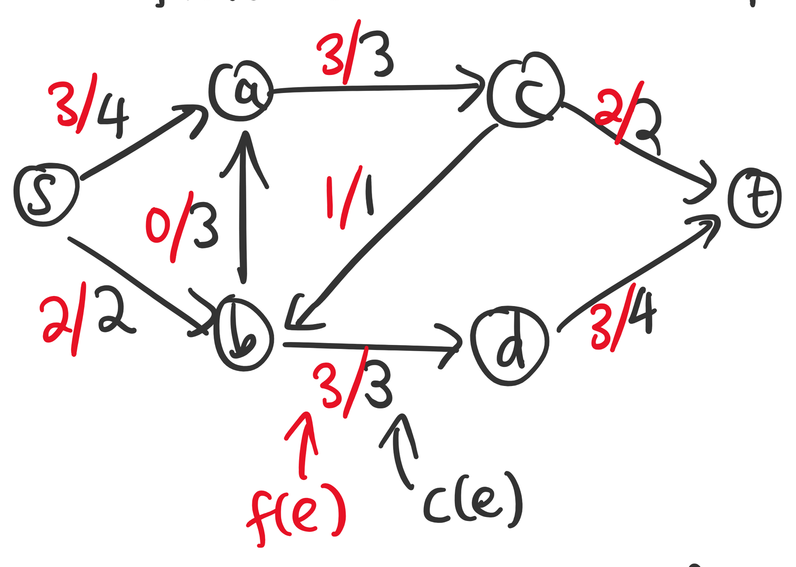
Maximum Flow and Matching

Tuesday, October 21, 2025 12:23 AM

Maximum Flow problem: Given s,t and edge capacities, send as much flow from $s \to t$ as possible.



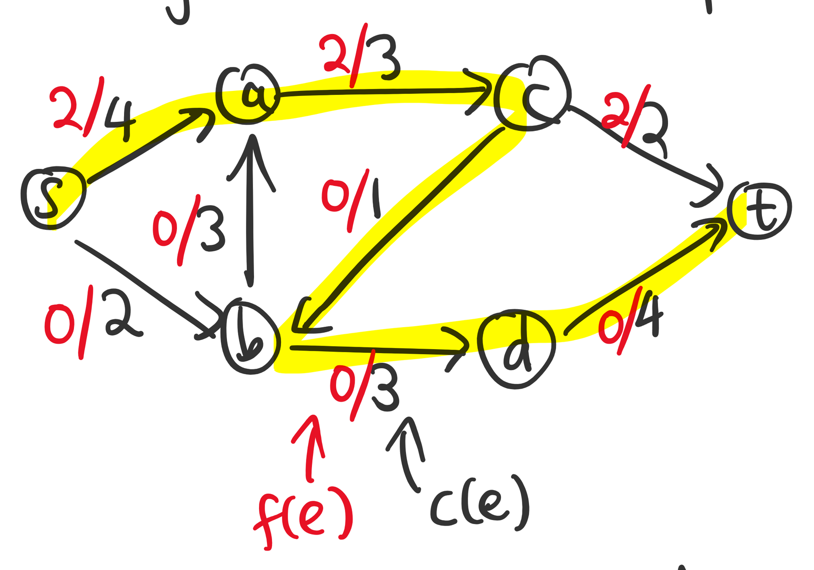
Capacity constraint: $0 \le f(e) \le c(e) \quad \forall e$

