Linear programming

Tuesday, October 28, 2025 1:40 AM

Example: a bakery sells cakes and cookies.

Each cake = \$3 profit, each cookie = \$2

Each cake uses 2 cups of flour, 1 cup of sugar

Each cookie uses 1 " " , 2 " "

Inventory:

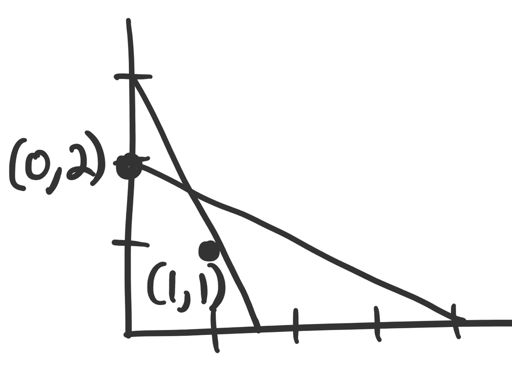
$x = \#$ cakes, $y = \#$ cookies

$\max\ 3x+2y$

s.t. $2x+y \le 3$

$x+2y \le 4$

$x \ge 0,\ y \ge 0$



Integral optimal solution: $(x,y)=(1,1)$, value 5

This is called Integer Programming. NP-hard!

What if units can be fractional (e.g. sell milk vs. juice)

Fractional optimal solution: $(x,y)=(\frac{2}{3},\frac{5}{3})$, value $\frac{16}{3}$

So fractional optimum $\ge$ integer optimum, can be $\ne$.

Theorem: can solve Linear Programming in polynomial time!

Max-Flow: represent each $f(u,v)$ as variable x_{uv}

$$\max \sum_v x_{sv} - \sum_v x_{vs}$$

$$\text{s.t. } \sum_u x_{uv} = \sum_u x_{vu} \quad \forall u$$

$$0 \le x_{uv} \le \underbrace{c_{uv}}_{\text{constant}} \quad \forall (u,v)$$

all linear functions on the variables x_{uv}

Recall from Ford-Fulkerson: if capacities are integral, then max flow is integral.

So integral maximum = fractional maximum.

Such an LP is called integral.

Duality: Recall that max-flow $\le$ min-cut.

In fact, any cut is proof that max-flow $\le$ value of that cut.

In general, the dual of an LP is "proof" that optimum $\le$ __.

Recall the example $\max\ 3x+2y$

s.t. $2x+y \le 3$

$x+2y \le 4$

$x \ge 0,\ y \ge 0$

Suppose I sum the inequalities:

$3x+3y \le 7$

So for any solution (x,y), value $= 3x+2y \le 3x+3y \le 7$.

What if I sum $2\times$ first inequality + second inequality:

$5x+4y \le 10 \quad \times\frac{3}{5}: \ 3x+2.4y \le 6$

So for any solution (x,y), value $= 3x+2y \le 3x+2.4y \le 6$

I can make this systematic:

$\max\ 3x+2y$

s.t. $2x+y \le 3$ $\quad \times a$: multiply by $a \ge 0$

$x+2y \le 4$ $\quad \times b$: multiply by $b \ge 0$

$x \ge 0,\ y\ge 0$

$a(2x+y)+b(x+2y) \le 3a+4b$

$\iff (2a+b)x+(a+2b)y \le 3a+4b$

For this to upper bound all $x \ge 0,\ y \ge 0$, I need

$2a+b \ge 3$

$a+2b \ge 2$

For any satisfying (a,b), value $= 3x+2y \le 3a+4b$

For the tightest bound, I want to minimize $3a+4b$.

The dual LP becomes

$\min\ 3a+4b$

s.t. $2a+b \ge 3$

$a+2b \ge 2$

$a \ge 0,\ b \ge 0$

Weak duality theorem: dual optimum $\ge$ primal optimum.

Strong duality theorem: dual optimum = primal optimum.

Matrix formulation of LPs:

$\max\ 3x+2y$ $\quad \longrightarrow \quad$ $\max \begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}$

s.t. $2x+y \le 3$ $\quad$ s.t. $\begin{bmatrix}2&1\\1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} \ge \begin{bmatrix}3\\4\end{bmatrix}$ (entrywise $\ge$)

$x+2y \le 4$

$x \ge 0,\ y \ge 0$ $\quad$ $\begin{bmatrix}x\\y\end{bmatrix} \ge \begin{bmatrix}0\\0\end{bmatrix}$

In general: $\max\ \vec{c}\cdot\vec{x}$

s.t. $A\vec{x} \le \vec{b}$

$\vec{x} \ge \vec{0}$

if $\vec{a}\cdot\vec{x} \ge b$ constraint, convert to $-\vec{a}\cdot\vec{x} \le -b$.

if $\vec{a}\cdot\vec{x} = b$ constraint, convert to both $\ge, \le$.

Example: flow LP

$$\max \sum_v x_{sv} - \sum_v x_{vs}$$

$$\text{s.t. } \sum_u x_{uv} = \sum_u x_{vu} \leftarrow \text{write as } \begin{cases} \sum_u x_{uv} - \sum_u x_{vu} \le 0 \\ -\sum_u x_{uv} + \sum_u x_{vu} \le 0 \end{cases}$$

$$0 \le x_{uv} \le c_{uv}$$

The dual LP is $\min\ \vec{b}\cdot\vec{y}$

$A^T\vec{y} \ge \vec{c}$

$\vec{y} \ge 0$

Weak Duality Theorem: for any primal solution $\vec{x}$ and dual solution $\vec{y}$,

$\vec{c}\cdot x \le \vec{b}\cdot\vec{y}$.

Proof: From $A^Ty \ge c$ and $x \ge 0$, $c^Tx \le (A^Ty)^Tx = y^TAx$

From $Ax \le b$ and $y \ge 0$, $b^Ty \ge (Ax)^Ty = x^TAy$

are equal

Strong Duality is a deep theorem that generalizes max-flow/min-cut.