad Ax = b \]

as \( \eta \downarrow 0 \)

actual solution of LP

the optimal

solution goes from \( x_0 \rightarrow x_\infty \)

along this “central path”.

Algorithmic idea: Solve for optimal \( x_0 \).

- reduce \( \eta \) slightly
  - solve for new optimal \( x_\eta \)
  - repeat

until very close to OPT. So that can “round”.

Each time: get a primal solution
  and a dual solution

when they are close enough, must be both close to OPT.

In fact that is what CVXOPT uses! 😊