Flow conservation: $\sum_u f(u,v) = \sum_u f(v,u) \quad \forall v$

flow in, flow out

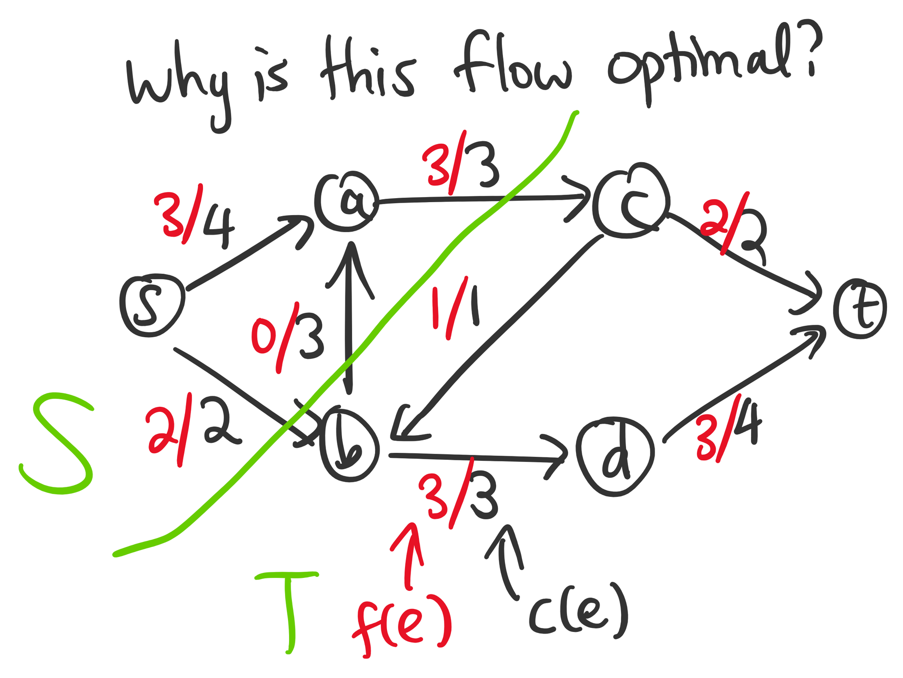
Maximize net flow out of s: $\sum_v f(s,v) - \sum_v f(v,s)$

Optimality condition: Why is this flow sub-optimal?



can find an s-t path of unsaturated edges.

Why is this flow optimal?



Consider an s-t cut (S,T) $\quad (s \in S,\ t \in T,\ T = V \setminus S)$

Any unit of s-t flow must take up ≥ 1 capacity of $S \to T$ edges.

Define $cap(S,T) = \sum_{\substack{u \in S \\ v \in T}} c(u,v)$.

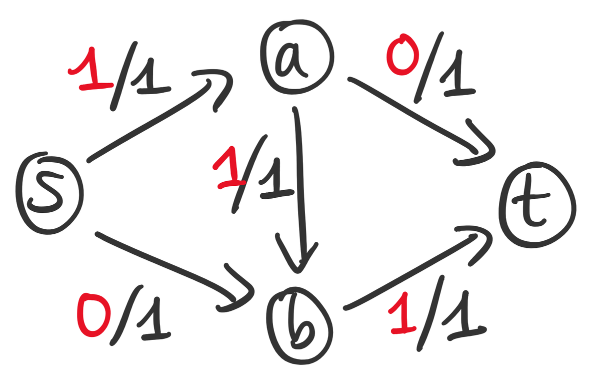
Define $f(S,T) = \sum_{\substack{u \in S \\ v \in T}} f(u,v) - \sum_{\substack{u \in S \\ v \in T}} f(v,u)$, net flow $S \to T$

By flow conservation, net flow out of s = net flow $S \to T$ for any s-t cut (S,T)

Then, $f(S,T) \le$ max-flow $\le cap(S,T)$.

But $f(S,T) = cap(S,T)$, so it's optimal.

Max-flow: greedy algorithm?



We're stuck, but flow is not optimal.

Ford-Fulkerson: allow "undoing" past flows.

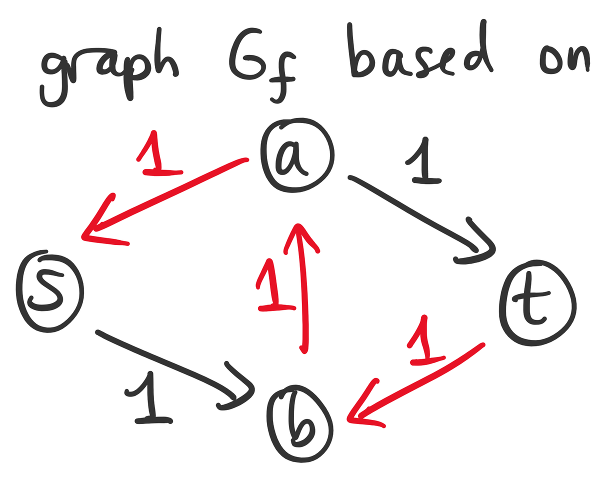
For simplicity, assume $c(u,v)$, $c(v,u)$ defined for all edges (u,v).
(In general, $c(u,v) \ne c(v,u)$, and possibly $c(u,v) = 0$.)

For a given flow f, define the residual capacity c_f as:
for each edge $(u,v) \in E$ with $f(u,v) > 0$, define

$c_f(u,v) = c(u,v) - f(u,v)$ // can still push $c(u,v) - f(u,v)$ forward

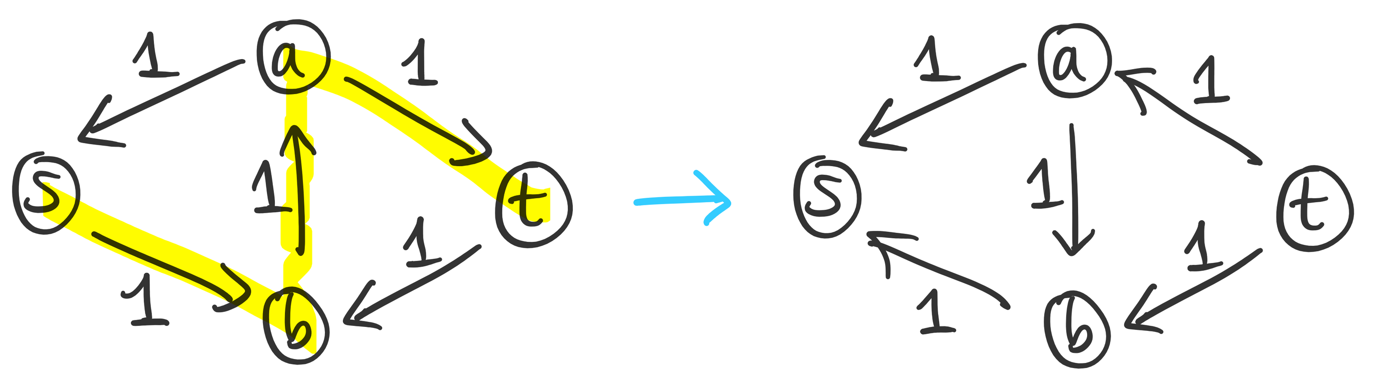
$c_f(v,u) = c(v,u) + f(u,v)$ // can undo $f(u,v)$ flow and still push $c(v,u)$ in reverse

Define the residual graph G_f based on these capacities.



(capacity $\emptyset$ is not drawn)

Ford-Fulkerson algorithm: while exists s-t path of positive residual capacity, add maximum possible flow along that path

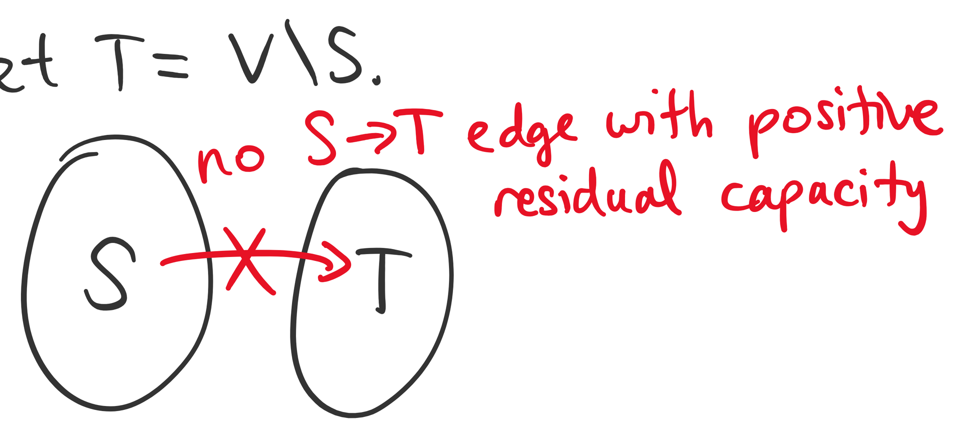


no more s-t path

Analysis: We will find a cut (S,T) s.t. flow $= cap(S,T)$
$\implies$ flow is max-flow.

Let S = vertices reachable from s along edges of positive residual capacity. Since no s-t path, $t \notin S$.

Let $T = V \setminus S$.

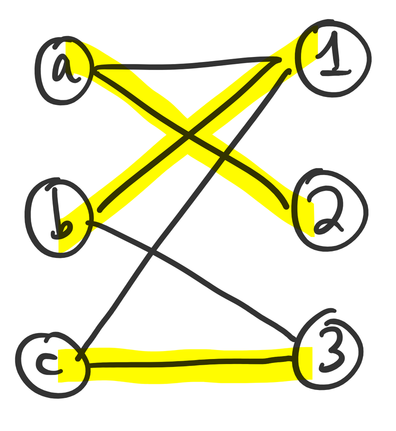


All $S \to T$ edges (u,v) are saturated: $f(u,v) = c(u,v)$

All $T \to S$ edges (v,u) have no flow: $f(v,u) = \emptyset$

So, $f(S,T) = cap(S,T)$ and flow is optimal.

Application: Bipartite Matching



Formulate as s-t flow: